Some Probability Inequalities for the Function of Continuous Parameter Demimartingales

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Abstract—**In this paper, we give some probability inequalities for the function of continuous parameter demimartingales based on probability inequalities for discrete parameter demimartingales and demisubmartingales.**

***Keywords*—continuous parameter demimartingales; continuous parameter demisubmartingales; probability inequalities**

I. Introduction

Notation and conventions. In this paper, let $\{S_n, n \geq 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. $S_0 = 0$, and $I(A)$ be the indicator function of the set A.

The concept of demimartingales was first introduced by Newman and Wright [1].

Definition 1.1 [1] Let $\{S_n, n \geq 1\}$ be an L^1 sequence of random variables. Assume that for $n = 1, 2, \ldots$

$$E[(S_{n+1} - S_n) f(S_1, \ldots, S_n)] \geq 0,$$

for all componentwise nondecreasing functions $f(\cdot)$ such that the expectation is defined. Then $\{S_n, n \geq 1\}$ is called a demimartingale. If in addition the function $f(\cdot)$ is assumed to be nonnegative, the sequence $\{S_n, n \geq 1\}$ is called a demisubmartingale.

After the concept of demimartingales was introduced, many scholars established some interesting conclusions for demimartigales，one can refer to [1-9], which promotes the development of the dependence sequences. For instance, Newman and Wright[1] obtained Doob-type maximal inequalities and upcrossing inequalities for demimartingales. Christofides[2] provided Chow-type maximal inequalities and some properties of demimartingales . Prakasa[3]showed maximal and minimal inequalities for demimartingales in. Hu et al. [4] presented maximal inequalities for stopping time sequences of demimartingales. Wood[5] provided maximal inequalities for separable demimartingales. Subsequently, Hadjikyriakou[10] introduced the definition of demimartingales with continuous parameters.

Definition 1.2 [10] The stochastic process $\{S_t, t \in [0, T]\}$ is called a demimartingale if for all $s, t \in [0, T]$ and for all $s \leq t$,

$$E[(S_t - S_s) f(S_{u_1}, \ldots, S_{u_k}, u_i \leq s, i = 1, \ldots, k)] \geq 0,$$

for all componentwise nondecreasing functions $f(\cdot)$ such that the expectation is defined. Then $\{S_t, t \in [0, T]\}$ is called a continuous parameter demimartingale. If in addition the function $f(\cdot)$ is assumed to be nonnegative, the sequence $\{S_t, t \in [0, T]\}$ is called a continuous parameter demisubmartingale.

In reference [11], Prakasa further provided an alternative definition for continuous parameter demisubmartingales.

Definition1.3[11] Let the process $\{S_t, t \in [0, T]\}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. It is called a demisubmartingale if for $0 = t_0 < t_1 < \ldots, t_k = T, k \geq 1$, the sequence $\{S_{t_j}, j = 0, 1, \ldots, k-1\}$ is a demisubmartingale.

Definition1.4[12] A process $\{S_t, t \in [0, T]\}$ is said to be separable if there is a measurable set B with $P(B) = 0$ and a countable subset $\tau \subseteq [0, T]$ such that for every closed interval $A \subseteq \mathbb{R}$ and any open interval $(a, b) \subseteq [0, T]$, the sets

$$\{\omega : S_t(\omega) \in A, t \in (a, b)\},$$

and

$$\{\omega : S_t(\omega) \in A, t \in (a,b) \cap \tau\},$$

differ at most by a subset of B.

Inspired by references [2] and [5], this paper presents probability inequalities for continuous parameter demisubmartingale functions.

II. MAIN RESULTS

Theorem 2.1 Let stochastic process $\{S_t, t \in [0,T]\}$ be a continuous parameter demimartingale, $g(\cdot)$ be a nondecreasing convex function. Then the stochastic process $\{S_t, t \in [0,\mathrm{T}]\}$ is called a continuous parameter demimartingale.

Proof. we need to prove that $\{g(S_{t_j}), j = 0,1,\cdots,k-1\}$ is a demisubmartingale for any $0 = t_0 < t_1 < \cdots < t_k \in [0,T], k \geq 1$.

Note that $\{S_{t_j}, j = 0,1,\cdots,k-1\}$ is a demisubmartingale by using Definition 1.4.

Define that $h(x) = \lim\limits_{x \to y-0} \dfrac{g(y)-g(x)}{y-x}$

is the left derivative of function $g(\cdot)$. It follows From the convexity of the function $g(\cdot)$ that the function $h(\cdot)$ is a nonnegative and nondecreasing function.

Thus

$$g(y) - g(x) \geq (y-x)h(x). \quad (2.1)$$

It can be verified that for any nonnegative and componentwise nondecreasing function $g(\cdot)$ by using Equation (2.1), where

$$\begin{aligned} & E[(g(S_{t_{j+1}}) - g(S_{t_j}))f(g(S_{t_1}),\cdots,g(S_{t_j}))] \\ & \geq E[(S_{t_{j+1}} - S_{t_j})h(S_{t_j})f(g(S_{t_1}),\cdots,g(S_{t_j}))] \\ & = E[(S_{t_{j+1}} - S_{t_j})f^*(S_{t_1},\cdots,S_{t_j})], \end{aligned}$$

where

$$f^*(S_{t_1},\cdots,S_{t_j}) = h(S_{t_j})f(g(S_{t_1}),\cdots,g(S_{t_j})).$$

Since $f^*(\cdot)$ is a arbitrary non-negative function of nonnegative and componentwise nondecreasing, the sequence $\{S_{t_j}, j = 0,1,\cdots,k-1\}$ is a demisubmartingale.

Thus

$$E[(S_{t_{j+1}} - S_{t_j})f^*(S_{t_1},\cdots,S_{t_j})] \geq 0.$$

Then,

$$E[(g(S_{t_{j+1}}) - g(S_{t_j}))f(g(S_{t_1}),\cdots,g(S_{t_j}))] \geq 0.$$

Hence, the sequence $\{g(S_{t_j}), j = 0,1,\cdots,k-1\}$ is a demisubmartingale.

By Definition 1.4 and the fact that t_j is arbitrary, $\{g(S_t), t \in [0,T]\}$ is a continuous parameter demisubmartingale.

As an application of Theorem 2.1, we can obtain the following conclusion.

Corollary 2.1 Let stochastic process $\{S_t, t \in [0,T]\}$ be a continuous parameter demimartingale, Then $\{S_t^+, t \in [0,T]\}$ is a continuous parameter demimartingale and $\{S_t^-, t \in [0,T]\}$ is also a continuous parameter demisubmartingale.

Proof. Since the function $g(x) = x^+$ is a non-decreasing convex function, by Theorem 2.1, we know that $\{S_t^+, t \in [0,T]\}$ is a continuous parameter demisubmartingale. Let $Y_t = -S_t, t \in [0,T]$. It is easy to see that $\{Y_t, t \geq 1\}$ is a continuous parameter demimartingale. Since $Y_t^+ = S_t^-$, by Theorem 1, we get that $\{S_t^-, t \in [0,T]\}$ is also a continuous parameter demisubmartingale.

Theorem 2.2 Let $\{S_n, n \geq 1\}$ be a demisubmartingale, and $g(\cdot)$ be a nondecreasing convex function. Then, for any $\lambda \in \mathrm{R}$, we have

$$\lambda P(\max_{1 \leq i \leq k} g(S_i) \leq \lambda) > \int_{\{\max_{1 \leq i \leq k} g(S_i) > \lambda\}} g(S_k)dP \quad (2.2)$$

and

$$\begin{aligned} & \lambda P(\min_{1 \leq i \leq k} g(S_i) \leq \lambda) \geq E[g(S_1)] - \int_{\{\min_{1 \leq i \leq k} g(S_i) > \lambda\}} g(S_k)dP \\ & \geq E[g(S_1)] - E|\ g(S_k)|\ . \end{aligned} \quad (2.3)$$

Proof. Let i be the smallest index such that it holds $g(S_i) > \lambda$. When $i = n$, we have $g(S_k) \le \lambda, 1 \le k \le n-1$ and $g(S_n) > \lambda$.

Let $A_k = \{\max_{1\le i\le k} g(S_i) > \lambda\}$.

Then

$$\begin{aligned}\int_{A_k} g(S_k)dP &= \sum_{n=1}^{k}\int_{\{i=n\}} g(S_k)dP \\ &= \sum_{n=1}^{k}[\int_{\{i=n\}} g(S_n)dP + \int_{\{i=n\}}(g(S_k)-g(S_n))dP] \\ &= \sum_{n=1}^{k}\int_{\{i=n\}} g(S_n)dP + \sum_{n=1}^{k-1}\int_{\{i=n\}}(g(S_k)-g(S_n))dP \\ &= \lambda P(A_k) + \\ &\sum_{n=1}^{k-1}\int_\Omega\left[\left(g\left(S\_\{n+1\}\right)-g\left(S\_\{n\}\right)\right)I\left(A\_\{n\}\right)\right]dP.\end{aligned}$$

Define that $h(x) = \lim_{x\to y-0}\frac{g(y)-g(x)}{y-x}$ is the left derivative of function $g(\cdot)$. From the convexity and nondecreasing of the function $g(\cdot)$, it follows that the function $h(\cdot)$ is a nonnegative nondecreasing function, and

$$\int_\Omega[(g(S_{n+1})-g(S_n))I(A_n)]dP \ge \int_\Omega[(S_{n+1}-S_n)h(S_n)I(A_n)]dP.$$

Since $I(A_n)$ is a nonnegative and componentwise nondecreasing function with respect to $\{S_1, \cdots, S_n\}$, $h(S_n)I(A_n)$ is also a nonnegative and coordinate-wise nondecreasing function with respect to $\{S_1, \cdots, S_n\}$. Since $\{S_n, n\ge 1\}$ is a demisubmartingale, we have

$$\int_\Omega[(S_{n+1}-S_n)h(S_n)I(A_n)]dP \ge 0.$$

Thus

$$\int_{A_k} g(S_k)dP \ge \lambda P(A_k).$$

Hence, equation (2.2) is proved.

Let $B_k = \{\min_{1\le i\le k} g(S_i) > \lambda\}$,

and i be the smallest index such that it holds $g(S_i) \le \lambda$. Therefore, by the same reasoning, we can similarly deduce that

$$\int_{B_k^c} g(S_k)dP \le \lambda P(B_k^c) + \sum_{n=1}^{k-1}\int_{B_n^c}(g(S_{n+1})-g(S_n))dP.$$

Thus

$$\lambda P(B_k^c) \ge \int_{B_k^c} g(S_k)dP - \sum_{n=1}^{k-1}\int_{B_n^c}(g(S_{n+1})-g(S_n))dP.$$

Since $h(x)$ is the left derivative of $g(\cdot)$ and $g(\cdot)$ is a nonnegative function with nondecreasing components, we can conclude from the definition of the demisubmartingale that:

$$\begin{aligned}&\sum_{n=1}^{k-1}\int_\Omega\left[\left(g\left(S\{n+1\}\right)-g\left(S\{n\}\right)\right)I\left(B\{n\}\right)\right]dP \\ &\ge \sum_{n=1}^{k-1}\int_\Omega[(S_{n+1}-S_n)h(S_n)I(B_n)]dP \ge 0.\end{aligned}$$

And

$$\begin{aligned}\lambda P(B_k^c) &\ge \int_{B_k^c} g(S_k)dP - \sum_{n=1}^{k-1}\int_{B_n^c}(g(S_{n+1})-g(S_n))dP \\ &\quad -\sum_{n=1}^{k-1}\int_{B_n}(g(S_{n+1})-g(S_n))dP \\ &= \int_{B_k^c} g(S_k)dP - \sum_{n=1}^{k-1}E[g(S_{n+1})-g(S_n)] \\ &= \int_{B_k^c} g(S_k)dP - E[g(S_k)] + E[g(S_1)] \\ &= E[g(S_1)] - \int_{B_k} g(S_k)dP \\ &\ge E[g(S_1)] - E|\ g(S_k)|.\end{aligned}$$

Hence, equation (2.3) is proved.

Theorem 2.3. Let stochastic process $\{S_t, t\in[0,T]\}$ be a separable continuous parameter demisubmartingale and $g(\cdot)$ be a nondecreasing convex function. For any $\lambda\in\mathrm{R}$, let

$$B_T = \{\omega\in\Omega : \inf_{t\in[0,T]} g(S_t) \le \lambda\}$$

and

$$A_T = \{\omega\in\Omega : \sup_{t\in[0,T]} g(S_t) > \lambda\}.$$

Then

$$\lambda P(A_T) \le \int_{A_T} g(S_T)dP \qquad (2.4)$$

and

$$\lambda P(B_T) \ge E[g(S_0)] - E|\ g(S_T)|. \qquad (2.5)$$

Proof. Since $\{S_t, t\in[0,T]\}$ is a demisubmartingale, $\{g(S_t), t\in[0,\mathrm{T}]\}$ is also a demisubmartingale by Theorem 1. Furthermore, from the separability of $\{S_t, t \in [0,T]\}$ and Theorem 2, it is evident that (2.4) and (2.5) hold. Thus, the conclusion is established.

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