The Minimal Inequality and Conditional Moment Inequality for Nonnegative Conditional

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Abstract — **In this paper, we give a class of minimal inequality for nonnegative conditional demisubmartingales, and obtain the conditional moment inequality for nonnegative conditional demimartingales by using the conditional Fubini theorem and the conditional Hölder inequality.**

Keywords—**Conditional demisubmartingale; Conditional moment inequality; Minimal inequality**

I. INTRODUCTION

The partial sums sequence of a conditional associated sequence with zero mean is a conditional demimartingales. For any satisfying $E|Z|<\infty$ random variable Z, we have $E(E(Z|\mathrm{F}))=\mathrm{E}(Z)$ by the property of conditional expectation. Therefore conditional demimartingales and conditional demisubmartingales defined on the probability space $(\Omega,\ \mathrm{A},\ \mathrm{P})$ are demimartingales and demisubmartingales on the probability space $(\Omega,\ \mathrm{A},\ \mathrm{P})$, respectively, but the converse is not true. Christofides and Hadjikyriakou [2] established some maximal inequalities and related results for conditional demimartingales. Wang and Wang [3] further improved these results, and obtained some maximal inequalities for conditional demi(sub)martingales, minimal inequalities and moment inequalities for nonnegative conditional demimartingales. Xing-Hui Wang [4] established some probability inequalities for conditional demimartingales, such as Doob-type inequalities, maximal inequalities based on concave Young functions and maximal $\phi-$ inequalities for nonnegative conditional demimartingales.

Wang and Hu [5] obtained some maximal $\phi-$ inequalities and some maximal inequalities based on concave Young functions for conditional demimartingales. In [6], the γ type probability inequalities for conditional demimartingales were obtained by using the maximal and minimal inequalities and a strong law of large numbers. for conditional demimartingales. Some minimal inequalities for conditional demimartingales and nonnegative conditional demimartingales were given in [7]. Wang et al. [8] established Chow-type maximal inequalities for conditional demimartingales and used them to obtain concave Young functions maximal inequalities for conditional demimartingales. In this paper, we obtain a class of minimal inequalities for nonnegative conditional demisubmartingales, and use the conditional Fubini theorem and the conditional Hölder inequality to obtain conditional moment inequalities for conditional demimartingales. Our conclusions extend the related results in [3].

Notation and conventions. In this paper, let $\{S_n, n\ge 1\}$ be a sequence of random variables defined on the probability space $(\Omega,\ \mathrm{A},\ \mathrm{P})$. Let $E^{\mathrm{F}}X=E[X|\mathrm{F}]$, Where F is a sub-σ algebra of A, and $I(A)$ be the indicator function of the set A.

II. DEFINITION OF CONDITIONAL DEMIMARTINGALES

Definition 1 Let $\{S_n, n\ge 1\}$ be L^1 a sequence of random variables. Assume that for $1\le i\le j<\infty$,

$$E^{\mathrm{F}}\left[\left(S_j-S_i\right)f\left(S_1,S_2,\cdots,S_i\right)\right]\ge 0\ \ a.s.,\ (1.1)$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_n, n\ge 1\}$ is called a conditional demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_n, n\ge 1\}$ is called a conditional demisubmartingale.

It is easy to check that for all $i\ge 1$, equation (1.1) is equivalent to

$$E^{\mathrm{F}}\left[\left(S_{i+1}-S_i\right)f\left(S_1,S_2,\cdots,S_i\right)\right]\ge 0\ \ a.s..$$

III. MAIN RESULTS

Lemma 1[2] Let $\{S_n, n\ge 1\}$ be a conditional demi(sub)martingale, $\mathrm{g}(\cdot)$ a nondecreasing convex function, and $\mathrm{g}(S_n)\in L^1$, $\mathrm{n}\ge 1$, then the sequence of random variables $\{\mathrm{g}(S_n), n\ge 1\}$ is a conditional demisubmartingales.

Theorem 1 Let $\{S_n, n\ge 1\}$ be a nonnegative conditional demisubmartingale and assume that $\{c_i, i\ge 1\}$ is a nondecreasing sequence of F-measurable positive numbers. For any F-measurable random variable $\varepsilon>0$ a.s., then

$$\begin{aligned} &eP^{\mathrm{F}}\left(\min_{1\le i\le n} c_iS_i\le e\right)\ge \\ &c_1E^{\mathrm{F}}S_1-c_nE^{\mathrm{F}}\left(S_nI\left(\min_{1\le i\le n}c_iS_i>e\right)\right)\ a.s.. \end{aligned} \quad (1)$$

Proof. Let $A=\left\{\min_{1\le i\le n}c_iS_i\le\varepsilon\right\}$,

where $A_1=\{c_1S_1\le\varepsilon\}$,

$A_i = \{c_k S_k > \varepsilon,\ 1 \le k \le i-1,\ c_i S_i \le \varepsilon\}, \quad 2 \le i \le n,$ and $A_i \cap A_j = \varnothing,\ i \ne j.$ Note that $I(A_2) = I(A_1^c) - I(A_1^c A_2^c)$ since $A_2 \subset A_1^c$. Thus

$$eP^{\mathcal{F}}(A) = e\sum_{i=1}^{n} P^{\mathcal{F}}(A_i)$$
$$= \sum_{i=1}^{n} E^{\mathcal{F}}(eI(A_i)) \ge \sum_{i=1}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$= c_1 E^{\mathcal{F}} S_1 - c_1 E^{\mathcal{F}}(S_1 I(A_1^c)) + \sum_{i=2}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$\ge c_1 E^{\mathcal{F}} S_1 - c_2 E^{\mathcal{F}}(S_1 I(A_1^c)) + \sum_{i=2}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$= c_1 E^{\mathcal{F}} S_1 - c_2 E^{\mathcal{F}}(S_1 I(A_1^c))$$
$$+ c_2 E^{\mathcal{F}}(S_2 I(A_2)) + \sum_{i=3}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$= c_1 E^{\mathcal{F}} S_1 + c_2 E^{\mathcal{F}}((S_2 - S_1) I(A_1^c))$$
$$- c_2 E^{\mathcal{F}}(S_2 I(A_1^c A_2^c)) + \sum_{i=3}^{n} E^{\mathcal{F}}(c_i S_i I(A_i)) \quad a.s..$$

Since $A_1 = \{c_1 S_1 \le \varepsilon\}, A_1^c = \{c_1 S_1 > \varepsilon\}, I(A_1^c)$ is a nondecreasing functional of S_1. By the definition of conditional demimartingales, we have

$$c_2 E^{\mathcal{F}}((S_2 - S_1) I(A_1^c)) \ge 0 \quad a.s..$$

Note that $I(A_3) = I(A_1^c A_2^c) - I(A_1^c A_2^c A_3^c)$ since $A_2 \subset A_1^c$. Thus,

$$eP^{\mathcal{F}}(A) \ge c_1 E^{\mathcal{F}} S_1 - c_2 E^{\mathcal{F}}(S_2 I(A_1^c A_2^c)) + \sum_{i=3}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$\ge c_1 E^{\mathcal{F}} S_1 - c_3 E^{\mathcal{F}}(S_2 I(A_1^c A_2^c)) + \sum_{i=3}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$= c_1 E^{\mathcal{F}} S_1 - c_3 E^{\mathcal{F}}(S_2 I(A_1^c A_2^c))$$
$$+ c_3 E^{\mathcal{F}}(S_3 I(A_3)) + \sum_{i=4}^{n} E^{\mathcal{F}}(c_i S_i I(A_i))$$
$$= c_1 E^{\mathcal{F}} S_1 + c_3 E^{\mathcal{F}}((S_3 - S_2) I(A_1^c A_2^c))$$
$$- c_3 E^{\mathcal{F}}(S_3 I(A_1^c A_2^c A_3^c)) + \sum_{i=4}^{n} E^{\mathcal{F}}(c_i S_i I(A_i)) \ a.s..$$

Observe that
$A_2^c = \{c_1 S_1 \le \varepsilon\} \cup \{c_2 S_2 > \varepsilon\}$ and $A_1^c A_2^c = \{c_1 S_1 > \varepsilon, c_2 S_2 > \varepsilon\}$.
Ii is easy to verify that $I(A_1^c A_2^c)$ is a componentwise nondecreasing function with respect to $\{S_1, S_2\}$. By the definition of conditional demimartingales, we have

$$c_3 E^{\mathcal{F}}((S_3 - S_2) I(A_1^c A_2^c)) \ge 0 \quad a.s..$$

By iterations, we get

$$eP^{\mathcal{F}}(A) \ge c_1 E^{\mathcal{F}} S_1 - c_n E^{\mathcal{F}}(S_n I(A_1^c A_2^c \cdots A_n^c))$$
$$= c_1 E^{\mathcal{F}} S_1 - c_n E^{\mathcal{F}}(S_n I(\min_{1 \le i \le n} c_i S_i \ge e)) \ a.s..$$

The equation (1) is obtained and the proof is over.
According to Lemma 1, we have the following corollary.

Corollary 1 Let $\{S_n, n \ge 1\}$ be a nonnegative conditional demimartingale, assume $\{c_i, i \ge 1\}$ is a non-decreasing sequence of $\mathcal{F}$-measurable positive numbers, and $g(\cdot)$ is a non-decreasing convex function. For any $\mathcal{F}$-measurable random variable $\varepsilon > 0$ $a.s.$, then

$$eP^{\mathcal{F}}\left(\min_{1 \le i \le n} c_i g(S_i) \le e\right) \ge c_1 E^{\mathcal{F}} g(S_1) - c_n E^{\mathcal{F}}\left(g(S_n) I\left(\min_{1 \le i \le n} c_i g(S_i) > e\right)\right) \ a.s.. \quad (2)$$

Taking $c_k \equiv 1,\ k \ge 1$ and $g(x) = x$ in Corollary 1, we can get the following corollary.

Corollary 2 Let $\{S_n, n \ge 1\}$ be a nonnegative conditional demimartingale. For any $\mathcal{F}$ measurable random variable $\varepsilon > 0$ $a.s.$, then

$$eP^{\mathcal{F}}\left(\min_{1 \le i \le n} S_i \le e\right) \ge E^{\mathcal{F}}\left(S_n I\left(\min_{1 \le i \le n} S_i \le e\right)\right) \ a.s.. \quad (3)$$

Corollary 3 Let $\{S_n, n \ge 1\}$ be a nonnegative conditional demisubmartingale. For any $\mathcal{F}$-measurable random variable $\varepsilon > 0$ $a.s.$, then

$$eP^{\mathcal{F}}\left(\min_{1 \le i \le n} S_i \le e\right) \ge E^{\mathcal{F}}\left(S_n I\left(\min_{1 \le i \le n} S_i \le e\right)\right) \ a.s..$$

Remark 1 Corollary 3 is similar to Theorem 3.3 in [3]. Therefore Theorem 1 in this paper extends Theorem 3.3 in [3].

Theorem 2 let $\{S_n, n \ge 1\}$ be a nonnegative conditional demimartingales and suppose $S_1 \equiv 1$. For any $n \ge 1$, then

$$E^{\mathcal{F}}\left(\min_{1 \le i \le n} S_i\right)^p \le \left(\frac{p}{p-1}\right)^p E^{\mathcal{F}}(S_n)^p \ a.s., p > 1, \quad (4)$$

$$E^{\mathcal{F}}\left(\min_{1 \le i \le n} S_i\right) \le 1 + E^{\mathcal{F}}(S_n \log(S_n)) \ a.s.. \quad (5)$$

Proof. Obviously, $0 \le \min_{1 \le i \le n} S_i \le S_1 = 1$, according to equation (3), Fubini theorem and Hölder inequality, we have

$$
\begin{aligned}
E^{\mathrm{F}}\left(\min_{1\le i\le n} S_i\right)^p &= p\int_0^{\infty} x^{p-1}P^{\mathrm{F}}\left(\min_{1\le i\le n} S_i \ge x\right)dx \\
&= p\int_0^{1} x^{p-1}P^{\mathrm{F}}\left(\min_{1\le i\le n} S_i \ge x\right)dx \\
&= p\int_0^{1} x^{p-1}dx - p\int_0^{1} x^{p-1}P^{\mathrm{F}}\left(\min_{1\le i\le n} S_i < x\right)dx \\
&\le 1 - p\int_0^{1} x^{p-2}E^{\mathrm{F}}\left(S_n I\left(\min_{1\le i\le n} S_i < x\right)\right)dx \\
&= 1 - pE^{\mathrm{F}}\left[S_n\int_0^{1} x^{p-2} I\left(\min_{1\le i\le n} S_i < x\right)dx\right] \\
&= 1 - pE^{\mathrm{F}}\left[S_n\int_{\min_{1\le i\le n} S_i}^{1} x^{p-2}dx\right] \\
&= 1 - \frac{p}{p-1}E^{\mathrm{F}}\left(S_n\right) + \frac{p}{p-1}E^{\mathrm{F}}\left(S_n\left(\min_{1\le i\le n} S_i\right)^{p-1}\right) \\
&\le \frac{p}{p-1}E^{\mathrm{F}}\left(S_n\left(\min_{1\le i\le n} S_i\right)^{p-1}\right) \\
&= \frac{p}{p-1}\left[E^{\mathrm{F}}\left(S_n\right)^p\right]^{\frac{1}{p}}\left[E^{\mathrm{F}}\left(\min_{1\le i\le n} S_i\right)^p\right]^{1-\frac{1}{p}} \quad a.s.,
\end{aligned}
$$

Thus

$$
\left[E^{\mathrm{F}}\left(\min_{1\le i\le n} S_i\right)^p\right]^{\frac{1}{p}} \le \frac{p}{p-1}\left[E^{\mathrm{F}}\left(S_n\right)^p\right]^{\frac{1}{p}} \quad a.s..
$$

Therefore, equation (4) follows from the above inequality.

Note that

$$
\begin{aligned}
E^{\mathrm{F}}\left(\min_{1\le i\le n} S_i\right) &= \int_0^{\infty} P^{\mathrm{F}}\left(\min_{1\le i\le n} S_i \ge x\right)dx \\
&= \int_0^{1} P^{\mathrm{F}}\left(\min_{1\le i\le n} S_i \ge x\right)dx \\
&= 1 - \int_0^{1} P^{\mathrm{F}}\left(\min_{1\le i\le n} S_i < x\right)dx \\
&\le 1 - \int_0^{1} \frac{1}{x}E^{\mathrm{F}}\left(S_n I\left(\min_{1\le i\le n} S_i < x\right)\right)dx \\
&= 1 - E^{\mathrm{F}}\left[S_n\int_{\min_{1\le i\le n} S_i}^{1} \frac{1}{x}dx\right] \\
&= 1 + E^{\mathrm{F}}\left(S_n \log\left(\min_{1\le i\le n} S_i\right)\right) \\
&\le 1 + E^{\mathrm{F}}\left(S_n \log\left(S_n\right)\right) \quad a.s..
\end{aligned}
$$

Then equation (5) is obtained and the proof is over.

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