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Position-dependent Quantum Walks Based on Rigged Hilbert Space

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Abstract—**In this paper, we discuss how to construct the rigged Hilbert space of the time evolution operator of the position-dependent quantum walks, and obtain the properties of the time evolution operator on the rigged Hilbert space.**

***Keywords*—Rigged Hilbert space; Quantum walk**

I. Introduction

Recently, position-dependent quantum walks has attracted extensive attention (see, e.g., [1-3]). In 2015, Asch et al. [4] proved spectral stability and propagation properties for general asymptotically uniform models by means of unitary Mourre theory, and proved the existence of singular continuous spectrum of the time evolution operator. In 2016, Suzuki [5] proved that the position-dependent Heisenberg operator converges to the asymptotic velocity operator. Richard and Suzuki [6,7] obtained the weak limit theorem of quantum walks based on the new exchange method of independent unitary operators in two-Hilbert spaces setting. Canter et al. [8] and Segawa-Suzuki [5,9] proved that if the initial state has an overlap with an eigenspace of the time evolution operator, then the associated quantum walks has localization.

In quantum walks model, the eigenvalues of time evolution operators play an important role in localization. For the unitary time evolution operator $\mathbb{U}$ of one-dimensional two-state quantum walk, its eigenvalues and eigenspaces are defined as follows. If equation $\mathbb{U}\psi = e^{i\theta}\psi,\ \theta \in [0,2\pi)$ has nontrivial solution $\psi \in l^2(\mathbb{Z},\mathbb{C}^2)$, then $e^{i\theta}$ is called the eigenvalue of $\mathbb{U}$. Moreover, the eigenspace $\mathcal{E}(\theta)$ formed by the eigenvector is a subspace of $\psi \in l^2(\mathbb{Z},\mathbb{C}^2)$.

In 2019, Segawa and Morioka [13] obtained a detection method for edge defects by embedded eigenvalues of its time evolution operator. Using Sobolev and Besov spaces, Morioka Hden [12] studied the spectral analysis and scattering theory of the time evolution operator of position-dependent quantum walks, and constructed its generalized eigenfunction. Recently, Maeda M et al. [10] proposed a generalized eigenfunction problem for a quantum walk depending on position, as follows

$$\mathcal{U}\varphi = e^{i\lambda}\varphi,\ \ \lambda \in \mathbb{C} / 2\pi\mathbb{Z}, \tag{1.1}$$

where $\varphi : \mathbb{Z} \to \mathbb{C}^2$. Roughly speaking, the generalized eigenfunction is not in $l^2(\mathbb{Z},\mathbb{C}^2)$, but in $l^2(\mathbb{Z},\mathbb{C}^\infty)$. This is a generalization of tunneling solutions of the discrete time quantum walks given in [11]. Böhm [15] proposed that if the time evolution operator has continuous spectrum, the rigged Hilbert space can be used to describe the generalized eigenfunction space. In 2013, Liu and Huang [14] et al. studied the generalized eigenvector expansion of the Liouville operator and constructed the corresponding rigged Liouville space. Motivated by [14], we discuss how to construct the suitable rigged Hilbert space for a given quantum walk and examine properties of its evolution operator accordingly.

II. Preliminaries

This section briefly recalls some necessary concepts and facts about position-dependent quantum walks as well as Gel'fand triples.

A. Position-Dependent Quantum Walks

Let $l^2(\mathbb{Z},\mathbb{C}^2)$ be the space of square summable function defined on the integer set $\mathbb{Z}$ and valued in $\mathbb{C}^2$, namely

$$l^2(\mathbb{Z},\mathbb{C}^2) = \{\phi : \mathbb{Z} \to \mathbb{C}^2 \mid \textstyle\sum_{x\in\mathbb{Z}} ||\phi(x)||^2 < \infty\}. \tag{2.1}$$

Its inner product $\langle\cdot,\cdot\rangle_{l^2(\mathbb{Z},\mathbb{C}^2)}$ is given by

$$\langle\phi,\psi\rangle_{l^2(\mathbb{Z},\mathbb{C}^2)} = \textstyle\sum_{x\in\mathbb{Z}}\langle\phi(x),\psi(x)\rangle,\ \ \phi,\psi \in l^2(\mathbb{Z},\mathbb{C}^2). \tag{2.2}$$

We use $||\cdot||_{l^2(\mathbb{Z},\mathbb{C}^2)}$ to represent the norm generated by the inner product $l^2(\mathbb{Z},\mathbb{C}^2)$. Moreover, its orthonormal basis is $\{\phi_{x,j} \mid x \in \mathbb{Z}, j = 1,2\}$, where $\phi_{x,j}$ is the function defined by $\phi_{x,j}(z) = \delta_x(z)\, e_j, z \in \mathbb{Z}$, with $\{e_j \mid j = 1,2\}$ being the canonical orthonormal basis of $\mathbb{C}^2$ and δ_z denotes Dirac symbolic function at x defined by

$$\delta_x(z) = \begin{cases} z, & z = x; \\ 0, & z \neq x,\ z \in \mathbb{Z}. \end{cases} \tag{2.3}$$

For position-dependent quantum walks, let $l^2(\mathbb{Z},\mathbb{C}^2)$ be the space of state, the time evolution operator $\mathcal{U}$ is given by

$$[\mathcal{U}\phi](x) = P(x+1)\phi(x+1) + Q(x-1)\phi(x-1),\quad x \in \mathbb{Z},$$

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(2.4)
where $\phi \in l^2(\mathbb{Z}, \mathbb{C}^2)$ and

$$P(x) = \begin{bmatrix} a(x) & b(x) \\ 0 & 0 \end{bmatrix},\ Q(x) = \begin{bmatrix} 0 & 0 \\ c(x) & d(x) \end{bmatrix}.$$

Here we assume $C(x)) := P(x) + Q(x) \in U(2)$ for every $x \in \mathbb{Z}$, where $U(2)$ is the set of all 2×2 unitary matrices. We denote by C the operator of multiplication by $C(x)$ for each $x \in \mathbb{Z}$, i.e., $(C\phi)(x) = C(x)\phi(x)$ for $\phi \in l^2(\mathbb{Z}, \mathbb{C}^2)$. we call C the coin operator. The operator $\mathcal{U}$ is written by $\mathcal{U} = SC$ where $S: l^2(\mathbb{Z}, \mathbb{C}^2) \to l^2(\mathbb{Z}, \mathbb{C}^2)$ is the shift operator defined by

$$S(x) = \begin{bmatrix} T_+ & 0 \\ 0 & T_- \end{bmatrix},\ (T_\pm\phi)(x) := \phi(x \mp 1).$$

By definition, S and C preserve the l^2 norm, and so does $\mathcal{U}$. If the initial state is $\phi_0 \in \mathcal{H}$, then the position related state at time $t \in \mathbb{Z}$ is $\phi(t,\cdot) = \mathcal{U}^t\phi_0$.

Dirac creates the Dirac bracket system to make the laws of quantum mechanics appear in the form of operators, quantum mechanics [17] can be presented to us in a more concise and universal form. The following mathematical principles must be satisfied [18]:

(1) Let A is a self-adjoint operator, then for $\forall \lambda \in \sigma(A)$, there exists an "eigenvector" F_λ such that

$$AF_\lambda = \lambda F_\lambda. \qquad (2.5).$$

(2) Every wave function φ can be expanded with the "eigenvectors"

$$\varphi = \int_{\sigma(A)} \langle \lambda, \varphi \rangle F_\lambda \, d\lambda. \qquad (2.6).$$

(3) "eigenvectors" are orthogonal, i.e.,

$$\langle \lambda, \lambda' \rangle = \delta(\lambda - \lambda'). \qquad (2.7).$$

Remark 2.1 The above description is only a special case. in general, differing with a weighted function $\mu(\lambda)$ is allowed, i,e.,

(2') Every wave function φ can be expanded with "eigenvectors"

$$\varphi = \int_{\sigma(A)} \langle \lambda, \varphi \rangle F_\lambda \, d\mu(\lambda). \qquad (2.8).$$

(3') "eigenvectors" are orthogonal, i.e.,

$$\langle \lambda, \lambda' \rangle d\mu(\lambda') = \delta(\lambda - \lambda')d\lambda'. \qquad (2.9).$$

The expanding form of a vector as (2.6) or (2.8) is called the generalized eigenvector expansions.

B. Gel'fand Triples and The Gel'fand-Maurin Theorem

Definition 2.1 [16] Let $\mathcal{H}$ be a Hilbert space and Φ a subspace of $\mathcal{H}$, if there exist countably monotone inner products on Φ, which are continuous with respect to the original inner product on $\mathcal{H}$, and Φ is complete with respect to the topology ι_Φ decided by the countable inner products, then we call Φ a countable Hilbert space.

Definition 2.2 [16] Suppose $\Phi \subset \mathcal{H}$ is a countable Hilbert space, the countably monotone inner products are

$$\langle\cdot,\cdot\rangle_1,\ \langle\cdot,\cdot\rangle_2, \cdots\ \langle\cdot,\cdot\rangle_n, \cdots$$

Let Φ_n be the completion of Φ respected to the inner product $\langle\cdot,\cdot\rangle_n$, if $\forall$ m, there exists, an $n > m$ such that the embedded map $T_{mn}: \Phi_n \to \Phi_m$ is nuclear, i.e., T_{mn} have the form

$$T_{mn}\varphi = \sum_{k=1}^{\infty} \lambda_k \langle \varphi_k, \varphi \rangle_n \psi_k, \quad \forall\, \varphi \in \Phi_n,$$

where $\{\varphi_k\}$ and $\{\psi_k\}$ are the orthogonal bases for Φ_n and Φ_m respectively, and the series $\sum_{k=1}^{\infty} \lambda_k$ is convergent with $\lambda_k \geq 0$, then we call Φ a nuclear space. Furthermore, we call $\Phi \subset \mathcal{H} \subset \Phi^*$ a Gel'fand triple or rigged Hilbert space, where Φ^* is the conjugate dual space of Φ with respect to the topology ι_Φ.

Lemma 2.1 [21] Let A be an operator in the nuclear space Φ, and Φ is invariant under A, then A is continuous with respect to the topology ι_Φ, that is, $\forall$ p, $\exists$ m > p, satisfying

$$||A\varphi||_p \leq M||\varphi||_m, \quad \forall\, \varphi \in \Phi, \qquad (2.10)$$

where $M = M(p, m)$.

Let A be a self-adjoint operator on the Hilbert space $\mathscr{H}$. If A has no continuous spectrum, then for $\forall \lambda \in \sigma_p(A)$, "eigenvector" F_λ is just the corresponding eigenvector, i.e., $F_\lambda \in \mathscr{H}$. Obviously, F_λ satisfies (2.5) and (2.7), and all the eigenvectors compose an orthogonal basis for the Hilbert space, i.e., (2.6) holds. Therefore, if A has only eigenvalues, then the Hilbert space is enough for studying the operator A.

However, according to document [12], when $\mathcal{U}$ has continuous spectrum, $\forall \lambda \in \sigma_c(\mathcal{U})$, we need to know what kind of space that the corresponding "eigenvector" locate.

Theorem 2.1 (Gel'fand – Maurin Theorem) [16] Let $\Phi \subset \mathcal{H} \subset \Phi^*$ be a rigged Hilbert space and let $\mathscr{H} \to L^2(\vartheta)$, $\varphi \to \varphi(\lambda)$ be an isometric isomorphism of $\mathscr{H}$ with the L^2 function space of a measure space $(\mathbb{R}, \vartheta)$, then for $\forall \lambda \in \mathbb{R}$, we can construct a continuous conjugate linear functional F_λ on Φ^*. Then we have

$$\varphi(\lambda) = F_\lambda(\varphi), \quad \varphi \in \Phi.$$

In rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^*$, $\Phi \subset \mathcal{D}(A)$, and for $\forall \lambda \in \sigma_c(A)$, we can find a corresponding "eigenvector" $F_\lambda \in \Phi^*$, which could be a generalized eigenvector. There is a vector $F \in \Phi^*$, which is called the generalized eigenvector corresponding to $\lambda \in \sigma(A)$. For $\forall \phi \in \Phi$, we have

$$F(A\varphi) = \lambda F(\varphi),$$

and F is denoted F_λ.

Theorem 2.2 (A special case of the Gel'fand – Maurin Theorem) [16,18] Let $\Phi \subset \mathcal{H} \subset \Phi^*$ be rigged Hilbert space, $A \in \mathscr{H}$ be a self-adjoint operator on $\mathscr{H}$, $\Phi \subset \mathcal{D}(A)$, and A has a cyclic vector, then $\forall \lambda \in \sigma_c(A)$, there exists a generalized eigenvector F_λ such that

$$A^*F_\lambda = \lambda F_\lambda,$$

i.e.,

$$\langle A\varphi, F_\lambda\rangle = \langle \varphi, A^*F_\lambda\rangle = \lambda\langle \varphi, F_\lambda\rangle, \quad \forall\, \varphi \in \Phi,$$

moreover, there exists a unique positive measure $d\mu(\lambda)$ on $\sigma_c(A)$ such that

$$\langle \varphi, \psi\rangle = \int_{\sigma(A)}\langle \varphi, \lambda\rangle\langle \lambda, \psi\rangle\, d\mu(\lambda). \tag{2.11}$$

If $\Phi \subset \mathcal{H} \subset \Phi^*$ is a rigged Hilbert space, where $\Phi \subset \mathcal{D}(A)$ and A has a cyclic vector, then for $\forall\, \lambda \in \sigma_c(A)$, $F_\lambda \in \Phi^*$ is not a usual eigenvector, but a generalized eigenvector. By Theorem 2.2, it is easy to prove that F_λ satisfies (2.4), (2.8) and (2.9).

Remark 2.2 If A doesn't have a cyclic vector, similarly, we can construct the generalized eigenvector expansions as (2.5), (2.8) and (2.9). But the technique of continuous direct sum [16] or direct integrals [19] of Hilbert spaces is needed.

III. Main Results

By using a specific orthonormal basis for the spaces $l^2(\mathbb{Z}, \mathbb{C}^2)$, we construct a sequence of dense subspaces $\mathcal{S}_p(\mathbb{Z}, \mathbb{C}^2)$, defined as follows

$$\mathcal{S}_p(\mathbb{Z}, \mathbb{C}^2) = \{\varphi: \mathbb{Z} \to \mathbb{C}^2 \,|\, \sum_{x\in\mathbb{Z}}(1 + |x|)^{2p}||\phi(x)||^2_{\mathbb{C}^2} < \infty\}, \tag{3.1}$$

where integer $p \geq 0$. In addition, it is easy to prove that $\mathcal{S}_p(\mathbb{Z}, \mathbb{C}^2)$ forms a complex Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_p$, where the inner product $\langle\cdot,\cdot\rangle_p$ is given by

$$\langle \varphi, \psi\rangle_p = \sum_{x\in\mathbb{Z}}(1 + |x|)^{2p}\langle \varphi(x), \psi(x)\rangle_{\mathbb{C}^2}, \quad \varphi, \psi \in \mathcal{S}_p(\mathbb{Z}, \mathbb{C}^2). \tag{3.2}$$

We use $||\cdot||_p$ the corresponding norm, which satisfies the following relationship

$$||\varphi||^2_p = \sum_{x\in\mathbb{Z}}(1 + |x|)^{2p}||\varphi(x)||^2_{\mathbb{C}^2}, \quad \varphi \in \mathcal{S}_p(\mathbb{Z}, \mathbb{C}^2). \tag{3.3}$$

In the following, in order to the convenience of expression, we simply denote by $\mathcal{S}_p$ the Hilbert space $\mathcal{S}_p(\mathbb{Z}, \mathbb{C}^2)$.

Proposition 3.1 For $p \geq 0$, one has $\{\phi_{x,j} \,|\, x \in \mathbb{Z}, j = 1, 2\} \subset \mathcal{S}_p$ and moreover the system $\{(1 + |x|)^{-p}\phi_{x,j} \,|\, x \in \mathbb{Z},\ j = 1, 2\}$ forms an orthonormal basis for $\mathcal{S}_p$.

Proof. For $x \in \mathbb{Z}$, $j = 1, 2$, a direct calculation gives

$$||\phi_{x,j}||_p = \sum_{y\in\mathbb{Z}}(1 + |y|)^{2p}||\phi_{x,j}(y)||^2_{\mathbb{C}^2}$$
$$= \sum_{y\in\mathbb{Z}}(1 + |y|)^{2p}||\delta_x(y)e_j||^2_{\mathbb{C}^2}$$
$$= (1 + |x|)^{2p} < \infty,$$

which means that $\phi_{x,j} \in \mathcal{S}_p$. Since $\{\phi_{x,j} | x \in \mathbb{Z}, j = 1, 2\}$ is an orthonormal system of $l^2(\mathbb{Z}, \mathbb{C}^2)$, then $\{(1 + |x|)^{-p}\phi_{x,j} | x \in \mathbb{Z},\ j = 1, 2\}$ is orthogonal in $\mathcal{S}_p$. we have

$$\langle(1 + |x|)^{-p}\phi_{x,j}, \varphi\rangle_p$$
$$= \sum_{y\in\mathbb{Z}}(1 + |y|)^{2p}\langle(1 + |x|)^{-p}\phi_{x,j}(y), \varphi(y)\rangle$$
$$= (1 + |x|)^p\langle\phi_{x,j}, \varphi\rangle, \quad \varphi \in \mathcal{S}_p.$$

So, if $\varphi \in \mathcal{S}_p$ satisfies that $\langle(1 + |x|)^{-p}\phi_{x,j}, \varphi\rangle_p = 0$, for all $x \in \mathbb{Z}$, $j = 1, 2$, then it has $\langle\phi_{x,j}, \varphi\rangle = 0$. It implies that $\varphi = 0$ because the system $\{\phi_{x,j} \,|\, x \in \mathbb{Z}, j = 1, 2\}$ is an orthonormal basis $l^2(\mathbb{Z}, \mathbb{C}^2)$. Thus $\{(1 + |x|)^{-p}\phi_{x,j} \,|\, x \in \mathbb{Z},\ j = 1, 2\}$ is an orthonormal basis $\mathcal{S}_p$.

It is easy that $(1 + |x|)^{2p} \geq 1$ for all $x \in \mathbb{Z}$. This implies that $\mathcal{S}_p \subset \mathcal{S}_q$ and $||\cdot||_p \leq ||\cdot||_q$ whenever $0 \leq p \leq q$. Thus we actually get a sequence of complex Hilbert spaces:

$$\cdots \subset \mathcal{S}_{p+1} \subset \mathcal{S}_p \cdots \subset \mathcal{S}_1 \subset \mathcal{S}_0 = l^2(\mathbb{Z}, \mathbb{C}^2).$$

We put

$$\mathcal{S} = \cap_{p=0}^{\infty}\mathcal{S}_p,$$

and endow $\mathcal{S}$ with the topology $\iota_{\mathcal{S}}$ generated by the norm sequence $\{||\cdot||_p\}_{p\geq 0}$. Note that, for each $p \geq 0$, $\mathcal{S}_p$ is exactly the complete space of $\mathcal{S}$ with respect to $||\cdot||_p$. Thus $\mathcal{S}$ is a countable Hilbert space. In addition, according to the definition of $\mathcal{U}$, we can get $\mathcal{S} \subset \mathcal{D}(\mathcal{U})$. Furthermore, the following proposition shows that $\mathcal{S}$ has a better property.

Proposition 3.2 The space $\mathcal{S}$ is a nuclear space, namely, for any $p \geq 0$, there exists $q > p$, such that the inclusion mapping $T_{pq}: \mathcal{S}_q \to \mathcal{S}_p$ defined by $T_{pq}(\varphi) = \varphi$ is a Hilbert-Schmidt operator.

Proof. Let $p \geq 0$ then there exists $q > p$ such that $2(q - p) > 1$. By Proposition 3.1, $\{(1 + |x|)^{-q}\phi_{x,j} \,|\, x \in \mathbb{Z},\ j = 1, 2\}$ is an orthonormal basis for $\mathcal{S}_q$. For $p \geq 1$, the positive term series $\sum_{x\in\mathbb{Z}}(1 + |x|)^{-p}$ converges. Thus, we have

$$||T_{pq}||^2_{HS} = \sum_{x\in\mathbb{Z}}||T_{pq}(1 + |x|)^{-q}\phi_{x,j}||^2_p$$
$$= \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}(1 + |y|)^{2p}||T_{pq}(1 + |x|)^{-q}\phi_{x,j}(y)||^2_{\mathbb{C}^2}$$
$$= \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}(1 + |y|)^{2p}||T_{pq}(1 + |x|)^{-q}\delta_x(y)e_j||^2_{\mathbb{C}^2}$$
$$= \sum_{x\in\mathbb{Z}}(1 + |x|)^{-2(q-p)} < \infty,$$

where $||\cdot||_{HS}$ denotes the Hilbert-Schmidt norm of an operator. Therefore the inclusion mapping $T_{pq}: \mathcal{S}_q \to \mathcal{S}_p$ is a Hilbert-Schmidt operator.

For $p \geq 0$, we denote by $\mathcal{S}^*_p$ the dual (conjugate) space of $\mathcal{S}_p$, and $||\cdot||_{-p}$ the norm of $\mathcal{S}^*_p$. Then $\mathcal{S}^*_q \subset \mathcal{S}^*_p$ and $||\cdot||_{-p} \leq ||\cdot||_{-q}$ whenever $0 \leq p \leq q$. According to the general theory of countable Hilbert spaces, the following lemma hold.

Lemma 3.1[16,20] Let $\mathcal{S}^*$ be the dual of $\mathcal{S}$ and endow it with the strong topology, then

$$\mathcal{S}^* = \bigcup_{p=0}^{\infty} \mathcal{S}_p^*$$

And, moreover, the inductive limit topology on $\mathcal{S}^*$ given by space sequence $\{\mathcal{S}_p^*\}_{p\geq 0}$ coincides with the strong topology.

Theorem 3.1 By identifying $l^2(\mathbb{Z},\mathbb{C}^2)$ with its dual naturally, then Gel'fand triple

$$\mathcal{S} \subset l^2(\mathbb{Z},\mathbb{C}^2) \subset \mathcal{S}^* \qquad (3.4)$$

can be constructed, which is referred to as the rigged Hilbert space.

Proof. It can be proved by Proposition 3.2 and Lemma 3.1.

Proposition 3.3 The system $\{\phi_{x,j} | x \in \mathbb{Z}, j = 1,2\}$ is contained in $\mathcal{S}$ and, moreover, it forms a basis for $\mathcal{S}$ in the sense that

$$\xi = \sum_{\substack{x\in\mathbb{Z},\\ j=1,2}} \langle \phi_{x,j}, \xi \rangle \phi_{x,j}, \quad \xi \in \mathcal{S},$$

where $\langle\cdot,\cdot\rangle$ is the inner product of $l^2(\mathbb{Z},\mathbb{C}^2)$ and the series converges in the topology of $\mathcal{S}$.

Proof. It follows from Proposition 3.1 and the definition of $\mathcal{S}$ that the system $\{\phi_{x,j} | x \in \mathbb{Z}, j = 1,2\}$ is contained in $\mathcal{S}$. Let $\xi \in \mathcal{S}$, for each $p \geq 0$, we have $\xi \in \mathcal{S}_p$, which together with proposition 3.1 gives

$$\xi = \sum_{\substack{x\in\mathbb{Z},\\ j=1,2}} \langle (1+|x|)^{-p}\phi_{x,j}, \xi \rangle_p (1+|x|)^{-p}\phi_{x,j}, \qquad (3.5)$$

where the series on the right-hand side converges in norm $||\cdot||_p$. In addition, we find

$$\langle (1+|x|)^{-p}\phi_{x,j}, \xi \rangle_p = (1+|x|)^{-p}\langle \phi_{x,j}, \xi \rangle,\ p \geq 0. \qquad (3.6)$$

Thus,

$$\xi = \sum_{\substack{x\in\mathbb{Z},\\ j=1,2}} \langle \phi_{x,j}, \xi \rangle_p \phi_{x,j},\ p \geq 0, \qquad (3.7)$$

where the series on the right-hand side converges in norm $||\cdot||_p$, namely, in the topology of $\mathcal{S}$.

Next, we naturally extend $\mathcal{U}$ to $\mathcal{S}^*$ space and the operators $\mathcal{U}$ is not unitary on $\mathcal{S}^*$. Moreover, for the above rigged Hilbert space $\mathcal{S} \subset l^2(\mathbb{Z},\mathbb{C}^2) \subset \mathcal{S}^*$, $\mathcal{S} \subset \mathcal{D}(\mathcal{U})$, for $\forall \lambda \in \sigma_c(\mathcal{U})$, it can be obtained that $\mathcal{U}$ has a generalized eigenvector $T_\lambda \in \mathcal{S}^*$, satisfying

$$\mathcal{U}T_\lambda = e^{i\lambda}T_\lambda, \quad \lambda \in \mathbb{C}/2\pi\mathbb{Z}. \qquad (3.8)$$

According to the Theorem 2.2, we can get T_λ satisfies the expansion of the generalized eigenfunction such as (2.8).

Theorem 3.2 Let the above triplet $\mathcal{S} \subset l^2(\mathbb{Z},\mathbb{C}^2) \subset \mathcal{S}^*$ is the rigged Hilbert space of $\mathcal{U}$, then the following statements holds

(1) $\mathcal{U}$ is continuous with respect to nuclear topology $\iota_{\mathcal{S}}$ on $\mathcal{S}$;

(2) The nuclear space $\mathcal{S}$ is invariant under the action of $\mathcal{U}$.

Proof. Since $\mathcal{S}_p$ is dense in $\mathcal{S}$ and (2.10) holds, then $\mathcal{U}$ is continuous with respect to the topology$\iota_{\mathcal{S}}$. Next, we prove that the nuclear space $\mathcal{S}$ is invariant under the action of $\mathcal{U}$. For $\forall\, \varphi \in \mathcal{S}$, by the definition of $\mathcal{S}$, $\exists\ \{\varphi_n\} \subset \mathcal{S}$, such that

$$\lim_{n\to\infty} ||\varphi_n - \varphi||_p = 0,\ p = 1,2,\cdots. \qquad (3.9)$$

By (2.10), $\{\mathcal{U}\varphi_n\}$ is a cauchy sequence in $\mathcal{S}_p$, $p \geq 0$. Clearly $\mathcal{S}$ is complete, then for $\varphi \in \mathcal{S}$, we have

$$\lim_{n\to\infty} ||\mathcal{U}\varphi_n - \varphi||_p = 0,\ p = 1,2,\cdots. \qquad (3.10)$$

On the other hand, $\mathcal{U}$ is closed, namely

$$\mathcal{U}\varphi = \psi \in \mathcal{S},\ \forall\, \varphi \in \mathcal{S}.$$

So $\mathcal{S}$ is invariant under $\mathcal{U}$.

Corollary 3.1 Let $\mathcal{S} \subset l^2(\mathbb{Z},\mathbb{C}^2) \subset \mathcal{S}^*$ be the above rigged Hilbert space . If $\mathcal{S} \subset \mathcal{D}(\mathcal{U})$ and the nuclear space $\mathcal{S}$ is invariant under the action of $\mathcal{U}$, then

$$\mathcal{S} \subset \bigcap_{n=1}^{\infty} \mathcal{D}(\mathcal{U}^n). \qquad (3.11).$$

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