Existence of attractor for wave equation with decay coefficient on R^n

Tingting Hu, Tingting Liu, Qiaozhen Ma

Abstract—**In this paper, by using the method called "asymptotic contractive process" on the timedependent entire space, the existence of time-dependent attractor for the wave equation with decay coefficient on $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ is obtained.**

Key words— **wave equation, time-dependent attractor, asymptotic contractive process, un-bounded domain.**
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1 Introduction

We onsider the asymptotic behavior of solutions to the following equations on $\mathbb{R}^3$:

$$\begin{cases}\varepsilon(t)u_{tt}+\alpha u_t-\Delta u+\lambda u+f(x,u)=g(x), x\in\mathbb{R}^3, t\ge\tau,\\ u(x,\tau)=u_0(x), u_t(x,\tau)=u_1(x), x\in\mathbb{R}^3,\end{cases} \quad (1.1)$$

where $u(x,t)$ is an unknown function, $\lambda>0, \varepsilon=\varepsilon(t)$ is a decreasing bounded function and

$$\lim_{t\to+\infty}\varepsilon(t)=0. \quad (1.2)$$

Especially, there exists a constant $L>0$ such that

$$\sup_{t\in\mathbb{R}}[|\varepsilon(t)|+|\varepsilon'(t)|]\le L. \quad (1.3)$$

The nonlinear term $f(x,s)\in C(\mathbb{R}^4)$ with $f(x,\cdot)\in C^2(\mathbb{R})$ for every fixed $x\in\mathbb{R}^3$, and satisfies

$$f(\cdot,0)\in L^2(\mathbb{R}^3), \left|\frac{\partial f}{\partial s}(x,0)\right|\le c_0, \forall x\in\mathbb{R}^3, \quad (1.4)$$

$$\left|\frac{\partial f}{\partial s}(x,\cdot)\right|\le c_1(1+|s|^2), \left|\frac{\partial f}{\partial x}(\cdot,s)\right|\le c_2, \forall s\in\mathbb{R}, x\in\mathbb{R}^3, \quad (1.5)$$

$$\liminf_{|s|\to\infty}\frac{\partial f(x,s)}{\partial s}>-\lambda_1, \forall s\in\mathbb{R}, x\in\mathbb{R}^3, \quad (1.6)$$

$$(f(x,s)-f(x,0))\ge c_3 s^2, \forall s\in\mathbb{R}, |x|\ge r_0>0, \quad (1.7)$$

where $c_i>0, i=0,1,2,3, \lambda_1$ is the first eigenvalue of the operator $A=-\Delta$.

Equation (1.1) can be seen as a nonlinear damped wave equation with time-dependent speed of propagation $\frac{1}{\varepsilon(t)}$. Besides, it can also be regarded as a model for the thermal evolution in ahomogeneous isotropic (rigid) heat conductor according to the Maxwell-Cattaneo law [8] (see also [12, AppendixB]), with $\varepsilon(t)$ representing a time-dependent relaxation parameter.

When ε is only a positive constant in (1.1), the asymptotic behavior of solutions to equation (1.1) on bounded domains has been the object of extensive studies since the eighties (see,e.g. [8 − 16]).

For equation (1.1), in [1], Conti, Pata and Temam presented a notion of time-dependent attractor exploiting the minimality with respect to the pullback attraction property, and constructed a sufficient condition proving the existence of time-dependent attractor based on the theory established by Plinio, Duane and Temam([5]). Meanwhile, within the new framework, on bounded domain, the authors studied the following weak damped wave equations with time-dependent speed of propagation

$$\varepsilon(t)u_{tt}+\alpha u_t-\Delta u+f(u)=g(x), x\in\Omega\subset\mathbb{R}^3, \quad (1.8)$$

in particular, they proved that the time-dependent global attractor of (1.8) converged in a suitable sense to the attractor of the parabolic equation $\alpha u_t-\Delta u+f(u)=g(x)$ when $\varepsilon(t)\to 0$ as $t\to+\infty$ ([2]). Successively, in[3], they continued to show the existence of an invariant time-dependent global attractor to the following specific one-dimensional wave equation $\varepsilon(t)u_{tt}-u_{xx}+[1+\varepsilon f'(u)]u_t+f(u)=h,$ which converges in suitable sense to the classical Fourier equation.

Recently, Meng et al. investigated the long-time behavior of the solution for the wave equation with nonlinear damping $g(u_t)$ on the time-dependent space, in which they found a new technical method verifying compactness of the process via defining the contractive functions, see [6]. In [7], Meng and Liu also showed the necessary and sufficient conditions of the existence of time-dependent global attractor borrowed from the ideas in [17].

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However, all researches about the time-dependent attractor were on bounded domain. In the recent, the authors gave a method called"asymptotic contractive process"to prove that the process is pullback asymptotic compactness on unbounded domian ([18]). So in this paper, we exploiting this new method to show the existence of time-dependent attractor for (1.1) on unbounded domain.

The rest of this article consists of four Sections. In the next Section, we define some functions sets and iterate some useful lemmas; in Section 3, we introduce the concept of asymptotic contractive process and a technique for verifying asymptotic compactness for the process is proposed; finally, dissipativity, tail estimate and the existence of the time-dependent global attractor is obtained on $\mathbb{R}^3$ in Section 4.

2 Preliminaries

Now we recall some basic notations and abstract results in[1,6,18], which are necessary for getting our main results.

2.1 Notations.

Without loss of generality, set $\mathrm{H}=L^2(\mathbb{R}^n)$, endowed with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. For $0\le s\le 2$, we define the hierarchy of nested Hilbert spaces

$$\mathrm{H}^s=\mathrm{H}^s(\mathbb{R}^n)=\mathrm{D}(A^{\frac{s}{2}}),\langle w,v\rangle_s=\langle A^{\frac{s}{2}}w,A^{\frac{s}{2}}v\rangle\|w\|_s^2=\langle A^{\frac{s}{2}}w,A^{\frac{s}{2}}w\rangle.$$

Now, for $t\in\mathbb{R}$ and $0\le s\le 2$, write the following symbols,

$$\mathrm{H}_t^s=\mathrm{H}^{s+1}\times\mathrm{H}^s,$$

with the norm

$$\|z\|_{\mathrm{H}_t^s}^2=\|(u,u_t)\|_{\mathrm{H}_t^s}^2=\|u\|_{s+1}^2+\varepsilon(t)\|u_t\|_s^2.$$

The letter s is always omitted whenever zero. Especially, we consider the time-dependent phase space

$$\mathrm{H}_t=\mathrm{H}^1\times\mathrm{H},$$

with the norm

$$\|z\|_{\mathrm{H}_t}^2=\|(u,u_t)\|_{\mathrm{H}_t}^2=\|u\|_2^2+\varepsilon(t)\|u_t\|^2=\|\nabla u\|^2+\|u\|^2+\varepsilon(t)\|u_t\|^2.$$

For every $t\in\mathbb{R}$, let X_t be a family of normed spaces, we introduce the $\rho-$ball of X_t

$$\mathrm{B}_t(\rho)=\{z\in X_t\|z\|_{X_t}\le\rho\}.$$

We denote the Hausdorff semi-distance of two (nonempty) sets $B,C\in X_t$ by

$$\delta_t(B,C)=\sup_{x\in B}dist_{X_t}(x,C)=\sup_{x\in B}\inf_{y\in C}\|x-y\|_{X_t}.$$

For any given $\epsilon>0$, the $\epsilon-$neighbourhood of a set $B\subset X_t$ is defined as

$$\mathrm{O}_t^{\epsilon}(B)=\bigcup_{x\in B}\{y\in X_t\|y-x\|_{X_t}<\epsilon\}=\bigcup_{x\in B}\{x+\mathrm{B}_t(\epsilon)\}.$$

2.2 Some concepts and abstract results.

Definition 2.1. ([1]). Let $\{X_t\}_{t\in\mathbb{R}}>0$ *be a family of normed spaces. A process is a two-parameter family of mappings* $\{U(t,\tau):X_\tau\to X_t,t\ge\tau\in\mathbb{R}\}$ *with properties*

$(i)U(\tau,\tau)=Id$ *is the identity on* $X_\tau,\tau\in\mathbb{R}$;

$(ii)U(t,s)U(s,\tau)=U(t,\tau),\forall t\ge s\ge\tau$

Definition 2.2. ([1]). A family $\mathcal{C}=\{C_t\}_{t\in\mathbb{R}}$ *of bounded sets* $C_t\subset X_t$ *is called uniformly bounded if there exists a constant* $\rho>0$ *such that* $C_t\subset\mathrm{B}_t(\rho),\forall t\in\mathbb{R}$.

Definition 2.4. ([1]). A (uniformly bounded) family $\mathcal{K}=\{K_t\}_{t\in\mathbb{R}}$ *is called pullback attracting if for all* $\epsilon>0$, *the family* $\{\mathrm{O}_t^{\epsilon}(K_t)\}_{t\in\mathbb{R}}$ *is pullback absorbing.*

Definition 2.5. ([1]). The time-dependent global attractor for $U(t,\tau)$ *is the smallest family* $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$ *with the following properties:*

(i) Each A_t *compact in* X_t;

(ii) $\mathcal{A}$ *is pullback attracting, namely, it is uniformly bounded and the limit*

$$\lim_{\tau\to-\infty}\delta_t(U(t,\tau)C_\tau,A_t)=0,$$

holds for every uniformly bounded family $\mathcal{C}=\{C_t\}_{t\in\mathbb{R}}$ *and every* $t\in\mathbb{R}$.

Theorem 2.6. ([1]). The time-dependent attractor $\mathcal{A}$ *exists and it is unique if and only if the process* $U(t,\tau)$ *is asymptotically compact, namely, the set*

$$\mathbb{K}=\{\mathcal{K}=\{K_t\}_{t\in\mathbb{R}}:K_t\subset X_t compact,\mathcal{K} pullback attracting\}\ne\varnothing.$$

Definition 2.7. ([17]). Let $\{X_t\}_{t\in\mathbb{R}}$ *be a Banach space and* $\mathcal{C}=\{C_t\}_{t\in\mathbb{R}}$ *be a family of uniformly bounded subset of* $\{X_t\}_{t\in\mathbb{R}}$. *We call a function* $\psi_\tau^t(\cdot,\cdot)$ *defined on* $\{X_t\}_{t\in\mathbb{R}}\times\{X_t\}_{t\in\mathbb{R}}$ *to be a asymptotic contractive function on* $C_t\times C_t$, *if for any* $t\in\mathbb{R}$ *and any sequence* $\{x_n\}_{n=1}^{\infty}\subset\mathrm{C}_t$, *such that for any* $\epsilon>0$, *there is a subsequence* $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ *satisfying:*

$$\psi_\tau^t(x_{n_k},x_{n_l})\le\epsilon+\phi_\tau^t(x_{n_k},x_{n_l}),$$

where

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi_\tau^t(x_{n_k},x_{n_l})=0,\tau\le t.$$

Definition 2.8. ([17]). Let $U(\cdot,\cdot)$ *be a process on* $\{X_t\}_{t\in\mathbb{R}}$ *and for any* $\epsilon>0$ *there exist* $\tau<T(\epsilon)\le t,\psi_T^t\in\mathcal{C}(C_T)$, *such that*

$$\|U(t,T)x-U(t,T)y\|_{X_t}\le\epsilon+\psi_T^t(x,y),\forall x,y\in C_T,$$

for any fixed $t\in\mathbb{R}$. *Then* $U(\cdot,\cdot)$ *is called asymptotic contractive process.*

Theorem 2.9. ([17]). Let $U(\cdot,\cdot)$ *be a process in a family of Banach space* $\{X_t\}_{t\in\mathbb{R}}$. *Then* $U(\cdot,\cdot)$ *has a time-dependent global attractor* $\mathrm{U}^*=\{A_t^*\}_{t\in\mathbb{R}}$ *in* $\{X_t\}_{t\in\mathbb{R}}$ *provided that the following conditions hold true:*

(i) $U(\cdot,\cdot)$ *has a pullback absorbing family* $\mathrm{B}=\{B_t\}_{t\in\mathbb{R}}$;

(ii) $U(\cdot,\cdot)$ *is pullback asymptotic contractive process on* B_t.

Let

$$F(x,u)=\int_0^u f(x,y)dy.$$

Lemma 2.10. ([10,11]). From (1.4), (1.6)−(1.7), for $0<\nu<\min\{1,\lambda\}$, *there exist* $\tilde{n}(\nu)>0, c(\nu)\geq 0 (i=1,2)$, *such that the following inequalities hold for every* $u\in \mathrm{H}^1$:

$$2\langle F(x,u),1\rangle \geq -\nu\|u\|^2 - c(\nu), \tag{2.1}$$

$$\langle f(x,u),u\rangle+\frac{\nu}{2}\|\nabla u\|^2\geq \tilde{n}(\nu)\|u\|^2-c(\nu). \tag{2.2}$$

3 Dissipativity

Global existence of solution u to (1.1) is classical, by using the standard Galerkin approximation method ([1, 12, 16]), so we only give the following results and omit the proof.

Lemma 3.1. Under the assumptions(1.2)−(1.7), for every pair of initial data $z_\tau\in \mathrm{H}_\tau$, $g\in L^2(\mathbb{R}^3)$, *there exists a unique solution* $z(t)=(u,u_t)$ *of problem (1.1) in space* H_t *and satisfy*

$$z\in C([\tau,t];\mathrm{H}_t)\cap L^\infty([\tau,t);\mathrm{H}_t^1).$$

Furthermore, let $z_i(\tau)\in \mathrm{H}_\tau$ *be the initial data such that* $\|z_i(\tau)\|_{\mathrm{H}_\tau}\leq\rho (i=1,2)$, *and* $z_i(t)$ *be the solution of problem (1.1). Then there exists* $K=K(\rho)>0$, *such that*

$$\|z_1(t)-z_2(t)\|_{\mathrm{H}_t}\leq e^{K(t-\tau)}\|z_1(\tau)-z_2(\tau)\|_{\mathrm{H}_\tau}, \forall t\geq\tau. \tag{3.1}$$

Lemma 3.2. Under the assumptions(1.2)−(1.7), $g\in L^2(\mathbb{R}^3)$, *for any initial data* $z_\tau\in \mathrm{B}_\tau(\rho_0)\subset \mathrm{H}_\tau$, *there exists* $\rho>0$, *such that*

$$\|U(t,\tau)z(\tau)\|_{\mathrm{H}_\tau}\leq\rho, \forall\tau\leq t.$$

Proof Multiplying (1.1) by $v=u_t+\delta u$ and integrating over $\mathbb{R}^3$, we obtain

$$\frac{d}{dt}E(t)+I(t)=0, \tag{3.2}$$

where

$$E(t)=\|\nabla u\|^2+\lambda\|u\|^2+\varepsilon(t)\|v\|^2+2\langle F(x,u),1\rangle-2\langle g,u\rangle,$$

$$I(t)=(2(\alpha-\delta\varepsilon(t))-\varepsilon'(t))\|v\|^2+2\delta\|\nabla u\|^2+2\delta\lambda\|u\|^2$$
$$-2\delta(\alpha-\delta\varepsilon(t))\langle u,v\rangle+2\delta\langle f(x,u),u\rangle-2\delta\langle g,u\rangle,$$

integrating (3.2) over $[\tau,t]$, there holds

$$E(t)=-\int_\tau^t I(s)ds+E(\tau). \tag{3.3}$$

Let $0<\delta<\min\{\frac{\alpha}{6L},\frac{\lambda+4\tilde{n}(\nu)}{2\alpha}\}$, such that

$$\frac{\alpha}{2}-2\delta\varepsilon(t)>\delta\varepsilon(t), \frac{3}{2}\delta\lambda+2\delta\tilde{n}(\nu)-\delta^2\alpha>\delta\lambda,$$

then by (2.2), Hölder, Young inequalities, we obtain

$$\begin{aligned}I(t)&\geq\delta(2-\nu)\|\nabla u\|^2+2\delta(\lambda+\tilde{n}(\nu))\|u\|^2+(2\alpha-2\delta\varepsilon(t))\|v\|^2\\&-(\alpha\|v\|^2+\delta^2\alpha\|u\|^2)-(\frac{\delta\lambda}{2}\|u\|^2+\frac{2\delta}{\lambda}\|g\|^2)-2\delta c_2\\&\geq\delta\nu\|\nabla u\|^2+(\frac{3}{2}\delta\lambda+2\delta\tilde{n}(\nu)-\delta^2\alpha)\|u\|^2+(\alpha-2\delta\varepsilon(t))\|v\|^2\\&-\frac{2\delta}{\lambda}\|g\|^2-2\delta c_2\\&\geq\delta\nu(\|\nabla u\|^2+\|u\|^2+\varepsilon(t)\|v\|^2)+\frac{\alpha}{2}\|v\|^2-\frac{2\delta}{\lambda}\|g\|^2-2\delta c_2,\end{aligned} \tag{3.4}$$

combining with (2.2) there holds

$$\begin{aligned}E(t)&\geq\|\nabla u\|^2+(\frac{3}{4}\lambda-\nu)\|u\|^2+\varepsilon(t)\|v\|^2-(\frac{4}{\lambda}\|g\|^2+c_1)\\&\geq\frac{\nu}{2}(\|\nabla u\|^2+\|u\|^2+\varepsilon(t)\|v\|^2)-(\frac{4}{\lambda}\|g\|^2+c_1),\end{aligned} \tag{3.5}$$

Together with (3.3), (3.4)−(3.5) it follows that

$$\begin{aligned}&\frac{\nu}{2}(\|\nabla u\|^2+\|u\|^2+\varepsilon(t)\|v\|^2)-m_1\\&\leq-\int_\tau^t[\delta\nu(\|\nabla u\|^2+\|u\|^2+\varepsilon(t)\|v\|^2)-m_2]dr+E(\tau),\end{aligned} \tag{3.6}$$

where $m_1=(\frac{4}{\lambda}\|g\|^2+c_1), m_2=\frac{2\delta}{\lambda}\|g\|^2+2\delta c_2$. So, for any $\rho_0>\frac{m_2}{\delta\nu}$, there exists $t_0>\tau$ such that

$$\|\nabla u(t_0)\|^2+\|u(t_0)\|^2+\varepsilon(t_0)\|v(t_0)\|^2\leq\rho_0.$$

As a result, let $B_t=\bigcup_{t\geq\tau}U(t,\tau)B_0$, where

$$B_0=\{(u_0,u_1)\in \mathrm{H}_\tau\,|\,\|u_0\|_1^2+\varepsilon(\tau)\|u_1\|^2\leq\rho_0\},$$

then B_t is a bounded absorbing set for process $\{U(t,\tau)\}$.

On the other hand, from the above discussion, there exist a positive constant ρ such that

$$\|u\|_1^2+\varepsilon(t)\|u_t\|^2\leq\rho, \forall t\geq t_0>\tau. \qquad \square$$

From Lemma 4.2, we can get the following conclusion:

Lemma 3.3. Under the assumptions(1.2)−(1.7), for $\rho_1>0$ *in Lemma 4.2, such that* $\mathrm{B}=\{\mathrm{B}_t(\rho_1)\}$ *is a time-dependent absorbing sets for the process* $\{U(t,\tau)\}$ *associated with (1.1) then for some* $R_0\geq\rho_0$, *there have*

$$\sup_{z\in \mathrm{B}_\tau(\rho_1)}\{\|U(t,\tau)z(\tau)\|_{\mathrm{H}_t}+\int_\tau^\infty\|v(y)\|^2dy\}\leq R_0, \forall t\in\mathbb{R} \tag{3.7}$$

Proof Combining with (3.2), (3.4) with $\delta=0$, we get

$$\frac{d}{dt}E(t)+\frac{\alpha}{2}\|v\|^2\le\frac{2\delta}{\lambda}\|g\|^2+2\delta c_2,$$

integrate on $[\tau,+\infty)$, we can easily conclude that (3.7) is true. □

Lemma 3.4. Under the assumptions (1.2)−(1.7), $g\in H^1(\mathbb{R}^3)$, for any initial data $z(\tau)\in B_\tau(R_0)\subset H^1_\tau$, there exists $\rho_2>0$, such that

$$\varepsilon(t)\|A^{\frac{1}{2}}v\|^2+\|Au\|^2\le\rho_1,\forall\tau\le t. \quad (3.8)$$

Proof Multiplying (1.1) by $Av=Au_t+\delta Au$ and integrating over $\mathbb{R}^n$, we obtain

$$\frac{d}{dt}(\varepsilon(t)\|A^{\frac{1}{2}}v\|^2+\|Au\|^2+\lambda\|A^{\frac{1}{2}}u\|^2)-\varepsilon'(t)\|A^{\frac{1}{2}}v\|^2$$
$$+2(\alpha-\delta\varepsilon(t))\|A^{\frac{1}{2}}v\|^2-2\delta(\alpha-\delta\varepsilon(t))\langle u,Av\rangle+2\delta\|Au\|^2$$
$$+2\delta\lambda\|A^{\frac{1}{2}}u\|^2+2\langle f(x,u),Av\rangle=2\langle g,Av\rangle.$$

By (3.8) Hölder, Young inequalities

$$2|\langle f(x,u),Av\rangle|=2\langle\frac{\partial f}{\partial x},A^{\frac{1}{2}}v\rangle+2\langle\frac{\partial f}{\partial u},A^{\frac{1}{2}}v\rangle$$
$$\le2|f'(u)|\cdot\|A^{\frac{1}{2}}u\|\cdot\|A^{\frac{1}{2}}v\|\le\frac{\alpha}{4}\|A^{\frac{1}{2}}u\|+\frac{4l^2}{\alpha}\|A^{\frac{1}{2}}u\|^2,$$

$$2|\langle g,Av\rangle|\le2\|g(x)\|_{H^1(\mathbb{R}^3)}\cdot\|A^{\frac{1}{2}}v\|\le\frac{\alpha}{4}\|A^{\frac{1}{2}}v\|^2+\frac{4}{\alpha}\|g(x)\|^2_{H^1(\mathbb{R}^3)},$$

and

$$2\delta\|Au\|^2+(2\alpha-2\delta\varepsilon(t))\|A^{\frac{1}{2}}v\|^2-\varepsilon'(t)\|A^{\frac{1}{2}}v\|^2-$$
$$2\delta(\alpha-\delta\varepsilon(t))\langle u,A^{\frac{1}{2}}v\rangle$$
$$\ge2\delta\|Au\|^2+(2\alpha-2\delta\varepsilon(t))\|A^{\frac{1}{2}}v\|^2-[\frac{\alpha}{2}\|A^{\frac{1}{2}}v\|^2+2\delta^2\alpha\|A^{\frac{1}{2}}u\|^2]$$
$$\ge2\delta\|Au\|^2+(\frac{3}{2}\alpha-2\delta\varepsilon(t))\|A^{\frac{1}{2}}v\|^2-2\delta^2\alpha\|A^{\frac{1}{2}}u\|^2$$
$$\ge\delta(\|Au\|^2+\varepsilon(t)\|A^{\frac{1}{2}}v\|^2)+\alpha\|A^{\frac{1}{2}}v\|^2-2\delta^2\alpha\|A^{\frac{1}{2}}u\|^2.$$

(3.9)

Combining with the above inequality, we get

$$\frac{d}{dt}[\varepsilon(t)\|A^{\frac{1}{2}}v\|^2+\|Au\|^2+\lambda\|A^{\frac{1}{2}}u\|^2]+\delta(\varepsilon(t)\|A^{\frac{1}{2}}v\|^2$$
$$+\|Au\|^2+\lambda\|A^{\frac{1}{2}}u\|^2)+\frac{\alpha}{2}\|A^{\frac{1}{2}}v\|^2 \quad (3.10)$$
$$\le\frac{4}{\alpha}\|g\|^2_{H^1(\mathbb{R}^3)}+(\frac{4l^2}{\alpha}+2\delta^2\alpha)\|A^{\frac{1}{2}}u\|^2,$$

applying the Gronwall Lemma to (3.10) over $[\tau,t]$ and combining with the above equality we ha

$$\varepsilon(t)\|A^{\frac{1}{2}}v\|^2+\|Au\|^2+\lambda\|A^{\frac{1}{2}}u\|^2$$
$$\le e^{-\delta(t-\tau)}[\varepsilon(t)\|A^{\frac{1}{2}}v_\tau\|^2+\|u_0\|^2+\lambda\|A^{\frac{1}{2}}u_0\|^2]+$$
$$(\frac{4}{\alpha}\|g\|^2_{H^1(\mathbb{R}^3)}+(\frac{4l^2}{\alpha}+2\delta^2\alpha)\cdot R_0^2)$$
$$\le\rho_1,$$

where $\rho_1=2(\frac{4}{\alpha}\|g\|^2_{H^1(\mathbb{R}^3)}+(\frac{4l^2}{\alpha}+2\delta^2\alpha)\cdot R_0^2)$. Then the proof is complete. □

Lemma 3.5. Under the assumptions (1.2)−(1.7), for $\rho_2>0$ in Lemma 3.4, there exists $R_1\ge\max\{\rho_2,\frac{2\rho_1}{\alpha}\}$, such that

$$\sup_{z\in B_\tau(R_0)}\{\varepsilon(t)\|A^{\frac{1}{2}}v\|^2+\|Au\|^2+\int_\tau^\infty\|A^{\frac{1}{2}}v(y)\|^2dy\} \quad (3.11)$$
$$\le R_1,\forall t\in\mathbb{R}.$$

Proof Integrating (3.10) on $[\tau,t]$ with $\delta=0$, we get $\int_\tau^\infty\|A^{\frac{1}{2}}v(y)\|^2dy\le\frac{2}{\alpha}\cdot\rho_1>0$. Then together with Lemma 3.4 we conclude that (3.11) is true. □

Lemma 3.6. Under the assumptions (1.2)−(1.7), $g\in L^2(\mathbb{R}^3)$, for any $\epsilon>0$, there exist $T_1=T_1(\epsilon)$, as $t\ge T_1$ and $K=K(\epsilon)>0$, such that

$$\int_{\Omega_k^c}(\varepsilon(t)|v|^2+|\nabla u|^2+|u|^2)dx\le C\epsilon,\forall t\ge T_1,k\ge K(\epsilon),$$

where $\Omega_k^c=\{x\in\mathbb{R}^n:|x|\ge k\}$, C is a positive constant.

Proof Choosing a smooth function θ such that $0\le\theta(s)\le1$, for any $s\in\mathbb{R}^+$, and

$$\theta(s)=0\ for\ 0\le s\le1,\theta(s)=1,\ for\ s\ge2.$$

Then there exists a positive constant $\tilde{C}_0$, such that

$\max\{|\theta'(s)|,|\theta''(\theta)|\}\le\tilde{C}_0$ for any $s\in\mathbb{R}^+$.

Multiplying (1.1) by $\theta(\frac{|x|^2}{k^2})v$ and integrating over $\mathbb{R}^3$, we obtain

$$\frac{d}{dt}[\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})\cdot(\varepsilon(t)|v|^2+\lambda|u|^2)dx]+(2(\alpha-\delta\varepsilon(t))$$
$$-\varepsilon'(t))\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})\cdot\varepsilon'(t)|v|^2dx-2\delta(\alpha-\delta\varepsilon(t))$$
$$\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})\cdot u\cdot vdx-2\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})\Delta u\cdot vdx \quad (3.12)$$
$$+2\delta\lambda\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2dx$$
$$=-2\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})f(x,u)vdx+2\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})g(x)vdx.$$

Now we deal with each term in the above equation:

First we have

$$
\begin{aligned}
&-2\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})\Delta u\cdot v dx \\
&=\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx+2\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx \\
&+2\int_{\mathbb{R}^n}\frac{2|x|}{k^2}\nabla u\cdot v\cdot\theta'(\frac{|x|^2}{k^2})dx\geq\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx \\
&+2\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx-\frac{4\sqrt{2}\tilde{C}_0}{k}\int_{k<|x|<\sqrt{2}k}|\nabla u|\cdot|v|\,dx \\
&\geq\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx+2\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx \\
&-\frac{2\sqrt{2}\tilde{C}_0}{k}[\int_{\mathbb{R}^n}|\nabla u|^2\,dx+\int_{\mathbb{R}^n}|v|^2\,dx] \\
&\geq\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx+2\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|\nabla u|^2\,dx \\
&-\frac{2\sqrt{2}\tilde{C}_0}{k}R_0^2-\frac{2\sqrt{2}\tilde{C}_0}{k}\|v\|^2,
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
&|2\delta(\alpha-\delta\varepsilon(t))\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})\cdot uv dx| \\
&\leq 2\delta\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|\cdot|v|\,dx \\
&\leq\frac{\alpha}{2}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|v|^2\,dx+2\delta^2\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2\,dx.
\end{aligned}
\tag{3.14}
$$

Moreover, we have that

$$
\begin{aligned}
&2\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})f(x,u)v dx \\
&=2\frac{d}{dt}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})F(x,u)dx+2\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})f(x,u)u dx,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&2|\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})f(x,u)v dx| \\
&\leq\frac{\alpha}{2}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|v|^2+\frac{2}{\alpha}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|g|^2\,dx.
\end{aligned}
\tag{3.16}
$$

Combining with the above estimates, we obtain

$$
\begin{aligned}
&\frac{d}{dt}[\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u|^2+\lambda|u|^2+\varepsilon(t)|v|^2+2F(x,u))dx] \\
&+\delta\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u|^2+\lambda|u|^2+\varepsilon(t)|v|^2+2\delta f(x,u)u)dx \\
&\leq\frac{2\sqrt{2}\tilde{C}_0}{k}R_0^2+2\delta^2\alpha\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2\,dx+\frac{2}{\alpha}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|g|^2\,dx \\
&+\frac{2\sqrt{2}\tilde{C}_0}{k}\|\nabla v\|^2.
\end{aligned}
$$

Let $k_1(\varepsilon)>0$, and any $0<\varepsilon<1$, such that if $k\geq k_1(\varepsilon)$, then

$$\frac{2\sqrt{2}C_0}{k}R_0^2<\varepsilon,$$

holds, and there there exist $k_2(\varepsilon)>0$, such that if $k\geq k_2(\varepsilon)$, we have

$$C\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|u|^2\,dx<\varepsilon.$$

since $g\in H_0^1(\mathbb{R}^n)$, so there exist $k_3(\varepsilon)>0$, such that

$$\frac{2}{\alpha}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})|g|^2\,dx<\varepsilon,$$

hold for any $k\geq k_3(\varepsilon)$.

So there exist $k_0=\max\{k_1(\varepsilon),k_2(\varepsilon),k_3(\varepsilon)\}$, if $k\geq k_0$, we infer

$$
\begin{aligned}
&\frac{d}{dt}[\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u|^2+\lambda|u|^2+\varepsilon(t)|v|^2)dx]+ \\
&\delta[\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u|^2+\lambda|u|^2+\varepsilon(t)|v|^2)dx] \\
&\leq 3\varepsilon+\frac{2\sqrt{2}\tilde{C}_0}{k}\|\nabla v\|^2.
\end{aligned}
$$

By the Gronwall Lemma on $[\tau,t]$, and Lemma 3.3, Lemma3.5, we can get

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u|^2+\lambda|u|^2+\varepsilon(t)|v|^2)dx \\
&\leq e^{-\delta(t-\tau)}\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u_\tau|^2+\lambda|u_\tau|^2+\varepsilon(t)|v_\tau|^2)dx \\
&+\frac{3\varepsilon}{\nu}+\frac{2\sqrt{2}\tilde{C}_0}{k}\int_\tau^t e^{-\delta(t-s)}\|\nabla v\|^2 ds \\
&\leq\rho_0 e^{-\delta(t-\tau)}+\frac{3\varepsilon}{\nu}+\frac{2\sqrt{2}\tilde{C}_0}{k}R_0^2,
\end{aligned}
\tag{3.17}
$$

so for given $\varepsilon>0$ above, setting $K=K(\varepsilon)$ there exist $T_1=T_1(\varepsilon)$, as $t\geq T_1$ and $k\geq K(\varepsilon)$, we have

$$\int_{\mathbb{R}^n}\theta(\frac{|x|^2}{k^2})(|\nabla u|^2+\lambda|u|^2+\varepsilon(t)|v|^2)dx\leq\frac{6\varepsilon}{\nu}.$$

Then we obtain

$$\int_{\Omega_k^c}(|\nabla u|^2+|u|^2+\varepsilon(t)|v|^2)dx\leq\frac{6\varepsilon}{\hat{\sigma}\nu}=C\varepsilon,$$

where $\hat{\sigma}=\min\{1,\lambda\}$. □

4 Existence of the time-dependent global attractor on $\mathcal{H}_t$

Theorem 4.1. Under the conditions(1.2)−(1.7), the process $U(t,\tau):\mathcal{H}_\tau\to\mathcal{H}_t$ *generated by problem (1.1) has a invariant time-dependent global attractor* $\mathcal{A}=\{A_t\}_{t\in\mathbb{R}}$ *in* $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$.

In the following, we will obtain the existence of the time-dependent attractor for system (1.1) by using the method of asymptotic contractive function. □

Lemma 4.2. Under the assumptions(1.2)−(1.7), $g\in H^1(\mathbb{R}^3),(u_n,u_{n_t})$ *be the solution corresponding to initial data* $(u_0^n,v_0^n)\in B_T$ *for the problem (1.1). Then for any* $k>0$ *and* $T(\varepsilon)>0$ *be given, let*

$$\Omega_k=\{x\in\mathbb{R}^3:|x|<k\},$$

we can have:

(i) $u_n\to u^*-$ *weakly in* $L^\infty(T,t;L^6(\Omega_k))$;

(ii) $u_{n_t}\to u_t^*-$ *weakly in* $L^\infty(T,t;L^2(\Omega_k))$;

(iii) $u_n \to u$ *strong in* $L^2(T,t;L^2(\Omega_k))$;

(iv) $u_n(T) \to u(T)$ *and* $u_n(t) \to u(t)$ *strong in* $L^2(\Omega_k), L^4(\Omega_k)$.

Proof From (4.7), (4.11), $\| u_n\|_2^2 + \varepsilon(\xi)\| u_{n_t}\|^2$ is bounded, where the bound depends on the T , furthermore, $\| u_n\|_2^2$ is bounded. Moreover, by (1.2) for fixed $T, \xi \in [T,t], \varepsilon(\xi)$ is bounded, hence $\| u_{n_t}\|^2$ is bounded. Then according to Alaoglu Theorem, and use the continuous emdedding $H_0^1(\Omega_k) \hookrightarrow L^6(\Omega_k)$, compact embedding $H_0^1(\Omega_k) \hookrightarrow L^2(\Omega_k)$, $H_0^1(\Omega_k) \hookrightarrow L^4(\Omega_k)$, the result can be obtained. □

4.1 A priori estimates

Let $(u_i(t), u_{i_t}(t))$ be the corresponding solution of (1.1) with initial datum $(u_0^i(\tau), v_0^i(\tau)) \in \{B_\tau\}_{\tau\in \mathbb{R}}$, and

$$w = u_1(t) - u_2(t),$$

then $w(t)$ satisfies

$$\begin{cases}\varepsilon(t)\omega_{tt} + \alpha\omega_{tt} - \Delta\omega + \lambda\omega + f(x,u_1) - f(x,u_2) = 0, t > T,\\ \omega(x,T) = u_0^1(T) - u_0^2(T), \omega_t(x,T) = v_0^1(T) - v_0^2(T),\end{cases} \quad (4.1)$$

Denote

$$E_w(t) = \frac{1}{2}[\| \nabla w\|^2 + \lambda\| w\|^2 + \varepsilon(t)\| w_t\|^2].$$

Taking the inner product (4.1) with w_t in $L^2(\mathbb{R}^3)$ we find

$$\frac{d}{dt}E_w(t) + (\alpha - \frac{\varepsilon'(t)}{2})\| w_t\|^2 + \langle f(x,u_1) - f(x,u_2), w_t(\xi)\rangle = 0. \quad (4.2)$$

Integrating (4.2) over $[s,t]$, we have

$$\begin{aligned}&E_w(t) - E_w(s) + \int_s^t (\alpha - \frac{\varepsilon'(\xi)}{2})\| w_t(\xi)\|^2 d\xi\\ &+\int_s^t \langle f(x,u_1) - f(x,u_2), w_t(\xi)\rangle d\xi = 0,\end{aligned} \quad (4.3)$$

where $T \le s \le t$. For $L < 2\alpha$, where L is the bound of $\varepsilon(t), \varepsilon'(t)$, then we can get

$$\int_T^t \varepsilon(\xi)\| \omega_t(\xi)\|^2 d\xi \le E_\omega(T) - \int_T^t \langle f(x,u_1) - f(x,u_2), \omega_t(\xi)\rangle ds. \quad (4.4)$$

Then multiplying (4.1) by ω , and integrating over $\mathbb{R}^3 \times [T,t]$, we obtain

$$\begin{aligned}&\int_T^t (\| \nabla\omega(s)\|^2 + \lambda\| \omega(s)\|^2)ds + \varepsilon(t)\langle\omega_t(t), \omega(t)\rangle\\ &= \langle\varepsilon(T)\omega_t(T), \omega(T)\rangle - \int_T^t (\alpha - \varepsilon'(s))\langle\omega_t(s), \omega(s)\rangle ds\\ &+\int_T^t \varepsilon(s)\| \omega_t(s)\|^2 ds - \int_T^t \langle f(x,u_1) - f(x,u_2), \omega(s)\rangle ds.\end{aligned} \quad (4.5)$$

Therefore, from (4.4) and (4.5) , yields

$$\begin{aligned}&2\int_T^t E_\omega(s) \le 2E_\omega(T) - 2\int_T^t \langle f(x,u_1) - f(x,u_2), \omega_t(s)\rangle ds\\ &-\int_T^t (\alpha - \varepsilon'(s))\langle\omega_t(s), \omega(s)\rangle ds - \varepsilon(t)\langle\omega_t(t), \omega(t)\rangle\\ &+\varepsilon(T)\langle\omega_t(T), \omega(T)\rangle - \int_T^t \langle f(x,u_1) - f(x,u_2), \omega(s)\rangle ds\end{aligned} \quad (4.6)$$

Integrating (4.3) over $[T,t]$, we have

$$\begin{aligned}&(t-T)E_\omega(t) + \int_T^t\int_s^t (\alpha - \frac{\varepsilon'(\xi)}{2})\| \omega_t(\xi)\|^2 d\xi ds\\ &= -\int_T^t\int_s^t \langle f(x,u_1(\xi)) - f(x,u_2(\xi)), \omega_t(\xi)\rangle d\xi ds + \int_T^t E_\omega(s)ds,\end{aligned} \quad (4.7)$$

together with (4.6), there holds

$$\begin{aligned}&(t-T)E_\omega(t) \le E_\omega(T) + \frac{1}{2}\langle\varepsilon(T)\omega_t(T), \omega(T)\rangle\\ &-\frac{1}{2}\langle\varepsilon(t)\omega_t(t), \omega(t)\rangle - \frac{1}{2}\int_T^t (\alpha - \varepsilon'(s))\langle\omega_t(s), \omega(s)\rangle ds\\ &-\frac{1}{2}\int_T^t \langle f(x,u_1) - f(x,u_2), \omega(s)\rangle ds\\ &-\int_T^t \langle f(x,u_1) - f(x,u_2), \omega_t(s)\rangle ds\\ &-\int_T^t\int_s^t \langle f(x,u_1(\xi)) - f(x,u_2(\xi)), \omega_t(\xi)\rangle d\xi ds.\end{aligned} \quad (4.8)$$

Set

$$\begin{aligned}&\psi_T^t((u_0^1(T), v_0^1(T)), (u_0^2(T), v_0^2(T)))\\ &= -\frac{1}{2(t-T)}\langle\varepsilon(t)\omega_t, \omega\rangle - \frac{1}{2(t-T)}\int_T^t (\alpha - \varepsilon'(s))\langle\omega_t(s), \omega(s)\rangle ds\\ &-\frac{1}{2(t-T)}\int_T^t \langle f(x,u_1) - f(x,u_2), \omega(s)\rangle ds\\ &-\frac{1}{t-T}\int_T^t \langle f(x,u_1) - f(x,u_2), \omega_t(s)\rangle ds\\ &-\frac{1}{t-T}\int_T^t\int_s^t \langle f(x,u_1(\xi)) - f(x,u_2(\xi)), \omega_t(\xi)\rangle d\xi ds,\end{aligned} \quad (4.9)$$

and

$$C_M = E_\omega(T) + \frac{1}{2}\langle\varepsilon(T)\omega_t(T), \omega(T)\rangle, \quad (4.10)$$

then we have

$$E_\omega(t) \le \frac{C_M}{t-T} + \psi_T^t((u_0^1(T), v_0^1(T)), (u_0^2(T), v_0^2(T))). \quad (4.11)$$

4.2 Asymptotically compact

Theorem 4.3. Under the assumption (1.2) − (1.7), then the process $\{U(t,\tau)\}$ *is a asymptotic contractive process, that is, for any fixed* $t \in \mathbb{R}$ *, bounded sequence* $\{x_n\}_{n=1}^\infty \subset X_{\tau_n}$ *and any* $\{\tau_n\}_{n=1}^\infty \subset \mathbb{R}^{-t}$ *, with* $\tau_n \to -\infty$ *as* $n \to \infty$ *, sequence* $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ *is precompact in* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Proof Let

$$\begin{aligned}&\psi_T^t((u_0^1(T), v_0^1(T)), (u_0^2(T), v_0^2(T))) = -\frac{1}{2(t-T)}\int_{\Omega_k} \varepsilon(t)\omega_t \cdot \omega dx\\ &-\frac{1}{2(t-T)}\int_T^t (\alpha - \varepsilon'(s))\int_{\Omega_k} \omega_t(s) \cdot \omega(s)dxds\\ &-\frac{1}{2(t-T)}\int_T^t\int_{\Omega_k} (f(x,u_1) - f(x,u_2)) \cdot \omega(s)dxds\\ &-\frac{1}{t-T}\int_T^t\int_{\Omega_k} (f(x,u_1) - f(x,u_2)) \cdot \omega_t(s)dxds\\ &-\frac{1}{t-T}\int_T^t\int_s^t\int_{\Omega_k} (f(x,u_1(\xi)) - f(x,u_2(\xi))) \cdot \omega_t(\xi)dxd\xi ds.\end{aligned} \quad (4.12)$$

From Lemma 3.6, as m,n large enough, and by Hölder, Young inequalities, we can have

$$\int_{\Omega_k^c}\varepsilon(t)(u_{n_t}(t)-u_{m_t}(t))(u_n(t)-u_m(t))dx \le L\| u_{n_t}(t)-u_{m_t}(t)\|_{L^2(\Omega_k^c)}\| u_n(t)-u_m(t)\|_{L^2(\Omega_k^c)}\le\frac{\epsilon}{4}, \quad (4.13)$$

and

$$\int_T^t\int_{\Omega_k^c}(u_{n_t}(s)-u_{m_t}(s))(u_n(s)-u_m(s))dxds \le \sup_{T\le s\le t}\| u_{n_t}(s)-u_{m_t}(s)\|_{L^2(\Omega_k^c)}\| u_n(s)-u_m(s)\|_{L^2(\Omega_k^c)}\le\frac{\epsilon}{4}. \quad (4.14)$$

Similar as m,n large enough

$$\int_T^t\int_{\Omega_k^c}(f(x,u_n)-f(x,u_m))(u_{n_t}(s)-u_{m_t}(s))dxds \le l\sup_{T\le s\le t}\| u_{n_t}(s)-u_{m_t}(s)\|_{L^2(\Omega_k^c)}\| u_n(s)-u_m(s)\|_{L^2(\Omega_k^c)}\le\frac{\epsilon}{4}. \quad (4.15)$$

Also we can obtain that as m,n large enough

$$\int_T^t\int_s^t\int_{\Omega_k^c}(u_{n_t}(\xi)-u_{m_t}(\xi))(f(x,u_n(\xi))-f(x,u_m(\xi)))dxd\xi ds \le\frac{\epsilon}{4}. \quad (4.16)$$

So we can get

$$\psi_T^t((u_n^1(T),v_n^1(T)),(u_m^2(T),v_m^2(T))) \le \epsilon+\phi_T^t((u_n^1(T),v_n^1(T)),(u_m^2(T),v_m^2(T))) \quad (4.17)$$

Next, for any fixed $\epsilon>0$, and some fixed t, let $T<t$ such that $t-T$ so large that

$$\frac{C_M}{t-T}<\frac{\epsilon}{2}.$$

Hence, from definition 3.1, 3.2, we only need to verify that ϕ_T^t in (4.17) is the contractive function for each fixed T.

Now, we will deal with each term in (4.12) one by one.

Firstly, from Lemma 3.2 and (i)−(iv) in Lemma 4.1, we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{\Omega_k}\varepsilon(t)(u_{n_t}(t)-u_{m_t}(t))(u_n(t)-u_m(t))dx=0, \quad (4.18)$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_{\Omega_k}L(u_{n_t}(s)-u_{m_t}(s))(u_n(s)-u_m(s))dxds=0, \quad (4.19)$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_{\Omega_k}(f(x,u_n)-f(x,u_m))(u_n(s)-u_m(s))dxds=0. \quad (4.20)$$

Similar to [4,16], we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_{\Omega_k}(f(x,u_n)-f(x,u_m))(u_{n_t}(s)-u_{m_t}(s))dxds=0,$$

At the same time, for each fixed t, $|\int_s^t\int_{\Omega_k}(u_{n_t}(\xi)-u_{m_t}(\xi))(f(x,u_n(\xi))-f(x,u_m(\xi)))dxd\xi|$ is bounded, then by the Lebesgue dominated convergence theorem we have

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_T^t\int_s^t\int_{\Omega_k}(u_{n_t}(\xi)-u_{m_t}(\xi))(f(x,u_n(\xi))-f(x,u_m(\xi)))dxd\xi ds\\&=\int_T^t(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^t\int_{\Omega_k}(u_{n_t}(\xi)-u_{m_t}(\xi))(f(x,u_n(\xi))-f(x,u_m(\xi)))dxd\xi)ds\\&=0.\end{aligned} \quad (4.21)$$

Hence, collecting all (4.18)−(4.21), we get that ϕ_T^t is the contractive function, so ϕ_T^t is asymptotic contractive function, then from (4.21) we know that the process is asymptotical contractive process. □

Proof of Theorem 4.1 From Lemma 3.3−Lemma 3.6, and Theorem 4.3,Theorem 2.9 we can easily obtain the result. □

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