A Class of Marshall Type Maximal Inequalities for Conditional Demimartingales

Yali Lu, Decheng Feng, Xia Lin

***Abstract* — In this paper, we got a further studying of Marshall type inequalities. We obtained a class of Marshall type maximal inequalities for conditional demimartingales {S_n, n>1} by using conditional Hölder inequalities and some inequalities related to conditional demimartingales.**

***Keywords*—Conditional Demimartingales; Marshall Type Inequalities; Maximal Inequalities**

I. INTRODUCTION

In this paper, let {S_n, n > 1} be a random variables sequence defined on (Ω,F,P). Denote S_0=0, I(A) is indicator function of set A, p > 0, p≠1 and $\frac{1}{p}+\frac{1}{q}$ =1. Let $\Lambda=\left\{\max_{1\le k\le n} S_k \ge \varepsilon\right\}$.

Definition 1 Let {S_n, n >1} is a random variables sequence on L^1(Ω,F,P). If for all $1 \le i < j < \infty$,

E[(S_j - S_i)f (S_1, . . . , S_j)] > 0

for every componentwise nondecreasing function f where expectation is defined, then the sequence {S_n, n>1} is called a demimartingale. In addition, if assume f is nonnegative, then {S_n, n>1} is called a demisubmartingale.

The concept of demimartingale was introduced by Newman et al[1], after that many scholars got further studying about it and investigated some interesting results and applications[2-11].

For zero mean square integrable random variable X ,we have

$$P\{X \ge \varepsilon\} \le \frac{EX^2}{\varepsilon^2 + EX^2}, \forall \varepsilon > 0.$$

Marshall[2] generalized above inequality as following form

$$P\left\{\max_{1\le k\le n}(X_1+X_2+\cdots+X_k)\right\} \le \frac{\sum_{k=1}^{n} EX^2}{\varepsilon^2+\sum_{k=1}^{n} EX^2}, \forall \varepsilon > 0$$

(1)

where $EX_k = 0$, $E(X_k|X_1, X_2, \ldots, X_{k-1}) = 0$ a.e., k > 2 and $EX_k^2 < \infty$, $k > 1$.

Under above condition, if assume

$$S_n = \sum_{k=1}^{n} X_k$$

then {S_n, n >1} would be a demimartingale. Mu et al[4] generalized inequality (1) under the condition of $E|X_i|^p < \infty$,i > 1 and p > 2, and got Marshall type inequality as following form

$$P\left\{\max_{1\le k\le n} S_k \ge \varepsilon\right\} \le \frac{E|S_n|^p}{\alpha^{1-p}\varepsilon^p + E|S_n|^p}, \forall \varepsilon > 0$$

where α is maximum value of following function

$h(x) = 1 - x + (1-x)^{2-q} x^{q-1}, x \in [0,1]$.

After that, Hu et al[5] generalized results in literature [4] to the case of condition demi-martingales and obtained the Marshall type probability inequalities for demimartingales.

Assume X and Y are random variables defined on probability space (Ω,F,P) and $EX^2 < \infty, EY^2 < \infty$.

Let F be a sub σ-algebra of A,

Prakasa Rao[14] defined conditional covariance of X and Y(F-covariance) as follow

$$Cov^F = E^F((X - E^F X)(Y - E^F Y))$$

where $E^F Z$ is conditional expectation of random variable Z which has been given condition *F*.

Christofides et al[12] introduced following definition.

Definition 2 Let {S_n, n > 1} be a random variables

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sequence on $L^1(\Omega,F,P)$, if for all $1 \le i < j < \infty$,

$$E^F[(S_j - S_i) f(S_1, \cdots S_j)] \ge 0 \text{ a.s.}$$

for every componentwise nondecreasing function f such that above expectation is defined, then $\{S_n, n > 1\}$ is called a F-demimartingale. In addition, if assume f is nonnegative, then $\{S_n, n > 1\}$ is called a F-demisubmartingale.

Inspired by literature[13], we extend Marshall type maximal inequalities for nonnegative demimartingales $\{S_n, n > 1\}$ in literature [4] to the case of conditional demimartingales $\{S_n, n > 1\}$.

II. MAXIMAL INEQUALITIES FOR CONDITIONAL DEMIMARTINGALES

Lemma 3[15] If $E^F|X|^p < \infty$ a.s., $E^F|Y|^p < \infty$ a.s., then

$$E^F|XY| \le (E^F|X|^p)^{\frac{1}{p}} (E^F|Y|^q)^{\frac{1}{q}} \text{ a.s. }, p>1, \quad (2)$$

$$E^F|XY| \ge (E^F|X|^p)^{\frac{1}{p}} (E^F|Y|^q)^{\frac{1}{q}} \text{ a.s. }, 0<p<1. \quad (3)$$

Lemma 4[12] Let $\{S_n, n>1\}$ be a F-demi(sub)martingale and $g(\cdot)$ be a nonnegative nondecreasing convex function, then $\{g(S_n), n>1\}$ is a F-demisubmartingale.

Lemma 5[13] Let $\{S_n, n>1\}$ be a F-demisubmartingale, then for any F-measurable random variable $\varepsilon > 0$ a.s.

$$\varepsilon P^F(\max_{1\le k\le n} S_k \ge \varepsilon) \le E^F(S_n I(\max_{1\le k\le n} S_k \ge \varepsilon)) \text{ a.s. } (4)$$

Lemma 6 Let $\{S_n, n > 1\}$ be a F-demisubmartingle and $E^F S_1 \le 0$ a.s., assume there is a $1 < p < 2$, such that $E^F|S_n|^p < \infty$ a.s. for all $n > 1$ established, then for every F-measurable random variable $\varepsilon > 0$ a.s. while $P(\Lambda) \in [\frac{1}{2}, 1]$ a.s.

$$[P^F(\Lambda)(1-P^F(\Lambda))^q + (1-P^F(\Lambda))P^F(\Lambda)^q]^{\frac{1}{q}} (E^F|S_n|^p)^{\frac{1}{p}} \ge \varepsilon P^F(\Lambda)$$

a.s.

Proof. Denote $Y = I(\Lambda)$. By using conditional Hölder inequality (2) and lemma 5, we can get

$$\begin{aligned}
&(E^F|Y - E^F Y|^q)^{\frac{1}{q}} (E^F|S_n|^p)^{\frac{1}{p}} \\
&\ge E^F[(Y - E^F Y) S_n] \\
&= E^F(Y S_n) - E^F Y E^F S_n \\
&\ge E^F(I(\Lambda) S_n) \\
&\ge E^F(\varepsilon I(\Lambda)) \\
&= \varepsilon P^F(\Lambda)
\end{aligned}$$

a. s.

since

$$P^F(\Lambda) \le P^F(\Lambda)(1-P^F(\Lambda))^q + (1-P^F(\Lambda))P^F(\Lambda)^q$$

a.s.

where $P^F(\Lambda) \in [\frac{1}{2}, 1]$ a.s., thus the proposition is proved.

Theorem 7 Let $\{S_n, n > 1\}$ be a F-demimartingale, $E^F S_1 < 0$. If there is a $1 < p < 2$, such that for all $n > 1$, have $0 < E^F|S_n|^p < \infty$ a.s., then for every F-measurable random variable $\varepsilon > 0$ a.s. while $P^F(\Lambda) \in [\frac{1}{2}, 1]$ a.s., we have

$$P^F(\Lambda) \le \frac{1}{1+M} \text{ a.s.}$$

where M is positive solution of following function,

$$x^q = (\beta - 1)x + \beta, \quad x \in (0, +\infty). \quad (5)$$

where $\beta = \dfrac{\varepsilon^q}{(E^F|S_n|^p)^{\frac{q}{p}}}$ a.s.

Proof. By lemma 6 we can get

$$[P^F(\Lambda)(1-P^F(\Lambda))^q + (1-P^F(\Lambda))P^F(\Lambda)^q] \ge \varepsilon^q P^F(\Lambda)^q$$

a.s.

divide both side by $P^F(\Lambda)^q$

$$[P^F(\Lambda)\frac{(1-P^F(\Lambda))^q}{P^F(\Lambda)^q} + (1+P^F(\Lambda))](E^F|S_n|^p)^{\frac{q}{p}} \ge \varepsilon^q \text{ a.s.}$$

let $x_0 = \dfrac{1-P^F(\Lambda)}{P^F(\Lambda)}$ a.s., $\beta = \dfrac{\varepsilon^q}{(E^F|S_n|^p)^{\frac{q}{p}}}$ a.s., then

$$P^F(\Lambda) = \frac{1}{1+x_0} \text{ a.s.}$$

since

$P^F(\Lambda)x_0{}^q + 1 - P^F(\Lambda) \ge \beta$ a.s.

then

$$\frac{x_0{}^q}{1+x_0} + \frac{x_0}{1+x_0} \ge \beta \text{ a.s.}$$

namely

$x_0{}^q \ge (\beta - 1)x_0 + \beta$ a.s.

(6)

let $g(x) = x^q - (\beta - 1)x_0 + \beta$, M is positive solution of equation(5). Since $g''(x) = q(q-1)x^{q-2} > 0$, $x \in (0, +\infty)$, we can easily get g(x) is a convex function on interval $(0, +\infty)$, which mean for $\forall x \in (0, M)$

$$\frac{g(x) - g(0)}{x - 0} \le \frac{g(M) - g(x)}{M - x},$$

since $g(0) = -\beta < 0$ and $g(M) = 0$. Then for $x \in (0, M)$ can get $g(x) < 0$, namely M is minimum value of equation (6).

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