A Class of Marshall Type Maximal Inequalities for Conditional Demimartingales

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Abstract — In this paper, we got a further studying of Marshall type inequalities. We obtained a class of Marshall type maximal inequalities for conditional demimartingales \( \{S_n, n>1\} \) by using conditional Hölder inequalities and some inequalities related to conditional demimartingales.

Keywords—Conditional Demimartingales; Marshall Type Inequalities; Maximal Inequalities

1. INTRODUCTION

In this paper, let \( \{S_n, n>1\} \) be a random variables sequence defined on \((\Omega,F,P)\). Denote \( S_0=0, I(A) \) is indicator function of set \( A, p>0, p\neq 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( \Lambda = \lim_{q \to 0} \max_{i \in \omega} S_i \geq \varepsilon \).

Definition 1 Let \( \{S_n, n>1\} \) be a random variables sequence on \( L^1(\Omega,F,P) \). If for all \( 1 \leq i < j < \infty \), \( E[(S_j - S_i)f(S_1, \ldots, S_i)] > 0 \) for every componentwise nondecreasing function \( f \) where expectation is defined, then the sequence \( \{S_n, n>1\} \) is called a demimartingale. In addition, if assume \( f \) is nonnegative, then \( \{S_n, n>1\} \) is called a demisubmartingale.

The concept of demimartingale was introduced by Newman et al[1], after that many scholars got further studying about it and investigated some interesting results and applications[2-11].

For zero mean square integrable random variable \( X \), we have

\[
P(X \geq \varepsilon) \leq \frac{EX^2}{\varepsilon^2 + EX^2}, \forall \varepsilon > 0.
\]


\[
P\left(\max_{1 \leq n \leq m}(X_1 + X_2 + \cdots + X_k) \leq \sum_{k=1}^{n} \frac{EX^2}{\varepsilon^2 + \sum_{k=1}^{n} EX^2}, \forall \varepsilon > 0 \right)
\]

(1)

where \( EX_k = 0, E(X_k|X_1, X_2, \ldots, X_{k-1}) = 0 \) a.e., \( k > 2 \) and \( EX_k^2 < \infty, k > 1 \).

Under above condition, if assume

\[
S_n = \sum_{k=1}^{n} X_k
\]

then \( \{S_n, n>1\} \) would be a demimartingale. Mu et al[4] generalized inequality (1) under the condition of \( E|X_i|^p < \infty, i > 1 \) and \( p > 2 \), and got Marshall type inequality as following form

\[
P\left(\max_{1 \leq n \leq m} S_n \geq \varepsilon\right) \leq \frac{E|S_n|^p}{\alpha^{1-p} \varepsilon^p + E|S_n|^p}, \forall \varepsilon > 0
\]

where \( \alpha \) is maximum value of \( f \) for \( |X| \).

Prakasa Rao[14] defined conditional covariance of \( X \) and \( Y(F\text{-covariance}) \) as follow

\[
Cov^F = E^F((X - E^F X)(Y - E^F Y))
\]

where \( E^F Z \) is conditional expectation of random variable \( Z \) which has been given condition \( F \).

Christofides et al[12] introduced following definition.

Definition 2 Let \( \{S_n, n>1\} \) be a random variables
sequence on L^1(Ω,F,P), if for all 1 ≤ i < j < ∞,
\[ E^F [(S_j - S_i) f(S_i, \ldots, S_j)] \geq 0 \text{ a.s.} \]
for every componentwise nondecreasing function f such that above expectation is defined, then \( \{S_n, n > 1\} \) is called a F-demimartingale. In addition, if assume f is nonnegative, then \( \{S_n, n > 1\} \) is called a F-demisubmartingale.

Inspired by literature[13], we extend Marshall type maximal inequalities for nonnegative demimartingales \( \{S_n, n > 1\} \) in literature [4] to the case of conditional demimartingales \( \{S_n, n > 1\} \).

II. MAXIMAL INEQUALITIES FOR CONDITIONAL DEMIMARTINGALES

Lemma 3[15] If \( E^F |X|^p < \infty \) a.s., \( E^F |Y|^p < \infty \) a.s., then
\[ E^F |XY| \leq (E^F |X|^p)^{\frac{1}{p}} (E^F |Y|^q)^{\frac{1}{q}} \text{ a.s., } p > 1. \]
(2)
\[ E^F |XY| \geq (E^F |X|^p)^{\frac{1}{p}} (E^F |Y|^q)^{\frac{1}{q}} \text{ a.s., } 0 < p < 1. \]
(3)

Lemma 4[12] Let \( \{S_n, n > 1\} \) be a F-demi(sub)martingale and \( g(\cdot) \) be a nonnegative nondecreasing convex function, then \( \{g(S_n), n > 1\} \) is a F-demi(sub)martingale.

Lemma 5[13] Let \( \{S_n, n > 1\} \) be a F-demi(sub)martingale, then for any F-measurable random variable \( \varepsilon > 0 \) a.s.
\[ \varepsilon P^F (\max_{1 \leq k \leq n} S_k \geq \varepsilon) \leq E^F (S_n I(\max_{1 \leq k \leq n} S_k \geq \varepsilon)) \text{ a.s.} \]
(4)

Lemma 6 Let \( \{S_n, n > 1\} \) be a F-demi(sub)martingale and \( E^F S_1 \leq 0 \) a.s., assume there is a \( 1 < p < 2 \), such that \( E^F |S_n|^p < \infty \) a.s. for all \( n > 1 \) established, then for every F-measurable random variable \( \varepsilon > 0 \) a.s. while \( P(\Lambda) \in [\frac{1}{2}, 1] \) a.s.
\[ P^F (\Lambda)(1 - P^F (\Lambda))^q + (1 - P^F (\Lambda))P^F (\Lambda)^q \] \[ \geq (E^F |S_n|^p)^{\frac{1}{p}} \varepsilon P^F (\Lambda) \]
a.s. Proof. Denote \( Y = I(\Lambda) \). By using conditional Hölder inequality (2) and lemma 5, we can get
\[ (E^F |Y - E^F Y|^q)^{\frac{1}{q}} (E^F |S_n|^p)^{\frac{1}{p}} \]
\[ \geq E^F [(Y - E^F Y)S_n] \geq E^F (YS_n) - E^F YE^F S_n \]
\[ \geq E^F (I(\Lambda)S_n) \geq E^F (\varepsilon I(\Lambda)) = \varepsilon P^F (\Lambda) \]
a.s.

where \( P^F (\Lambda) \in [\frac{1}{2}, 1] \) a.s., thus the proposition is proved.

Theorem 7 Let \( \{S_n, n > 1\} \) be a F-demimartingale, \( E^F S_1 < 0 \). If there is a \( 1 < p < 2 \), such that for all \( n > 1 \), have \( 0 < E^F |S_n|^p < \infty \) a.s., then for every F-measurable random variable \( \varepsilon > 0 \) a.s. while \( P^F (\Lambda) \in [\frac{1}{2}, 1] \) a.s., we have
\[ P^F (\Lambda) \leq \frac{1}{1 + M} \]
a.s.

where \( M \) is positive solution of following function,
\[ x^q = (\beta - 1)x + \beta \quad x \in (0, +\infty). \]
(5)

where \( \beta = \frac{\varepsilon^q}{P^F (\Lambda)} \) a.s.

Proof. By lemma 6 we can get
\[ [P^F (\Lambda)(1 - P^F (\Lambda))^q + (1 - P^F (\Lambda))P^F (\Lambda)^q] \geq \varepsilon P^F (\Lambda)^q \]
a.s. divide both side by \( P^F (\Lambda)^q \)
\[ (1 - P^F (\Lambda))^q + (1 + P^F (\Lambda))\left(\frac{E^F |S_n|^p}{P^F (\Lambda)^q}\right) \geq \varepsilon \]
a.s.

let \( x_0 = \frac{1 - P^F (\Lambda)}{P^F (\Lambda)} \) a.s., \( \beta = \frac{\varepsilon^q}{P^F (\Lambda)^q} \) a.s., then
\[ P^F (\Lambda) = \frac{1}{1 + x_0} \]
a.s.
\[ P^F (\Lambda) x_0^g + 1 - P^F (\Lambda) \geq \beta \text{ a.s.} \]

then

\[ \frac{x_0^g}{1 + x_0} + \frac{x_0}{1 + x_0} \geq \beta \text{ a.s.} \]

namely

\[ x_0^g \geq (\beta - 1)x_0 + \beta \text{ a.s.} \]

(6)

Let \( g(x) = x^q - (\beta - 1)x_0 + \beta \), \( M \) is positive solution of equation (5). Since \( g''(x) = q(q - 1)x^{q - 2} > 0 \), \( x \in (0, +\infty) \), we can easily get \( g(x) \) is a convex function on interval \( (0, +\infty) \), which mean for \( \forall x \in (0, M) \)

\[ \frac{g(x) - g(0)}{x - 0} \leq \frac{g(M) - g(x)}{M - x} , \]

since \( g(0) = -\beta < 0 \) and \( g(M) = 0 \). Then for \( x \in (0, M) \) can get \( g(x) < 0 \), namely \( M \) is minimum value of equation (6).

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