Inventory Management Model for Deteriorating and Ameliorating items with Cubic Demand under Salvage Value and Permissible Delay in Payments.

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Abstract **- It is seen widely that the demand of items like fruits, flowers, green vegetables, dairy products etc is very high in our daily life and at the same time it is also decreased owing to spoilage or decay as these are kept in farms, in flower shops, in supermarkets or in cold storages. So we cannot ignore the effect both of amelioration and deterioration in the inventory management system. The assumption of constant demand rate may not be always appropriate for many inventory goods like milk, vegetables etc., the age of inventory has negative impact on demand due to loss of consumer confidence on quality of such products. Here demand rate is considered as a cubic function of time and shortages are allowed which are fully backlogged. The model is solved with salvages value associated to the units deteriorating during the cycle. Moreover, collaborative business policy in an inventory management system between the venders and the customers is most important. There are many inventory situations where the buyer is allowed a permissible period to pay back the cost of goods bought without paying any interest. This permissible delay in payment is a win–win strategy for sharing profit in the collaborative system. However, the purchaser can earn interest on the sales of the inventory during the payment period.**

The present paper deals with an inventory model assuming time-varying deteriorating items and two-parameter Weibull distributed ameliorating items with cubic demand under salvage allowing permissible delay in payments.

Finally the model is illustrated with the help of numerical examples and some particular cases are illustrated numerically.

Index Terms **- Inventory, deteriorating items, ameliorating items, Weibull distributed, time-varying, cubic demand, salvages value, shortages and delay payment.**

AMS **Mathematics Subject Classification (2010):** 90B05

I. Introduction

It is very often that many items like fruits, flowers, green vegetables, dairy products, radioactive substances etc deteriorate over time. Several researchers have addressed the importance of the deterioration phenomenon in their field of applications; as a result, many inventory models with deteriorating items have been developed. But the effect of ameliorating items like broiler, ducks, pigs etc. in the poultry farm, highbred fishes in berry (pond) is increasing gradually in the inventory management system. It is a natural phenomenon observing in much life stock models. A few researchers have focused on ameliorating system. When these items are in storage, the stock increases (in weight) due to growth of the items and also decrease due to death, various diseases or some other factors. At the point when these things are away, the stock increases (in weight) because of development of the things. Furthermore the stock diminishes because of death, different illnesses or due to some different components. Hwang [1997] developed an inventory model for ameliorating items only. Again Hwang [2004] added to a stock model for both ameliorating and deteriorating things independently. Mallick et al. [2018] has considered a creation inventory model for both ameliorating and deteriorating items. Many other researchers like Moon et al[2005], L-Q ji [2008], Valliathal et al [2010], Nodoust [2017], Dr. Biswaranjan Mandal [2010],] are mentioned a few. In this paper, we consider the model where the deterioration is time-varying and the environment of Amelioration followed by Weibull Distribution to describe the different life spans effectively by utilizing the changes of parameters.

There are many inventory models assuming Weibull distributed deteriorating items under ramp type demand and shortages. Sahoo et al. [2010] formulated an EOQ model for price dependent demand rate and time-varying holding cost. Hung [2011] have used generalized type demand, deterioration and backorder rates. Mishra and Singh [2011] find an inventory model for ramp type demand, and time dependent deteriorating items with salvage value and shortages. According to Mishra and Singh [2011] an inventory model for deteriorating items with uniform replenishment rate with power form demand, the rate of deterioration is cubic polynomial. Anil Kumar Sharma et.al. [2012] are considered an inventory model with time dependent holding cost. Pratibha Yadav [2013] have used cubic demand rate and production rate is variable with Weibull distribution. Sharma et al [2015], Biswaranjan Mandal [2020] and many others developed inventory models assuming demand rate as cubic function of time. Karthikeyan et al [2015] developed a model to determine the optimum order quantity for constant deteriorating items with cubic demand and salvage value.

The effect of a permissible delay in payment plays an important role in an inventory management system. While developing a mathematical model in inventory control, the classical approach is that the payment would be made to the supplier for the goods soon after receiving the consignment. However, in day-to-day dealings, it is often found that a supplier allows certain fixed period of time for settling the amount owed to him for the items supplied. During this period no interest is charged, but beyond this period of time interest is charged by the supplier under the terms and conditions agreed upon. This gives an advantage to the customers in the sense that they do not have to pay the supplier immediately after receiving the consignment, but instead, can delay their payment until the end of the allowed time period. Moreover, the customer can earn interest on the revenues accumulated from the sale of the product received. Inventory problems with such permissible credit period was first considered by Goyal [1995]. Later, Jaggi and Aggarwa [1995] have developed a model to determine the optimum order quantity for deteriorating items under a permissible delay in payment by rectifying certain flaws in the paper of Goyal. After that many researchers Mandal and Phaujdar [1989], Chang et al [2002], Huang [2007], Cheng et al [2011], Li et al [2014], Behara et al [2017], Dr. Biswaranjan Mandal [2021] derived an ordering policy of a deteriorating inventory system under conditions of trade credit policy.

For these sort of situations, efforts have been made to develop a realistic inventory model with time-varying deterioration rate and two-parameter Weibull distributed ameliorating rate. The demand rate is considered as a cubic function of time. The model is solved with salvages value associated to the units deteriorating during the cycle under permissible delay in payments. Shortages are allowed and fully backlogged. Finally the model is illustrated with the help of numerical examples and some particular cases are illustrated numerically.

II. Notations and Assumptions

The mathematical models are developed under the following notations and assumptions:

Notations:

(i) I(t) : On hand inventory level at time t,
(ii) Q : The maximum inventory level during the cycle,
(iii) t_c : Permissible delay in settling the account,
(iv) t_1 : Length of the period with positive stock of the item,
(v) T : The fixed length of each production cycle,
(vi) A_0 : The ordering cost per order during the cycle period,
(vii) p_c : The purchasing cost per unit item,
(viii) d_c : The deterioration cost per unit item,
(ix) a_c : The cost of amelioration per unit item,
(x) c_s : The shortage cost per unit item,
(xi) I_e : The interest earned per unit time,
(xii) I_p : The interest paid per unit time,
(xiii) OC : Ordering cost per order,
(xiv) HC : Holding cost over the cycle period,
(xv) CD : Cost due to deterioration over the cycle period,
(xvi) AMC : Amelioration cost over the cycle period,
(xvii) SV : Salvage value over the cycle period,
(xviii) CS : Cost due to shortage over the cycle period,
(xix) IP : Total Interest payable over the cycle period,
(xx) IE : Total Interest earned over the cycle period,
(xxi) TC : The average total cost per unit time.

Assumptions:

(i). The inventory system included only one item.
(ii). The demand rate is time dependent cubic function.

R(t) = a + bt + ct^2 + dt^3, a, b, c, d $\geq$ 0 where a is the initial demand rate, b is the initial rate of change of demand, c is the rate at which the demand rate increases and d is the rate at which the change in the demand rate itself increases.

(iii). The time-varying deterioration rate is given by

$$\theta(t) = \theta_o t, 0 \le \theta_o << 1.$$

(iv). A(t) is the amelioration rate following Weibull distributed

$A(t) = \alpha\beta t^{\beta-1}, 0 \le \alpha << 1, \beta \ge 1$, where α is the shape parameter and β is the scale parameter.

(v). The holding cost is time dependent i.e h(t) = $\delta t, \delta > 0$.

(vi). The salvage value kd_c , $0 \le k < 1$ is associated with deteriorated units during a cycle time.

(vii). Lead time is zero.

(viii). Shortages are allowed and fully backlogged.

(ix). Replenishment rate is infinite.

(x). The time horizon is infinite.

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III. Formulation and Solution of the Model

Let I(t) be the on-hand inventory level at time t. Depletion in inventory occurs due to demand, deterioration and amelioration as a whole For this, the inventory level gradually diminishes during the period (0, t_1) and ultimately falls to zero at t = t_1. Shortages occur during time period (t_1, T) which are fully backlogged. The differential equations which the on-hand inventory I(t) over entire cycle time (0,T) governed by the following :

$$\frac{dI(t)}{dt} + (\theta(t) - A(t))I(t) = -R(t), 0 \le t \le t_1 \quad (3.1)$$

And $$\frac{dI(t)}{dt} = -R(t), t_1 \le t \le T \quad (3.2)$$

The initial condition is $I(0) = Q$ and $I(t_1) = 0$ (3.3)

Putting the values of $\theta(t) = \theta_o t, 0 \le \theta_o << 1$, $A(t) = \alpha\beta t^{\beta-1}, 0 \le \alpha << 1, \beta \ge 1$
and R(t) = a + bt + ct^2 + dt^3, a, b, c, d $\ge$ 0, we get

$$\frac{dI(t)}{dt} + (\theta_o t - \alpha\beta t^{\beta-1})I(t) = -(a + bt + ct^2 + dt^3), 0 \le t \le t_1 \quad (3.4)$$

And $$\frac{dI(t)}{dt} = -(a + bt + ct^2 + dt^3), t_1 \le t \le T \quad (3.5)$$

Now solving the equations (3.4) and (3.5) using the initial condition (3.3) and neglecting the second and higher powers of θ_o and α [since $O(\theta_o{}^2)$ and $O(\alpha^2)$ are very small as $0 \le \theta_o, \alpha << 1$], we get

$$\begin{aligned} I(t) = {} & a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{c}{3}(t_1^3 - t^3) + \frac{d}{4}(t_1^4 - t^4) + \frac{a\theta_o}{6}(t_1^3 - 3t_1 t^2 + 2t^3) \\ & + \frac{b\theta_o}{8}(t_1^4 - 2t^2{}_1 t^2 + t^4) + \frac{c\theta_o}{30}(3t_1^5 - 5t^3{}_1 t^2 + 2t^5) + \frac{d\theta_o}{24}(2t_1^6 - 3t^4{}_1 t^2 + t^6), \\ & - \frac{a\alpha}{\beta+1}(t_1^{\beta+1} - t^{\beta+1}) - \frac{b\alpha}{\beta+2}(t_1^{\beta+2} - t^{\beta+2}) - \frac{c\alpha}{\beta+3}(t_1^{\beta+3} - t^{\beta+3}) - \frac{d\alpha}{\beta+4}(t_1^{\beta+4} - t^{\beta+4}) \\ & + a\alpha t^{\beta}(t_1 - t) + \frac{b\alpha}{2}t^{\beta}(t_1{}^2 - t^2) + \frac{c\alpha}{3}t^{\beta}(t_1{}^3 - t^3) + \frac{d\alpha}{4}t^{\beta}(t_1{}^4 - t^4),\ 0 \le t \le t_1 \end{aligned} \quad (3.6)$$

And $$I(t) = a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{c}{3}(t_1^3 - t^3) + \frac{d}{4}(t_1^4 - t^4),\ t_1 \le t \le T \quad (3.7)$$

Since I(0) = Q, we get from equation (3.6) the following expression

$$\begin{aligned} Q = {} & at_1 + \frac{b}{2}t_1^2 + (\frac{c}{3} + \frac{a\theta_o}{6})t_1^3 + (\frac{d}{4} + \frac{b\theta_o}{8})t_1^4 + \frac{c\theta_o}{10}t_1^5 + \frac{d\theta_o}{12}t_1^6 \\ & - \frac{a\alpha}{\beta+1}t_1^{\beta+1} - \frac{b\alpha}{\beta+2}t_1^{\beta+2} - \frac{c\alpha}{\beta+3}t_1^{\beta+3} - \frac{d\alpha}{\beta+4}t_1^{\beta+4} \end{aligned} \quad (3.8)$$

IV. Inventory Scenarios

Since there is the total depletion of the on-hand inventory at time $t_1(< T)$, the following two distinct cases are observed.

(i). $t_c \le t_1 < T$ (payment at or before the total depletion of inventory)

(ii). $t_1 < t_c$ (payment after depletion of inventory)

The total interest payable over the entire cycle (0,T) is

$$IP = \begin{cases} IP_1, 0 < t_c \le t_1 \\ IP_2, t_1 < t_c < T \end{cases}$$

The total interest earned over the entire cycle (0,T) is

$$IE = \{ \begin{matrix} IE_1, 0 < t_c \le t_1 \\ IE_2, t_1 < t_c < T \end{matrix}$$

The total average cost of the system per unit time is given by

$$TC = \{ \begin{matrix} TC_1, 0 < t_c \le t_1 \\ TC_2, t_1 < t_c < T \end{matrix}$$

4.1 : Case 1: $t_c \le t_1 < T$ **(payment at or before the total depletion of inventory):**

Cost Components:

The total cost over the period [0,T] consists of the following cost components :

(i). Ordering cost **(OC)** over the period [0,T] = A_0 (fixed) (4.1.1)

(ii). Holding cost **(HC)** for carrying inventory over the period [0,T] = $\int_0^{t_1} \delta t I(t)dt$

Using the expression of I(t) given in (3.6) and then integrating we get

$$HC = \delta\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + (\frac{c}{10} + \frac{a\theta_o}{40})t_1^5 + (\frac{d}{12} + \frac{b\theta_o}{48})t_1^6 + \frac{c\theta_o}{56}t_1^7 + \frac{d\theta_o}{64}t_1^8 - \frac{a\alpha\beta}{2(\beta+2)(\beta+3)}t_1^{\beta+3} - \frac{b\alpha\beta}{2(\beta+2)(\beta+4)}t_1^{\beta+4} - \frac{c\alpha\beta}{2(\beta+2)(\beta+5)}t_1^{\beta+5} - \frac{d\alpha\beta}{2(\beta+2)(\beta+6)}t_1^{\beta+6}\} \quad (4.1.2)$$

(iii). Cost due to deterioration **(CD)** over the period [0,T] = $d_c \int_0^{t_1} \theta_o t I(t)dt$

$$= d_c\theta_o\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 + \frac{d}{12}t_1^6\} \quad (4.1.3)$$

(iv). Salvage cost **(SV)** over the period [0,T] = k $d_c\theta_o\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 + \frac{d}{12}t_1^6\}$ (4.1.4)

(v). The amelioration cost **(AMC)** over the period [0,T] = $a_c \int_0^{t_1} \alpha\beta t^{\beta-1} I(t)dt$

$$= a_c\alpha\{\frac{a}{\beta+1}t_1^{\beta+1} + \frac{b}{\beta+2}t_1^{\beta+2} + \frac{c}{\beta+3}t_1^{\beta+3} + \frac{d}{\beta+4}t_1^{\beta+4}\} \quad (4.1.5)$$

(vi). Cost due to shortage **(CS)** over the period [0,T] = $c_s \int_{t_1}^{T} (T-t)R(t)dt$

$$= \int_{t_1}^{T} (T-t)R(t)dt = \int_{t_1}^{T} (T-t)(a+bt+ct^2+dt^3)dt$$

$$= c_s\{\frac{a}{2}(T^2 - 2Tt_1 + t_1^2) + \frac{b}{6}(T^3 - 3Tt_1^2 + 2t_1^3) + \frac{c}{12}(T^4 - 4Tt_1^3 + 3t_1^4) + \frac{d}{20}(T^5 - 5Tt_1^4 + 4t_1^5)\} \quad (4.1.6)$$

(vii). The total interest payable **(** IP_1 **)** over the entire cycle (0,T) = $p_c I_p \int_{t_c}^{t_1} I(t)dt$

Using the expression of I(t) given in (3.6) and then integrating we get the following

$$IP_1 = p_c I_p [\frac{a}{2}(t_1^2 - 2t_c t_1 + t_c^2) + \frac{b}{6}(2t_1^3 - 3t_c t_1^2 + t_c^3) + \frac{c}{12}(3t_1^4 - 4t_c t_1^3 + t_c^4) + \frac{d}{20}(4t_1^5 - 5t_c t_1^4 + t_c^5)$$
$$+ \frac{a\theta_0}{6}(-t_c t_1^3 + t_c^3 t_1 + \frac{1}{2}t_1^4 - \frac{1}{2}t_c^4) + \frac{b\theta_0}{8}(-t_c t_1^4 + \frac{2}{3}t_c^3 t_1^2 + \frac{8}{15}t_1^5 - \frac{1}{5}t_c^5)$$
$$+ \frac{c\theta_0}{30}(-3t_c t_1^5 + \frac{5}{3}t_c^3 t_1^3 + \frac{5}{3}t_1^6 - \frac{1}{3}t_c^6) + \frac{d\theta_0}{24}(-2t_c t_1^6 + t_c^3 t_1^4 + \frac{8}{7}t_1^7 - \frac{1}{7}t_c^7)$$
$$-\frac{a\alpha}{\beta+1}\{(t_1 - t_c)t_1^{\beta+1} - \frac{1}{\beta+2}(t_1^{\beta+2} - t_c^{\beta+2})\} - \frac{b\alpha}{\beta+2}\{(t_1 - t_c)t_1^{\beta+2} - \frac{1}{\beta+3}(t_1^{\beta+3} - t_c^{\beta+3})\}$$
$$-\frac{c\alpha}{\beta+3}\{(t_1 - t_c)t_1^{\beta+3} - \frac{1}{\beta+4}(t_1^{\beta+4} - t_c^{\beta+4})\} - \frac{d\alpha}{\beta+4}\{(t_1 - t_c)t_1^{\beta+4} - \frac{1}{\beta+5}(t_1^{\beta+5} - t_c^{\beta+5})\}$$
$$+a\alpha\{\frac{t_1}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+2}(t_1^{\beta+2} - t_c^{\beta+2})\} + \frac{b\alpha}{2}\{\frac{t_1^2}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+3}(t_1^{\beta+3} - t_c^{\beta+3})\}$$
$$+\frac{c\alpha}{3}\{\frac{t_1^3}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+4}(t_1^{\beta+4} - t_c^{\beta+4})\} + \frac{d\alpha}{4}\{\frac{t_1^4}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+5}(t_1^{\beta+5} - t_c^{\beta+5})\}]$$

(4.1.7)

(viii). The total interest earned (IE_1) over the entire cycle (0,T) = $p_c I_e \int_0^{t_1} t(a + bt + ct^2 + dt^3)dt$

$$\text{Or, } IE_1 = p_c I_e [\frac{a}{2}t_1^2 + \frac{b}{3}t_1^3 + \frac{c}{4}t_1^4 + \frac{d}{5}t_1^5] \quad (4.1.8)$$

The average total cost per unit time of the system during the cycle [0, T] will be

$$TC_1(t_1) = \frac{1}{T} [\textbf{OC + HC+CD-SV+AMC+CS+} IP_1 - IE_1]$$
$$= \frac{1}{T}[A_0 + \delta\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + (\frac{c}{10} + \frac{a\theta_o}{40})t_1^5 + (\frac{d}{12} + \frac{b\theta_o}{48})t_1^6 + \frac{c\theta_o}{56}t_1^7 + \frac{d\theta_o}{64}t_1^8$$
$$-\frac{a\alpha\beta}{2(\beta+2)(\beta+3)}t_1^{\beta+3} - \frac{b\alpha\beta}{2(\beta+2)(\beta+4)}t_1^{\beta+4} - \frac{c\alpha\beta}{2(\beta+2)(\beta+5)}t_1^{\beta+5} - \frac{d\alpha\beta}{2(\beta+2)(\beta+6)}t_1^{\beta+6}\}$$
$$+ (1-k)\, d_c\theta_o\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 + \frac{d}{12}t_1^6\} + c_s\{\frac{a}{2}(T^2 - 2Tt_1 + t_1^2) + \frac{b}{6}(T^3 - 3Tt_1^2 + 2t_1^3) +$$
$$\frac{c}{12}(T^4 - 4Tt_1^3 + 3t_1^4) + \frac{d}{20}(T^5 - 5Tt_1^4 + 4t_1^5)\} + p_c I_p [\frac{a}{2}(t_1^2 - 2t_c t_1 + t_c^2) + \frac{b}{6}(2t_1^3 - 3t_c t_1^2 + t_c^3) +$$
$$\frac{c}{12}(3t_1^4 - 4t_c t_1^3 + t_c^4) + \frac{d}{20}(4t_1^5 - 5t_c t_1^4 + t_c^5) + \frac{a\theta_0}{6}(-t_c t_1^3 + t_c^3 t_1 + \frac{1}{2}t_1^4 - \frac{1}{2}t_c^4)$$
$$+\frac{b\theta_0}{8}(-t_c t_1^4 + \frac{2}{3}t_c^3 t_1^2 + \frac{8}{15}t_1^5 - \frac{1}{5}t_c^5) + \frac{c\theta_0}{30}(-3t_c t_1^5 + \frac{5}{3}t_c^3 t_1^3 + \frac{5}{3}t_1^6 - \frac{1}{3}t_c^6) + \frac{d\theta_0}{24}(-2t_c t_1^6 + t_c^3 t_1^4 + \frac{8}{7}t_1^7 - \frac{1}{7}t_c^7)$$
$$-\frac{a\alpha}{\beta+1}\{(t_1 - t_c)t_1^{\beta+1} - \frac{1}{\beta+2}(t_1^{\beta+2} - t_c^{\beta+2})\} - \frac{b\alpha}{\beta+2}\{(t_1 - t_c)t_1^{\beta+2} - \frac{1}{\beta+3}(t_1^{\beta+3} - t_c^{\beta+3})\}$$
$$-\frac{c\alpha}{\beta+3}\{(t_1 - t_c)t_1^{\beta+3} - \frac{1}{\beta+4}(t_1^{\beta+4} - t_c^{\beta+4})\} - \frac{d\alpha}{\beta+4}\{(t_1 - t_c)t_1^{\beta+4} - \frac{1}{\beta+5}(t_1^{\beta+5} - t_c^{\beta+5})\}$$
$$+a\alpha\{\frac{t_1}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+2}(t_1^{\beta+2} - t_c^{\beta+2})\} + \frac{b\alpha}{2}\{\frac{t_1^2}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+3}(t_1^{\beta+3} - t_c^{\beta+3})\}$$
$$+\frac{c\alpha}{3}\{\frac{t_1^3}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+4}(t_1^{\beta+4} - t_c^{\beta+4})\} + \frac{d\alpha}{4}\{\frac{t_1^4}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1}) - \frac{1}{\beta+5}(t_1^{\beta+5} - t_c^{\beta+5})\}]$$
$$- p_c I_e \{\frac{a}{2}t_1^2 + \frac{b}{3}t_1^3 + \frac{c}{4}t_1^4 + \frac{d}{5}t_1^5\}] \quad (4.1.9)$$

For minimum, the necessary condition is $\frac{dTC_1(t_1)}{dt_1} = 0$

This gives $[\frac{\delta}{2}(t_1^2 + \frac{\theta_o}{4}t_1^4 - \frac{\alpha\beta}{\beta+2}t_1^{\beta+2}) + \frac{d_c\theta_o}{2}(1-k)t_1^2 + a_c\alpha t_1^\beta - c_s(T-t_1)$

$+ p_c I_p\{(t_1 - t_c) + \frac{\theta_0}{6}(2t_1^3 - 3t_c t_1^2 + t_c^3) - \alpha(t_1 - t_c)t_1^\beta + \frac{\alpha}{\beta+1}(t_1^{\beta+1} - t_c^{\beta+1})\}$

$- p_c I_e t_1] \times (a + bt_1 + ct_1^2 + dt_1^3) = 0$ (4.1.10)

For minimum the sufficient condition $\frac{d^2TC_1(t_1)}{dt_1^2} > 0$ would be satisfied.

Let $t_1 = t_1^*$ be the optimum value of t_1.

The optimal values of the maximum inventory level Q^* of Q, the holding cost HC^* of HC and the average total cost per unit time of the system TC_1^* of TC_1 are obtained by putting the value $t_1 = t_1^*$ from the expressions (3.8), (4.1.2) and (4.1.9).

A special case : $t_1 = t_c$

In this case, the ordering cost, holding cost, deterioration cost, Salvage cost**,** amelioration cost**,** interest earned and shortage cost remain same as in the previous section. Since the payment is done at the time $t_1 = t_c$**,** the interest payable IP_1 is zero. So replacing $t_c = t_1$ and $IP_1 = 0$, the expression (4.1.10) becomes

$$[\frac{\delta}{2}(t_1^2 + \frac{\theta_o}{4}t_1^4 - \frac{\alpha\beta}{\beta+2}t_1^{\beta+2}) + \frac{d_c\theta_o}{2}(1-k)t_1^2 + a_c\alpha t_1^\beta - c_s(T-t_1) - p_c I_e t_1] \times (a + bt_1 + ct_1^2 + dt_1^3) = 0 \quad (4.1.11)$$

which gives the optimum value of t_1.

4.2: Case 2: $t_1 < t_c$ **(payment after depletion of inventory)**

In this case, the customer earn interest on sales revenue up to the permissible credit period and pays no interest ($IP_2 = 0$) for the items kept in stock. The interest earned per cycle is the interest earned during the positive inventory period plus the interest earned from the cash invested during the time period (t_1, t_c) after the inventory **is** exhausted at time t_1**,** and it is given by

$$IE_2 = p_c I_e \int_0^{t_1} t(a + bt + ct^2 + dt^3)dt + p_c I_e (t_c - t_1)\int_0^{t_1}(a + bt + ct^2 + dt^3)dt$$

$$= p_c I_e \{t_c t_1(a + \frac{b}{2}t_1 + \frac{c}{3}t_1^2 + \frac{d}{4}t_1^3) - \frac{t_1^2}{2}(a + \frac{b}{3}t_1 + \frac{c}{6}t_1^2 + \frac{d}{10}t_1^3)\} \quad (4.2.1)$$

Hence the total variable cost per unit time during the cycle [0, T] is given by the following

$$TC_2(t_1) = \frac{1}{T}\,[\mathbf{OC + HC + CD - SV + AMC + CS} + IP_2 - IE_2]$$

Or $TC_2(t_1) = \frac{1}{T}[A_0 + \delta\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + (\frac{c}{10} + \frac{a\theta_o}{40})t_1^5 + (\frac{d}{12} + \frac{b\theta_o}{48})t_1^6 + \frac{c\theta_o}{56}t_1^7 + \frac{d\theta_o}{64}t_1^8$

$- \frac{a\alpha\beta}{2(\beta+2)(\beta+3)}t_1^{\beta+3} - \frac{b\alpha\beta}{2(\beta+2)(\beta+4)}t_1^{\beta+4} - \frac{c\alpha\beta}{2(\beta+2)(\beta+5)}t_1^{\beta+5} - \frac{d\alpha\beta}{2(\beta+2)(\beta+6)}t_1^{\beta+6}\}$

$+ (1-k)\, d_c\theta_o\{\frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \frac{c}{10}t_1^5 + \frac{d}{12}t_1^6\} + c_s\{\frac{a}{2}(T^2 - 2Tt_1 + t_1^2) + \frac{b}{6}(T^3 - 3Tt_1^2 + 2t_1^3) +$

$\frac{c}{12}(T^4 - 4Tt_1^3 + 3t_1^4) + \frac{d}{20}(T^5 - 5Tt_1^4 + 4t_1^5)\} - p_c I_e \{t_c t_1(a + \frac{b}{2}t_1 + \frac{c}{3}t_1^2 + \frac{d}{4}t_1^3) - \frac{t_1^2}{2}(a + \frac{b}{3}t_1 + \frac{c}{6}t_1^2 + \frac{d}{10}t_1^3)\}]$

(4.2.2)

By the similar procedure as in case 1, the optimality condition $\frac{dTC_2(t_1)}{dt_1} = 0$ yields

$$\{\frac{\delta}{2}(t_1^2 + \frac{\theta_o}{4}t_1^4 - \frac{\alpha\beta}{\beta+2}t_1^{\beta+2}) + \frac{d_c\theta_o}{2}(1-k)t_1^2 + a_c\alpha t_1^{\beta} - c_s(T-t_1) - p_c I_e t_c\} \times (a + bt_1 + ct_1^2 + dt_1^3)$$

$$+ \; p_c I_e \; (at_1 + \frac{b}{2}t_1^2 + \frac{c}{3}t_1^3 + \frac{d}{4}t_1^4) = 0 \qquad (4.2.3)$$

The above equation can be solved to find the optimal values of t_1, and then the optimal values of Q, HC and TC_2 can be obtained from the expressions (3.8), (4.1.2) and (4.2.2) respectively.

V. Numerical Examples

Case 1: $t_c \le t_1 < T$ (payment at or before the total depletion of inventory):

To illustrate the developed inventory model, let the values of parameters be as follows:

A_0 = \$500 per order; a = 30; b = 20; c = 10; d = 3; δ = 3; θ_o = 0.01; α = 0.001; β = 2; k = 0.1; p_c = \$30 per unit; d_c = \$4 per unit; a_c = \$7 per unit; c_s = \$5 per unit; I_p = \$0.15 per unit; I_e = \$0.13 per unit; t_c = 0.1 year; T = 1 year

Solving the equation (4.1.10) with the help of computer using the above parameter values, we find the following optimum outputs

t_1^* = 0.798 year; Q^* = 32.36 units, HC^* = \$ 11.86 and TC^* = \$ 212.73.

It is checked that this solution satisfies the sufficient condition for optimality.

Case 2: $t_1 < t_c$ (payment after depletion of inventory)

To illustrate the developed inventory model, let the values of parameters be as follows:

A_0 = \$500 per order; a = 30; b = 20; c = 10; d = 3; δ = 20; θ_o = 0.01; α = 0.001; β = 2; k = 0.1; p_c = \$30 per unit; d_c = \$4 per unit; a_c = \$7 per unit; c_s = \$5 per unit; I_p = \$0.15 per unit; I_e = \$0.13 per unit; t_c = 0.6 year; T = 1 year

Solving the equation (4.2.3) with the help of computer using the above parameter values, we find the following optimum outputs

t_1^* = 0.539 year; Q^* =19.66 units, HC^* = \$ 20.88 and TC^* = \$ 220.78.

It is checked that this solution satisfies the sufficient condition for optimality.

VI. Sensitivity Analysis

On the basis of the above parametric values in case 1 and case 2, we have discussed the sensitivity analysis under some particular cases.

Table A: When the payment at or before the total depletion of inventory (case 1)

Particular cases	Optimal values		
	Q^* (units)	HC^* (\$)	TC^* (\$)
Quadratic demand (d = 0)	32.06	11.67	212.13
Linear trended demand (c=d = 0)	30.36	10.69	210.53
Constant demand (b=c=d = 0)	23.97	7.64	206.48
Absence of deterioration (θ_0 =0)	32.45	11.97	212.52
Absence of amelioration (α =0)	32.39	11.88	212.69
Without backlogging (c_s =0)	13.07	1.01	197.40

Table B: When the payment after depletion of inventory (case 2)

	Optimal values

Particular cases	Q^* (units)	HC^* ($)	TC^* ($)
Quadratic demand (d = 0)	19.56	20.67	220.21
Linear trended demand (c=d = 0)	18.88	19.30	217.62
Constant demand (b=c=d = 0)	15.61	14.06	210.69
Absence of deterioration (θ_0 =0)	19.66	20.92	220.73
Absence of amelioration (α =0)	19.66	20.82	220.77
Without backlogging (c_s =0)	11.51	4.71	185.13

VII. Concluding Remarks

Based on the results in Table A and Table B, the following observations are briefly stated as follows:
a). Maximum inventory level (Q), the holding cost (HC) and he average total costs per unit time of the system (TC_1 and TC_2) change significantly when the demand rate is constant in nature. Whereas these change moderately when the demand rates have quadratic and linear trended behaviour.

b). In the absence of deterioration and amelioration items, it is observed that there is no significant changes in the optimal values of the maximum inventory level (Q), the holding cost (HC) and he average total costs per unit time of the system (TC_1 and TC_2).

c) When shortages are not considered in the model, it is seen from the above tables that the optimal values of Q, HC and TC (TC_1 & TC_2) change very significantly.

So demand parameters (b, c, d) and backlogging parameter (c_s) are very sensitive parameters for estimation of optimal solution of the inventory model and we need adequate attention to estimate these parameters.

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