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A Class of Maximal Inequalities for Conditional Demimartingales

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Abstract **—In this paper, based on conditional Fubini theorem and a maximal inequality for conditional demimartingales, we obtain a class of maximal inequalities for conditional demimartingales.**

Keywords**—demi(sub) martingales; conditional demi(sub) martingales; conditional Fubini theorem; maximal inequalities**

I. INTRODUCTION

Newman and Wright [1] introduced the concepts of demimartingale and demi(sub)martingale. Definition 2 is due to Hadjikyriakou [2], since then, many scholars have studied it. For example, Christofides and Hadjikyriakou [3] established some maximal inequalities and asymptotic results for conditional demimartingales. Wang and Wang [4] obtained some maximal inequalities for conditional demi(sub)martingales and minimal inequalities for nonnegative conditional demimartingales. Feng and Gao [5] obtained conditional moment inequalities of conditional demisubmartingales. Feng, Wen and Yang [6] established a class of maximum inequalities of conditional demi(sub)martingales.Wang and Hu [7] established some maximal ϕ-inequalities for nonnegative conditional demimartingales and some maximal inequalities for conditional demimartingales based on concave Young functions.

Inspired by reference [8], this paper first gave the definition of demi(sub)martingale and conditional demi(sub)martingale. On this basis, we further obtained a class of maximal inequalities for conditional demimartingales .

Notation and conventions. Throughout this paper, $\{X_n, n \ge 1\}$ or $\{S_n, n \ge 1\}$ denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathrm{G}, \mathrm{P})$, $E^{\mathrm{F}}(X)$ denote the conditional expectation of X, that is, $E^F(X) = E(X \mid F)$, where F be a sub-s algebra of G. Let $X^+ = \max(0, X)$, $a \vee b = \max(a, b)$, $I(A)$ denote the indicator function of the set A, $\mathrm{S}_0 = 0$, $\log x = \log_e x = \ln x$, $\log^+ x = \ln(x \vee 1)$. Let C denote the class of Orlicz functions, that is, unbounded, nondecreasing convex functions $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$. Let $C' = \{f \in C : f'(0) = 0, \frac{f'(x)}{x}$ is integrable on $(0, e)$ for some $e > 0\}$.

Given $f \in C$ and $a \ge 0$, define

$$\Phi_a(x) = \int_a^x \int_a^s \frac{f'(r)}{r} dr ds, \quad x > 0.$$

Set $\Phi(x) = \Phi_0(x)$, $x > 0$.

II. DEFINITION OF CONDITIONAL DEMIMARTINGALE

Definition 1 Let $\{S_n, n \ge 1\}$ be a sequence of random variables defined on $L^1(\Omega, G, P)$. Assume that for $\mathrm{j} = 1, 2, \ldots$,

$$E[(S_{j+1} - S_j) f(S_1, \ldots, S_j)] \ge 0 \qquad (1)$$

for all coordinatewise nondecreasing functions f such that the expectation is defined, Then $\{S_n, n \ge 1\}$ is called a demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_n, n \ge 1\}$ is called a demisubmartingale.

Definition 2 Let $\{S_n, n \ge 1\}$ be a sequence of random variables defined on $L^1(\Omega, G, P)$. Assume that for $1 \le i < j < \infty$,

$$E^F[(S_j - S_i) f(S_1, \ldots, S_i)] \ge 0 \quad \text{a.s.} \qquad (2)$$

for all coordinatewise nondecreasing functions f such that the expectation is defined, Then $\{S_n, n \ge 1\}$ is called a F $-$demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_n, n \ge 1\}$ is called a F $-$demisubmartingale.

It is easy to check that for all $i \ge 1$, (2) is equivalent to

$$E^F[(S_{i+1} - S_i) f(S_1, \ldots, S_i)] \ge 0 \quad \text{a.s.} \qquad (3)$$

From the property of conditional expectations that $E[E(X \mid F)] = E(X)$ for any random variable X with $E|X| < \infty$, it follows that F $-$ demi(sub)martingales defined on a proability space (Ω, G, P) are demi(sub)martingales on the proability space (Ω, G, P), but the converse is not true.

Ⅱ. MAIN RESULTS

Lemma 1[6] Let $\{S_n, n \geq 1\}$ be an F − demimartingale and g be a nonnegative convex function satisfying $g(0)=0$ and $Eg(S_n)<+\infty$ for every $n \geq 1$. Suppose that $\{c_n, n \geq 1\}$ is a positive nondecreasing sequence of F − measurable random variables, then for any F −measurable random variable $e>0$ a.s.,

$$eP^F\left(\max_{1\le k\le n} c_k g(S_k)\ge e\right)\le c_n E^F\left(g(S_n)I(\max_{1\le k\le n} c_k g(S_k)\ge e)\right) \text{a.s.}$$

Theorem 1 Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. Assume that $\{S_n, n\geq 1\}$ is an F −demimartingale and g is a nonnegative convex function satisfying $g(0)=0$ and $E(g(S_k))^p<+\infty$ for every $k\geq 1$. Suppose that $\{c_n, n\geq 1\}$ is a positive nondecreasing sequence of F − measurable random variables and define $S_n^{\max}=\max_{1\le k\le n} c_k g(S_k)$. Then for $f\in C'$,

$$E^F f(S_n^{\max})\le\left(E^F\left[c_n g(S_n)\right]^p\right)^{\frac{1}{p}}\left(E^F\left[\Phi'(S_n^{\max})\right]^q\right)^{\frac{1}{q}} \text{a.s.} (4)$$

Proof. Since from lemma 1 ,the conditional Fubini theorem and the conditional Hölder inequality, we can get that

$$\begin{aligned}
E^F f(S_n^{\max}) &= E^F\left(\int_0^{S_n^{\max}} f'(t)dt\right)\\
&= E^F\left(\int_0^{\infty} f'(t)I(S_n^{\max}\ge t)dt\right)\\
&= \int_0^{\infty} E^F\left(f'(t)I(S_n^{\max}\ge t)\right)dt\\
&= \int_0^{\infty} f'(t)P^F(S_n^{\max}\ge t)dt\\
&\le \int_0^{\infty} f'(t)\cdot\frac{1}{t}c_n E^F\left(g(S_n)I(S_n^{\max}\ge t)\right)dt\\
&= \int_0^{\infty}\frac{f'(t)}{t}E^F\left(c_n g(S_n)I(S_n^{\max}\ge t)\right)dt\\
&= E^F\left(\int_0^{S_n^{\max}}\frac{f'(t)}{t}c_n g(S_n)\right)dt\\
&= E^F\left(c_n g(S_n)\Phi'(S_n^{\max})\right)\\
&\le \left(E^F\left[c_n g(S_n)\right]^p\right)^{\frac{1}{p}}\left(E^F\left[\Phi'(S_n^{\max})\right]^q\right)^{\frac{1}{q}} \quad \text{a.s.}
\end{aligned}$$

Corollary 1 Assume that the condition for Theorem 1 is satisfied and $c_k\equiv 1$ for every $k\geq 1$, then

$$E^F f\left(\max_{1\le k\le n} g(S_k)\right)\le\left(E^F\left[g(S_n)\right]^p\right)^{\frac{1}{p}}\times\left(E^F\left[\Phi'\left(\max_{1\le k\le n} g(S_k)\right)\right]^q\right)^{\frac{1}{q}} \quad \text{a.s.} (5)$$

Corollary 2 Assume that the condition for Theorem 1 is satisfied. Then for every $p>1$,

$$E^F(S_n^{\max})^p\le\left(\frac{p}{p-1}\right)^p E^F\left[c_n g(S_n)\right]^p \text{ a.s.} (6)$$

$$E^F(S_n^{\max})\le\frac{e}{e-1}\left\{1+E^F\left(c_n g(S_n)\times\log^+(c_n g(S_n))\right)\right\}. \text{ a.s.} (7)$$

Proof. By taking $f(x)=x^p, p>1$ in Theorem 1 , we get $\Phi'(x)=\frac{p}{p-1}x^{p-1}$, Hence

$$E^F(S_n^{\max})^p\le\left(\frac{p}{p-1}\right)^p E^F\left[c_n g(S_n)\right]^p$$

Taking $f(x)=(x-1)^+$ in Theorem 1 , it follows that $f'(x)=1\cdot I(x\ge 1)$. Therefore

$$\begin{aligned}
E^F\left(S_n^{\max}-1\right)&\le E^F\left(S_n^{\max}-1\right)^+\\
&\le E^F\left(c_n g(S_n)\int_0^{S_n^{\max}}\frac{I(r\ge 1)}{r}dr\right)\\
&= E^F\left(c_n g(S_n)\log^+ S_n^{\max}\right) \quad \text{a.s.}
\end{aligned}$$

By the inequality

$$a\log^+ b\le a\log^+ a+be^{-1}, a>0, b>0,$$

we have

$$\begin{aligned}
E^F\left(S_n^{\max}-1\right)&\le E^F\left(c_n g(S_n)\log^+ c_n g(S_n)+e^{-1}S_n^{\max}\right)\\
&= E^F\left(c_n g(S_n)\log^+ c_n g(S_n)\right)+e^{-1}E^F\left(S_n^{\max}\right)
\end{aligned}$$

Hence

$$E^F(S_n^{\max})\le\frac{e}{e-1}\left\{1+E^F\left(c_n g(S_n)\times\log^+(c_n g(S_n))\right)\right\}. \text{ a.s.}$$

Let $c_k\equiv 1, k\ge 1$ in inequalities (6) and (7), we have

$$E^F\left[\max_{1\le k\le n} g(S_k)\right]^p\le\left(\frac{p}{p-1}\right)^p E^F\left[g(S_n)\right]^p \text{a.s.} (8)$$

$$E^F\left[\max_{1\le k\le n} g(S_k)\right]\le\frac{e}{e-1}\left\{1+E^F\left[g(S_n)\times\log^+(g(S_n))\right]\right\} \quad \text{a.s.} (9)$$

Remark 1 Letting $g(x)=|x|$ in inequality (7), we have

$$E^F\left[\max_{1\le k\le n} c_k|S_k|\right]\le\frac{e}{e-1}\left\{1+E^F\left[c_n|S_n|\times\log^+(c_n|S_n|)\right]\right\} \quad \text{a.s.} (10)$$

Taking $g(x)=|x|$ in inequalities (8) and (9), we can get the following corollary.

Corollary 3 Let $\{S_n, n\geq 1\}$ be an F $-$demimartingale and $p>1$. Suppose that $E|S_k|^p<\infty$ for each $k\geq 1$, then

$$E^F\ (\max_{1\leq k\leq n}|S_k|)^p \leq \left(\frac{p}{p-1}\right)^p E^F\ |S_n|^p \quad \text{a.s.} \qquad (11)$$

$$E^F\ (\max_{1\leq k\leq n}|S_k|) \leq \frac{e}{e-1}\left\{\begin{matrix} 1+E^F\ (|S_n| \\ \times\log^+|S_n|) \end{matrix}\right\} \quad \text{a.s.} \qquad (12)$$

Theorem 2 Let $\{S_n, n\geq 1\}$ be an F $-$demimartingale and g be an nonnegative function satisfying $g(0)=0$. Assume that $\{c_n, n\geq 1\}$ is a positive nondecreasing sequence of F $-$measurable random variable. Then for all $n\geq 1, t>0$ and $0<l<1$,

$$P^F\left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)\leq \frac{l}{(1-l)t}\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds$$

$$=\frac{l}{(1-l)t}E^F\ \left(\frac{c_n g(S_n)}{l}-t\right)^+ \quad \text{a.s.}$$

Furthermore, for $f\in C', n\geq 1, a>0, b>0$ and $0<l<1$,

$$E^F\ \left(f\,(\max_{1\leq k\leq n} c_k g(S_k))\right)=f\,(b)+\frac{l}{1-l}E^F\left\{\begin{matrix}\Phi_b\left(\frac{c_n g(S_n)}{l}\right) \\ I(c_n g(S_n)\geq l\ b)\end{matrix}\right\}$$

$$=f\,(b)+\frac{l}{1-l}E^F\left\{\begin{matrix}\left(\begin{matrix}\Phi_a\left(\frac{c_n g(S_n)}{l}\right)-\Phi_a(b) \\ -\Phi_a'(b)\left(\frac{c_n g(S_n)}{l}-b\right)\end{matrix}\right) \\ I(c_n g(S_n)\geq l\ b)\end{matrix}\right\} \quad \text{a.s.}$$

Proof. Since from lemma 1, we have

$$P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)\leq \frac{1}{t}c_n E^F\ \left(g(S_n)I(\max_{1\leq k\leq n} c_k g(S_k)\geq t)\right)$$

$$=\frac{1}{t}E^F\ \left(c_n g(S_n)I(\max_{1\leq k\leq n} c_k g(S_k)\geq t)\right)$$

$$=\frac{1}{t}\int_0^\infty P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t, c_n g(S_n)\geq s\right)ds$$

$$\leq \frac{1}{t}\int_0^{l\,t} P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)ds \quad \text{a.s.}$$

$$+\frac{1}{t}\int_{l\,t}^\infty P^F\ \left(c_n g(S_n)\geq s\right)ds$$

$$=l\ P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)$$

$$+\frac{l}{t}\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds$$

then

$$(1-l)P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)\leq \frac{l}{t}\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds \quad \text{a.s.}$$

So we have

$$P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)\leq \frac{l}{(1-l)t}\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds \quad \text{a.s.}$$

It can be checked that

$$\int_t^\infty I\left(\frac{c_n g(S_n)}{l}\geq s\right)ds=\left(\frac{c_n g(S_n)}{l}-t\right)^+, t\geq 0$$

Hence

$$\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds=E^F\ \left(\frac{c_n g(S_n)}{l}-t\right)^+ \quad \text{a.s.}$$

Furthermore, the first inequality in theorem 2 is proved.
By the inequality that we got before, so we have

$$E^F\ \left(f\left[\max_{1\leq k\leq n} c_k g(S_k)\right]\right)=E^F\ \left(\int_0^{\max_{1\leq k\leq n} c_k g(S_k)} f'(t)dt\right)$$

$$=E^F\ \left(\int_0^\infty f'(t)I(\max_{1\leq k\leq n} c_k g(S_k)\geq t)dt\right)$$

$$=\int_0^\infty E^F\ \left[f'(t)I(\max_{1\leq k\leq n} c_k g(S_k)\geq t)\right]dt$$

$$=\int_0^\infty f'(t)P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)dt$$

$$=\int_0^b f'(t)P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)dt$$

$$+\int_b^\infty f'(t)P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)dt$$

$$\leq f\,(b)+\int_b^\infty f'(t)P^F\ \left(\max_{1\leq k\leq n} c_k g(S_k)\geq t\right)dt$$

$$\leq f\,(b)+\int_b^\infty f'(t)\left[\frac{l}{(1-l)t}\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds\right]dt$$

$$=f\,(b)+\frac{l}{1-l}\int_b^\infty \frac{f'(t)}{t}\left[\int_t^\infty P^F\ \left(c_n g(S_n)\geq l\ s\right)ds\right]dt$$

$$=f\,(b)+\frac{l}{1-l}\int_b^\infty \frac{f'(t)}{t}E^F\ \left(\frac{c_n g(S_n)}{l}-t\right)^+dt$$

$$=f\,(b)+\frac{l}{1-l}E^F\ \left[\int_b^\infty \frac{f'(t)}{t}\left(\frac{c_n g(S_n)}{l}-t\right)^+dt\right]$$

$$=f\,(b)+\frac{l}{1-l}E^F\ \left(\int_b^\infty \frac{f'(t)}{t}\left[\int_t^\infty I\left(\frac{c_n g(S_n)}{l}\geq s\right)ds\right]dt\right)$$

$$=f\,(b)+\frac{l}{1-l}E^F\ \left(\int_b^{\frac{c_n g(S_n)}{l}}\left[\int_b^s \frac{f'(t)}{t}dt\right]ds\cdot I(c_n g(S_n)\geq l\ b)\right)$$

$$=f\,(b)+\frac{l}{1-l}E^F\ \left(\Phi_b\left[\frac{c_n g(S_n)}{l}\right]I(c_n g(S_n)\geq l\ b)\right)$$

Since

$$\Phi_a\left(\frac{c_n g(S_n)}{l}\right)-\Phi_a(b)-\Phi_b\left(\frac{c_n g(S_n)}{l}\right)=\int_a^{\frac{c_n g(S_n)}{l}}\int_a^s\frac{f'(r)}{r}drds$$

$$-\int_a^b\int_a^s\frac{f'(r)}{r}drds-\int_b^{\frac{c_n g(S_n)}{l}}\int_b^s\frac{f'(r)}{r}drds$$

$$=\int_b^{\frac{c_n g(S_n)}{l}}\int_a^s\frac{f'(r)}{r}drds-\int_b^{\frac{c_n g(S_n)}{l}}\int_b^s\frac{f'(r)}{r}drds$$

$$=\int_b^{\frac{c_n g(S_n)}{l}}\int_a^b\frac{f'(r)}{r}drds$$

$$=\int_a^b\frac{f'(r)}{r}dr\left(\frac{c_n g(S_n)}{l}-b\right)$$

$$=\Phi_a'(b)\left(\frac{c_n g(S_n)}{l}-b\right)$$

Hence, the second inequality in theorem 2 is proved
In conclusion, theorem 2 is proved.

Remark 2 Taking $c_k \equiv 1$ for each $k \geq 1$ in Theorem 2 , then

$$P^F\left(\max_{1\le k\le n} g(S_k)\ge t\right)\le \frac{l}{(1-l)t}\int_t^\infty P^F\left(g(S_n)\ge l\, s\right)ds$$

$$=\frac{l}{(1-l)t}E^F\left(\frac{g(S_n)}{l}-t\right)^+ \quad \text{a.s.}$$

Furthermore, for $f \in C', n\geq 1, a>0, b>0$ and $0<l<1$,

$$E^F\left(f(\max_{1\le k\le n} g(S_k))\right)=f(b)+\frac{l}{1-l}E^F\left(\Phi_b\left(\frac{g(S_n)}{l}\right)I(g(S_n)\ge l\, b)\right)$$

$$=f(b)+\frac{l}{1-l}E^F\left\{\begin{array}{l}\left(\Phi_a\left(\frac{g(S_n)}{l}\right)-\Phi_a(b)\right.\\ \left.-\Phi_a'(b)\left(\frac{g(S_n)}{l}-b\right)\right)\\ I(g(S_n)\ge l\, b)\end{array}\right\} \quad \text{a.s.}$$

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