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Continuous-time quantum walk on graph associated with quantum Bernoulli noises

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Abstract— **In this paper, we introduce a kind of graph relations based on quantum Bernoulli noises and investigate the corresponding continuous-time quantum walks on these graphs. Evolution properties are examined of the walks and first step results are obtained.**

Keywords— Quantum Bernoulli noise, graph relation, continuous-time quantum walk

I. Introduction

In recent years, quantum walks have been intensively investigated, with the hope that they may be useful in constructing new efficient quantum algorithms (see [1-3] for reviews of quantum walks). Quantum walks can be viewed as quantum analogs of the classical random walks. Whereas, studies have shown that quantum walks are distinct from their classical counterparts [12]. There are two different types of quantum walks: discrete-time ones and continuous-time ones. Here we focus only on continuous-time ones.

Continuous-time quantum walks (CTQWs) were introduced by Farhi and Gutmann [12] as a quantum-mechanical transport process on discrete structure, generally called graphs. For the past few years, CTQWs have been considered on diverse graphs such as integer lattice [14-16], distance regular graphs [17], n -cube [18], start-graphs [19-20], Apolloman network [17] and circulantBunkeds [22].

Quantum Bernoulli noises [10] are annihilation and creation operators acting on Bernoulli functionals, which satisfy the canonical anti-commutation relations (CAR) in equal time. It turns out that quantum Bernoulli noises can play an important role in the study of quantum Markov semigroups and stochastic Schrodinger equations [11]. Recently, Wang and Ye [9] have constructed a model of discrete-time quantum walk in terms of quantum Bernoulli noise and shown its interesting properties.

In this paper, we introduce a graph relation associated with quantum Bernoulli noises and construct a model of CTQW based on the graph relation. We examine properties of the model and obtain several interesting results. The paper is organized as follows. In Section 2, we describe the graph relation and define the model. Properties of the model are shown in Section 3.

II. Graph relation

In this section, we first briefly recall some notions and facts about quantum Bernoulli noises [9]. And then, we introduce a graph relation associated with quantum Bernoulli noises.

Let $\mathbb{N}$ be the set of all nonnegative integers and Γ the finite power set of $\mathbb{N}$, namely

$$\Gamma = \{\tau \mid \tau \subset \mathbb{N} \text{ and } \#\tau < \infty\}, \tag{1}$$

where $\#\tau$ denotes the cardinality of τ as a set. Throughout this section, we suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space and $\mathrm{Z} = \{\mathrm{Z}_n\}_{n\geq 0}$ is an independent sequence of random variables on $(\Omega, \mathcal{F}, \mathrm{P})$, which satisfies that:

$$\mathrm{P}\{\mathrm{Z}_n = \theta_n\} = \mathrm{p}_n, \quad \mathrm{P}\{\mathrm{Z}_n = -1/\theta_n\} = \mathrm{q}_n, \tag{2}$$

where $\theta_n = \sqrt{q_n/p_n}$, $0 < p_n < 1$. We denote by $\mathcal{F} = \sigma\{\mathrm{Z}_n, n \geq 0\}$ the σ-field generated by $\mathrm{Z} = \{\mathrm{Z}_n\}_{n\geq 0}$.

Let $L^2(Z)$ be the space of square integrable complex-valued random variables on $(\Omega, \mathcal{F}, \mathrm{P})$. It is known that $L^2(Z)$ has an orthonormal basis of form $\{Z_\tau | \tau \in \Gamma\}$, where $Z_\emptyset = 1$ and $Z_\tau = \prod_{i\in\tau} Z_i$ for $\tau \in \Gamma$, $\tau \neq \emptyset$

Lemma 1. [10] For $k \geq 0$, there exists a bounded operator ∂_k on $L^2(Z)$ such that

$$\partial_k Z_\tau = \chi_\tau(k) Z_{\tau\setminus k}, \quad \tau \in \Gamma, \tag{3}$$

where $\tau\backslash k = \tau\backslash\{k\}$, and $\chi_\tau(k)$ is the indicator of τ as a subset of $\mathbb{N}$.

$$\partial_k^* = (1 - \chi_\tau(k))Z_{\tau\cup k}, \quad \tau \in \Gamma, \tag{4}$$

where $\tau \cup k = \tau \cup \{k\}$.

We note that the sequence $\{\partial_k, \partial_k^*\}_{k\geq 0}$ is known as quantum Bernoulli noise (QBN, for short).

Let $N \in \mathbb{N}$ be a fixed nonnegative integer and Γ_N the power set of $\{0, 1, 2, 3, \cdots, N\}$, namely

$$\Gamma_N = \{\sigma \mid \sigma \subset \{0, 1, 2, 3, \cdots, N\}\}. \tag{5}$$

It is obviously that Γ_N just has 2^{N+1} elements. For $\sigma, \tau \in \Gamma_N$ with $\sigma \neq \tau$, we use $\sigma < \tau$ to mean one of the following two relations holds true:

Relation 1. $\#\sigma < \#\tau$, where $\#\sigma$ and $\#\tau$ denote the cardinalities of σ and τ, respectively;

Relation 2. $\#\sigma = \#\tau$, but

$$10^{k-1}i_1 + 10^{k-2}i_2 + \cdots + 10^0 i_k < 10^{k-1}j_1 + 10^{k-2}j_2 + \cdots + 10^0 j_k,$$

where $\sigma = \{i_1, i_2, \cdots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$, $\tau = \{j_1, j_2, \cdots, j_k\}$ with $j_1 < j_2 < \cdots < j_k$ and $k = \#\sigma = \#\tau$.

Lemma 2. [10] For $k \geq 0$, the adjont operator ∂_k^* of ∂_k has following property:

It can be seen that for $\sigma, \tau \in \Gamma_N$ with $\sigma \neq \tau$ one has $\sigma < \tau$ or $\tau < \sigma$, which actually defines an order relation in Γ_N. By using this order relation, we can enumerate elements of Γ_N as follows

$$\Gamma_N = \{\sigma_1, \sigma_2, \cdots, \sigma_{2^{N+1}}\}$$

with $\sigma_1 < \sigma_2 < \cdots < \sigma_{2^{N+1}}$. The following table shows the cases for $N = 0, 1$ and 2.

N \ σ	$N = 0$	$N = 1$	$N = 2$
σ_1	$\emptyset$	$\emptyset$	$\emptyset$
σ_2	$\{0\}$	$\{0\}$	$\{0\}$
σ_3		$\{1\}$	$\{1\}$
σ_4		$\{0,1\}$	$\{2\}$
σ_5			$\{0,1\}$
σ_6			$\{0,2\}$
σ_7			$\{1,2\}$
σ_8			$\{1,2,3\}$

Table 1.

We now introduce a graph relation "$\sim$" in Γ_N as follows. Two elements σ_i, σ_j of Γ_N are said to form an edge, written as $\sigma_i \sim \sigma_j$, if there exists a unique $k \in \{0, 1, 2, 3, \cdots, N\}$ such that

$$\partial_k Z_{\sigma_i} = Z_{\sigma_j} \text{ or } \partial_k^* Z_{\sigma_i} = Z_{\sigma_j}.$$

Let E be the set of edges defined by the above graph relation. Then we come to a graph

$$G = (V, E)$$

with $V = \Gamma_N$, which is called the graph of order N associated with quantum Bernoulli noises. The adjacent matrix A of G is given by

$$A_{ij} = \begin{cases} 1, & \text{if } \sigma_i \sim \sigma_j \\ 0, & \text{others.} \end{cases} \tag{6}$$

It is easy to see that the graph $G = (V, E)$ defined above is symmetric. Thus its Hamiltonian $H_N = D_N - A_N$ is Hermitian, where D_N is a diagonal matrix with its ith diagonal entry $D_{N_{ii}} = \deg(\sigma_i)$, namely, $D_{N_{ii}}$ is the degree of vertex σ_i. By lengthy but straightforward calculations, we can work out the matrices D_N, A_N and H_N of graph G for $N = 0, 1$ and 2.

Example 1. For $N = 0$, the corresponding matrices D_0, A_0 and H_0 of graph G have the following forms

$$D_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ H_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (7)$$

Example 2. For $N = 1$, the corresponding matrices D_1, A_1 and H_1 of graph G take the following forms

$$D_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, H_1 = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}. \quad (8)$$

Example 3. For $N = 2$, the corresponding matrices D_2, A_2 and H_2 of graph G are of the following forms

$$D_2 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad (9)$$

and $H_2 = D_2 - A_2$.

Proposition 1. For $N = 0$, 1 and 2, the eigenvalues and the corresponding eigenvectors of Hamiltonian H_N are as follows

	λ_n	ϵ_n
H_0	$\lambda_0 = 0, \lambda_1 = 2$	$\epsilon_0 = \frac{1}{\sqrt{2}}(1,1)^T$ $\epsilon_1 = \frac{1}{\sqrt{2}}(-1,1)^T$
H_1	$\lambda_0 = 0,\ \lambda_1 = \lambda_2 = 2,\ \lambda_3 = 4$	$\epsilon_0 = \frac{1}{\sqrt{4}}(1,1,1,1)^T$ $\epsilon_1 = \frac{1}{\sqrt{4}}(1,-1,1,-1)^T$ $\epsilon_1 = \frac{1}{\sqrt{4}}(1,1,-1,-1)^T$ $\epsilon_2 = \frac{1}{\sqrt{4}}(1,-1,-1,1)^T$
H_2	$\lambda_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 2,\ \lambda_4 = \lambda_5 = \lambda_6 = 4$, $\lambda_7 = 6$	...

Table 2. Eigenvalues λ_n and eigenvectors ϵ_n of H_N.

III. Continuous–time quantum walk

In the present section, we investigate the continuous-time quantum walk on the graph $G = (V, E)$ introduced in the previous section.

Recall that $V = \Gamma_N$, which exactly contains 2^{N+1} elements. In view of this, we can take the canonical complex Hilbert space $\mathcal{H}$ of dimension 2^{N+1} as the state space of the walk that we will consider below. By convention, the inner product $<,>$ of $\mathcal{H}$ is linear in its second variable and conjugate linear in its the first variable. Let

$$\{\omega_j \mid i = 1, 2, 3, \cdots, 2^{N+1}\} \quad (10)$$

be the canonical orthonormal basis for $\mathcal{H}$. Then, the Hamiltonian H_N of G acts just as an Hermitian operator on $\mathcal{H}$ in a natural way.

Definition 1. The evolution equation of the continuous-time quantum walk on the graph G takes the following form

$$i\frac{d}{dt}\varphi(t) = H_N\varphi(t), \quad (11)$$

where H_N is the Hamiltonian of the graph and $\varphi(t)$

denotes the state of the walk at time $t \geq 0$, which is a unit vector in $\mathcal{H}$. The probability $P(t)$ of the walk at vertex σ_j and time $t \geq 0$ is given by $P_j(t) = |<\omega_j, \varphi(t)>|^2$, where ω_j is the basis vector of the canonical orthonormal basis for $\mathcal{H}$.

$$\varphi(t) = \sum_{n=0}^{2N+1} e^{-it\lambda_n} |\epsilon_n><\epsilon_n| \varphi(0), \quad (12)$$

where λ_n is the eigenvalue of H_N and ϵ_n the corresponding eigenvector. By using this representation, we can get a formula for calculating the probability of the walk at vertex σ_j and time $t \geq 0$, which reads

$$P_j(t) = |\sum_{n=0}^{2N+1} e^{-it\lambda_n} < \epsilon_n, \varphi(0) >< \omega_j, \epsilon_n >|^2 . \quad (13)$$

The average probability $\overline{P_j}$ of the walk at vertex σ_j is defined as

$$\overline{P_j} = \lim_{T\to\infty} \frac{\int_0^T P_j(t)dt}{T}. \quad (14)$$

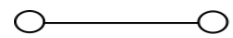
Applying the above formula to the case of $N = 0$,

Note that the Planck constant is taken to be 1 here. Equation (11) just has a unique solution $\varphi(t) = e^{-itH_N}\varphi(0)$, where $\varphi(0)$ denotes the initial state of the walk. It easily follows from the spectral decomposition of H_N that the state $\varphi(t)$ has a representation of the following form

we immediately come to the probability distribution of the walk at $t \geq 0$, which reads

$$P_1(t) = \frac{1}{2} + \frac{1}{2}\cos 2t, \quad P_2(t) = \frac{1}{2} - \frac{1}{2}\cos 2t. \quad (15)$$

This then gives the average probability $\overline{P_1} = \overline{P_2} = \frac{1}{2}$. Here we assume that the walk starts at vertex σ_1. Note that in case of $N = 0$ the graph $G = (V, E)$ looks like



Similarly, we can work out the probability distribution of the walk in the case of $N = 1$ as follows.

Proposition 2. Let $N = 1$. Then at time $t \geq 0$ the walk has a probability distribution given by

$$P_1(t) = \frac{6}{16} + \frac{4}{16}\cos 2t + \frac{2}{16}\cos 4t + \frac{4}{16}\cos 2t \cos 4t + \frac{4}{16}\sin 2t \sin 4t \quad (15)$$

$$P_2(t) = \frac{2}{16} - \frac{2}{16}\cos 4t \quad (16)$$

$$P_3(t) = \frac{2}{16} - \frac{2}{16}\cos 4t \quad (17)$$

$$P_4(t) = \frac{6}{16} - \frac{4}{16}\cos 2t + \frac{2}{16}\cos 4t - \frac{4}{16}\cos 2t \cos 4t - \frac{4}{16}\sin 2t \sin 4t \quad (18)$$

Here the walk is assumed to start at vertex σ_1.

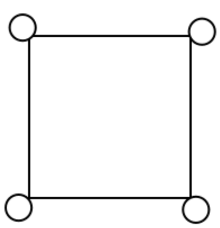


Fig 1. the graph in the case $N = 1$

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