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Existence and Uniqueness of the Solution to the 3D Navier-Stokes Equations in the Homogeneous Sobolev-Gevrey Space.

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Abstract—**We apply the principle of compression mapping to show existence and uniqueness of solutions for the classical Navier-Stokes equations in Sobolev-Gevrey space $\dot{H}^{\frac{1}{2}}_{a,\sigma}$.**

***Index Terms*—Navier-Stokes equations; Sobolev-Gevrey spaces ;existence and uniqueness.**

I. Introduction

The 3D generalized Navier-Stokes system is give by:

$$\begin{cases} \partial_t u - \nu\Delta u = Q(u,u), & x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u = 0, & x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u(0,x) = u^0(x), & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

with Q is the bilinear operator defined as:

$$Q^j(u,v) = \sum_{k,l,m} q^{j,m}_{k,l}\partial_m(u^k v^l), \quad j = 1,2,3,$$

where

$$q^{j,m}_{k,l} = \sum_{n,p=1}^{3} a^{j,m,p,n}_{k,l}\mathcal{F}^{-1}\left(\frac{\xi_n\xi_p}{|\xi|^2}\hat{u}(\xi)\right)$$

and $a^{j,m,p,n}_{k,l}$ are real numbers.

The particular case of the above system is the Navier-Stokes system for incompressible fluid:

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u = -\nabla p & x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u = 0, & x \in \mathbb{R}^3,\ t \in (0,\infty), \\ u(0,x) = u^0(x), & x \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

where $\upsilon > 0$ is the viscosity of the fluid, and $\alpha > 0$ is real parameters. $u = (t,x)$ is the velocity field of fluid, $p = p(t,x) \in \mathbb{R}$ denotes the unknow pressure of the fluid at the point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and $(u\cdot\nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, while $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is a given initial velocity. If u^0 is a quite regular, the divergence free condition determines the pressure p. Here ∂_t and $\Delta = \sum_{j=1}^{n}\partial^2_{x_j}$ are the partial derivative with respect to t and the Laplacian with respect to $x = (x_1\cdots,x_n)$. For simplicity, we will take $\upsilon=1$.

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We are mainly interested in studying existence and uniqueness of solutions for the classical Navier-Stokes equations. Here we extend the results obtained in [1] to the case of homogeneous Sobolev-Gevrey space. To define the spaces, for $s \in \mathbb{R}$, let us denote by $H^s(\mathbb{R}^3)$ the usual Sobolev spacs on $\mathbb{R}^3$, with respective inner product $<,>_{H^s(\mathbb{R}^3)}$, and $\dot{H}^s(\mathbb{R}^3)$ denote the usual homogeneous Sobolev spacs on $\mathbb{R}^3$, with respective inner product $<,>_{\dot{H}^s(\mathbb{R}^3)}$. We denote the Fourier transform, as in [3] by

$$F(f)(\xi) = \int_{\mathbb{R}^2}\exp(-ix\cdot\xi)f(x)dx,\ \xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3,$$

and its inverse by

$$F^{-1}(f)(x) = (2\pi)^{-2}\int_{\mathbb{R}^2}\exp(ix\cdot\xi)f(\xi)d\xi,\ x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$

The convolution product of a suitable pair of function f and g on $\mathbb{R}^3$ is given by

$$(f*g)(x) = \int_{\mathbb{R}^2} f(x-y)g(y)dy,\ x \in \mathbb{R}^3$$

If $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$ are two vector fields, we denote $f \otimes g := (g_1 f, g_2 f, g_3 f)$ and

$$\operatorname{div}(f\otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f), \operatorname{div}(g_3 f)).$$

The fractional Laplace operator $(-\Delta)^{\frac{s}{2}}$, $0 < s < 2$, is defined by

$$\mathcal{F}((-\Delta)^{\frac{s}{2}}f)(\xi)) = |\xi|^s\mathcal{F}(f)(\xi).$$

We write $|D| = -\Delta^{\frac{1}{2}}$.

Definition1.1

For $a, s > 0, \sigma \ge 1$ and $|D| = (-\Delta)^{\frac{1}{2}}$, The Sobolev-Gevery spaces $H^s_{a,\sigma}(\mathbb{R}^3)$ defined as follows:

$$H^s_{a,\sigma}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3); e^{a|D|^{\frac{1}{\sigma}}}f \in H^s(\mathbb{R}^3)\}$$

which is equipped with the norm

$$\|f\|_{H^s_{a,\sigma}} = \|e^{a|D|^{\frac{1}{\sigma}}}f\|_{H^s}$$

and the associated inner product

$$\langle f/g\rangle_{H^s_{a,\sigma}} = \langle e^{a|D|^{\frac{1}{\sigma}}}f/e^{a|D|^{\frac{1}{\sigma}}}g\rangle_{H^s}.$$

Analogously, the homogeneous Sobolev-Gevery spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ is

$$\dot{H}^s_{a,\sigma}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3); e^{a|D|^{\frac{1}{\sigma}}}f \in \dot{H}^s(\mathbb{R}^3)\}$$

which is equipped with the norm

$$\|f\|_{\dot{H}^s_{a,\sigma}} = \|e^{a|D|^{\frac{1}{\sigma}}}f\|_{\dot{H}^s}$$

and the associated inner product

$$\langle f/g\rangle_{\dot H^s_{a,\sigma}} = \langle e^{a|D|^{\frac{1}{\sigma}}} f/e^{a|D|^{\frac{1}{\sigma}}} g\rangle_{\dot H^s}.$$

Our main result is the following.

Theorem.1.2

Let $s>0, \sigma\geq 1$. If $u^0 \in \dot H^{\frac{1}{2}}_{a,\sigma}$ such that $\mathrm{div}\, u^0 = 0$, then there exists a positive time T such that (1.1) has unique solution $u\in L^4\left([0,T],\dot H^1_{a,\sigma}\right)$ which also belongs to

$$u\in C([0,T],\dot H^{\frac{1}{2}}_{a,\sigma})\cap L^2([0,T],\dot H^{\frac{3}{2}}_{a,\sigma}).$$

II. PRELIMINARIES

Lemma.2.1[5]

Let E be a Banach space, B a continuous bilinear map from $E\times E\longrightarrow E$, and a positive real number such that $\alpha<\dfrac{1}{4\|B\|}$, with

$$\|B\| = \sup_{\|u\|\leq 1,\|v\|\leq 1}\|B(u,v)\|.$$

For any a in the ball $B(0,\alpha)$ in E, then there exists a unique x in $B(0,2\alpha)$ such that

$$x = a + B(x,x)$$

Lemma.2.2 [3]

Let $a>0$, $\sigma\geq 1$ and $(s_1,s_2)\in\mathbb{R}^2$, such that $s_1<\frac{3}{2}$, $s_1+s_2>0$. Then there exists a constant $C=C(s_1,s_2)$, such that for all $u,v\in \dot H^{s_1}_{a,\sigma}(\mathbb{R}^3)\cap \dot H^{s_2}_{a,\sigma}(\mathbb{R}^3)$, we have

$$\|uv\|_{\dot H^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} \leq C(\|u\|_{\dot H^{s_1}_{a,\sigma}(\mathbb{R}^3)}\|v\|_{\dot H^{s_2}_{a,\sigma}(\mathbb{R}^3)} + \|u\|_{\dot H^{s_2}_{a,\sigma}(\mathbb{R}^3)}\|v\|_{\dot H^{s_1}_{a,\sigma}(\mathbb{R}^3)}).$$

If $a>0$, $s_1<\frac{3}{2}$, $s_2<\frac{3}{2}$ and $s_1+s_2>0$, then exists a constant $C=C(s_1,s_2)$, such that

$$\|uv\|_{\dot H^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} \leq C\|u\|_{\dot H^{s_1}_{a,\sigma}(\mathbb{R}^3)}\|v\|_{\dot H^{s_2}_{a,\sigma}(\mathbb{R}^3)}.$$

Lemma.2.3

Let $a>0$ and $\sigma\geq 1$. Let Q be a bilinear form as defined in (1.1). Then, there exists a constant $C>0$ such that for all $u,v\in \dot H^1_{a,\sigma}(\mathbb{R}^3)$ we have:

$$\|Q(u,v)\|_{\dot H^{-\frac{1}{2}}_{a,\sigma}} \leq C\|u\|_{\dot H^1_{a,\sigma}}\|v\|_{\dot H^1_{a,\sigma}}.$$

Proof Thanks to the inequality (2.1), we get:

$$\begin{aligned}\|Q(u,v)\|_{\dot H^{-\frac{1}{2}}_{a,\sigma}} &\leq C\sup_{k,l}(\|u^k\partial v^l\|_{\dot H^{-\frac{1}{2}}_{a,\sigma}} + \|v^l\partial u^k\|_{\dot H^{-\frac{1}{2}}_{a,\sigma}})\\ &\leq C(\|u\|_{\dot H^1_{a,\sigma}}\|\nabla v\|_{\dot H^0_{a,\sigma}} + \|v\|_{\dot H^1_{a,\sigma}}\|\nabla u\|_{\dot H^0_{a,\sigma}}).\\ &\leq 2C\|u\|_{\dot H^1_{a,\sigma}}\|v\|_{\dot H^1_{a,\sigma}}.\end{aligned}$$

Lemma.2.4

Let u be the solution in $C([0,T],S')$ of the Cauchy problem

$$\begin{cases}\partial_t u - \Delta u = f\\ u(0,x) = u^0(x)\end{cases}$$

with and $u^0\in \dot H^{\frac{1}{2}}_{a,\sigma}$. Then

$$u\in (\cap_{p=2}^{\infty} L^p([0,T],\dot H^{\frac{1}{2}+\frac{2}{p}}_{a,\sigma})\cap C([0,T],\dot H^{\frac{1}{2}}_{a,\sigma})).$$

Moreover, we have the following estimates:

$$(2.1)\|u\|^2_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \int_0^t\|\nabla u\|^2_{\dot H^{\frac{1}{2}}_{a,\sigma}}ds \leq \|u^0\|^2_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \int_0^t\|f(s)\|^2_{\dot H^{-\frac{1}{2}}_{a,\sigma}}ds$$

$$(2.2)[\int_{\mathbb{R}^3}|\xi|e^{2a|\xi|^{\frac{1}{\sigma}}}(\sup_{0\leq t'\leq t}|\hat u(t',\xi)|)^2d\xi]^{\frac{1}{2}} \leq \|u^0\|_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \|f\|_{L^2([0,t),\dot H^{-\frac{1}{2}}_{a,\sigma})}$$

$$(2.3)\|u\|_{L^p_T(\dot H^{\frac{1}{2}+\frac{2}{p}}_{a,\sigma})} \leq \|u^0\|_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \|f\|_{L^2([0,t),\dot H^{-\frac{1}{2}}_{a,\sigma})}$$

Proof The first estimate is just the energy estimate.

$$\partial_t u - \Delta u = f.$$

Let us take the inner product in $\dot H^{\frac{1}{2}}_{a,\sigma}$ with u, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \|\nabla u\|^2_{\dot H^{\frac{1}{2}}_{a,\sigma}} &\leqslant |<f,u>_{\dot H^{\frac{1}{2}}_{a,\sigma}}|\\ &\leqslant C\|u\|_{\dot H^{\frac{1}{2}}_{a,\sigma}}\|f\|^2_{\dot H^{-\frac{1}{2}}_{a,\sigma}}\end{aligned}$$

Taking the L^1 norm with respect to time and using Young's inequality , we deduce the results.

The proof of the second one is based around writing Duhamel's formula in Fourier space, namely,

$$\hat u(t,\xi) = e^{-t|\xi|^2}\widehat{u^0}(\xi) + \int_0^t e^{-(t-s)|\xi|^2}\hat f(s,\xi)ds$$

The Cauchy-Schwartz inequality implies that: For any $0<t'<T$, we get

$$\sup_{0\leq t'\leq t}|\hat u(t',\xi)| \leq |\widehat{u^0}(\xi)| + \frac{1}{\sqrt{2}|\xi|}\|\hat f(\xi,\cdot)\|_{L^2([0,t))}.$$

Multiplying the obtained inequality by $|\xi|^{\frac{1}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}$, we obtain

$$|\xi|^{\frac{1}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}\sup_{0\leq t'\leq t}|\hat u(t',\xi)| \leq |\xi|^{\frac{1}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{u^0}(\xi)| + \frac{|\xi|^{\frac{1}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}}{\sqrt{2}|\xi|}\|\hat f(\xi,\cdot)\|_{L^2([0,t))}$$

Taking the L^2 norm with respect to the frequency variable ξ, we conclude that:

$$[\int_{\mathbb{R}^3}|\xi|e^{2a|\xi|^{\frac{1}{\sigma}}}(\sup_{0\leq t'\leq t}|\hat u(t',\xi)|)^2d\xi]^{\frac{1}{2}} \leq \|u^0(\xi)\|_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \|f\|_{L^2([0,t),\dot H^{-\frac{1}{2}}_{a,\sigma})}$$

Since, for almost all fixed $\xi\in\mathbb{R}^3$, the map $t\mapsto \hat u(t,\xi)$ is continuous over $[0,T]$, the Lebesgue dominated convergence theorem ensures that $u\in C([0,T],\dot H^{\frac{1}{2}}_{a,\sigma}(\mathbb{R}^3))$. Similary, we have:

$$|\xi|^{\frac{3}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\hat u| \leq |\xi|^{\frac{3}{2}}e^{-t|\xi|^2}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{u^0}| + \int_0^t|\xi|^{\frac{3}{2}}e^{-(t-s)|\xi|^2}e^{a|\xi|^{\frac{1}{\sigma}}}|\hat f(s,\xi)|ds$$

Taking the L^2 norm with respect to time and using Young's inequality, we obtain:

$$\begin{aligned}[\int_0^t|\xi|^3e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat u(\xi,s)|^2ds]^{\frac{1}{2}} &\leqslant (\int_0^t|\xi|^2e^{-2s|\xi|^2}ds)^{\frac{1}{2}}|\xi|^{\frac{1}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{u^0}|\\ &+ \int_0^t|\xi|^2e^{-s|\xi|^2}ds(\int_0^t|\xi|^{-1}e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat f(s,\xi)|^2ds)^{\frac{1}{2}}\\ &\leqslant |\xi|^{\frac{1}{2}}e^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{u^0}| + (\int_0^t|\xi|^{-1}e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat f(s,\xi)|^2ds)^{\frac{1}{2}}\end{aligned}$$

Taking the L^2 norm with respect to the frequency variable ξ, we obtain :

$$(2.4)\qquad \|u\|_{L^2_T(\dot H^{\frac{3}{2}}_{a,\sigma})} \leq \|u^0\|_{\dot H^{\frac{1}{2}}_{a,\sigma}} + \|f\|_{L^2_T(\dot H^{-\frac{1}{2}}_{a,\sigma})}$$

Finally, the last inequality follows by interpolation:

$$\|u\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}_{a,\sigma}} \leq \|u\|^{1-\frac{2}{p}}_{\dot H^{\frac{1}{2}}_{a,\sigma}}\|u\|^{\frac{2}{p}}_{\dot H^{\frac{3}{2}}_{a,\sigma}}$$

and

$$\|u\|^p_{\dot H^{\frac{1}{2}+\frac{2}{p}}_{a,\sigma}} \leq \|u\|^{p-2}_{\dot H^{\frac{1}{2}}_{a,\sigma}}\|u\|^2_{\dot H^{\frac{3}{2}}_{a,\sigma}}$$

Taking the L^1 norm with respect to time and using the inequalities (2.2) and (2.4) lead to (2.3).

III. PROOF OF THEOREM 1.1

Proof Let $B(u,u)$ be the solution to the heat equations.

$$\begin{cases}\partial_t B(u,u) - \Delta B(u,u) = Q(u,u)\\ B(u,u) = 0\\ B(u,u)(0) = 0\end{cases}$$

with the bilinear operators Q defined as in (1.2) and

$$B(u,u) = -\int_0^t e^{-(t-s)(\Delta)}Q(u,u))ds$$

Thanks to Lemma 2.3, we have

$$\int_0^T \|Q(u,v)\|^2_{\dot{H}^{-\frac{1}{2}}_{a,\sigma}} ds \leq C\|u\|_{L^4_T(\dot{H}^1_{a,\sigma})}\|v\|_{L^4_T(\dot{H}^1_{a,\sigma})}$$

Thus, combining Duhamel's formula and the inequality (2.3), we have

$$\|B(u,u)\|_{L^4_T(\dot{H}^1_{a,\sigma})} \leq C\|u\|_{L^4_T(\dot{H}^1_{a,\sigma})}\|u\|_{L^4_T(\dot{H}^1_{a,\sigma})}$$

this implies:

$$\|B\|_{L^4_T(\dot{H}^1_{a,\sigma})} \leq C < C_0$$

thanks to Minkowski's inequality, we have

$$\|e^{t\Delta}u^0\|_{L^4_T(\dot{H}^1_{a,\sigma})} \leq \|u^0\|_{\dot{H}^{\frac{1}{2}}_{a,\sigma}}$$

thus, if $\|u^0\|_{\dot{H}^{\frac{1}{2}}_{a,\sigma}} \leq \frac{1}{4C_0}$, we get

$$\|e^{t\Delta}u^0\|_{L^4_T(\dot{H}^1_{a,\sigma})} \leq \frac{1}{4C_0} < \frac{1}{4\|B\|}.$$

According to Lemma 2.1, there exists a unique solution of (1.2) in the ball with center 0 and radius $\frac{1}{2C_0}$ in the space $L^4([0,T];\dot{H}^1_{a,\sigma})$ such that

$$u(t,x) = e^{t\Delta}u^0 + B(u,u).$$

we now consider the case of a large initial data $u^0 \in \dot{H}^{\frac{1}{2}}_{a,\sigma}$. Let $\rho_{u_0} > 0$, such that

$$\Big(\int_{|\xi|\geq\rho_{u_0}} e^{2a|\xi|^{\frac{1}{\sigma}}}|\xi||\widehat{u^0}|^2 d\xi\Big)^{\frac{1}{2}} < \frac{1}{8C_0}.$$

Using the inequality (3.2) and defining $v_0 = \mathcal{F}^{-1}(\chi_{|\xi|<\rho_{u_0}}\widehat{u^0})$, we get

$$\begin{aligned}\|e^{t\Delta}u^0\|_{L^4_T(\dot{H}^1_{a,\sigma})} &\leqslant \|e^{t\Delta}\mathcal{F}^{-1}(\chi_{|\xi|\geq\rho_{u_0}}\widehat{u^0})\|_{L^4_T(\dot{H}^1_{a,\sigma})} + \|e^{t\Delta}v_0\|_{L^4_T(\dot{H}^1_{a,\sigma})}\\ &\leqslant \frac{1}{8C_0} + \|e^{t\Delta}v_0\|_{L^4_T(\dot{H}^1_{a,\sigma})}\end{aligned}$$

From which we can deduce that

$$\begin{aligned}\|e^{t\Delta}v_0\|^4_{L^4_T(\dot{H}^1_{a,\sigma})} &\leqslant \int_0^T\Big[\int_{|\xi|<\rho_{u_0}}|\xi|^2 e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{u^0}|^2 d\xi\Big]^2 dt\\ &\leqslant \rho^2_{u_0}\int_0^T\Big[\int_{|\xi|<\rho_{u_0}}|\xi||^{\frac{1}{\sigma}}e^{2a|\xi|}|\widehat{u^0}|^2 d\xi\Big]^2 dt\\ &\leqslant T\rho^2_{u_0}\|u^0\|^4_{\dot{H}^{\frac{1}{2}}_{a,\sigma}}\end{aligned}$$

which yields

$$\|e^{t\Delta}v_0\|_{L^4_T(\dot{H}^1_{a,\sigma})} \leq (T\rho^2_{u_0})^{\frac{1}{4}}\|u^0\|_{\dot{H}^{\frac{1}{2}}_{a,\sigma}}$$

Thus, if

$$T \leq \Big(\frac{1}{8C_0\rho^{\frac{1}{2}}_{u_0}\|u^0\|_{\dot{H}^{\frac{1}{2}}_{a,\sigma}}}\Big)^4,$$

then we conclude the existence of a unique solution in the ball with center 0 and radius $\frac{1}{2C_0}$ in the space $L^4([0,T];\dot{H}^1_{a,\sigma})$ and we observe that if u is a solution of (1.2) in $L^4([0,T];\dot{H}^1_{a,\sigma})$, then, by Lemma 2.3 $Q(u,u)$ belongs to $L^2_T(\dot{H}^{-\frac{1}{2}}_{a,\sigma})$. Hence, Lemma 2.4 implies that the solution u belongs to $C([0,T];\dot{H}^1_{a,\sigma}) \cap L^2([0,T];\dot{H}^{\frac{3}{2}}_{a,\sigma})$.

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