Characterization of R-Annihilator –μ- Hollow Modules

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Abstract: In this paper we construct some examples for R-Ann-μ-hollow modules and add theorems, propositions. Related concept was given by Nicholsion and Zhou. Let M be an R-modules, then M is R-Ann-μ-hollow module if and only if every submodule A of M such that M/A Small in M. Every finitely generated proper submodule N of M is R-Ann-μ-small for M is a faithful and torsion-μ-small.

Keywords: Hollow module, annihilator-small, μ-small, cosingular module, R-Ann-hollow, R-Ann-μ-small submodule, torsion-μ-small.

INTRODUCTION

Throughout this paper all rings are associative ring with identity and modules are unitary left modules .Nicholson and Zhou defined annihilator-small right (left) ideals in [1] as follows: a left ideal A of a ring R is called annihilator-small if A + T = R, where T is a left ideal , implies that r(T) = 0 where r(T) indicates the right annihilator.

Kalati and Keskin consider this problem for modules in [2] as follows : let M be an R-module and S=End(M).A submodule K of M is called annihilator-small if $K + T = M$, T is a submodule of M ,implies that $r_s(T) = 0$, where r_s indicates the right annihilator of T over S=End(M), where $r_s(T) = \{f \in S; f(T) = 0 , \forall\, t \in T\}$.

A nonzero module M is called hollow module, if every proper submodule of M is small in M [1].A submodule A of M is called μ-small submodules of M ($A \ll_\mu M$) if whenever

$M = A + X, \frac{M}{X}$ is cosingular, then M=X. see [1]. A nonzero R-module M is called μ-Hollow module if every proper submodule of M is μ-small sub modules of M. A nonzero module M is called R-annihilator –hollow module (R-Ann-hollow) if every proper submodule of M is R-annihilator –small submodule of M. These observations lead us to introduce the following concept a nonzero module M is called R-Annihilator –μ- Hollow Module if every proper submodule of M is R-Annihilator-μ-small submodule of M.

Examples:

1) For an R-module M ,M is not R-Annihilator –μ-small submodule of M.

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where M = M + 0 and Ann(0) = {r ∈ R : r.0 = 0}= R ≠ 0

2) Z_4 as Z-module is R-Ann-μ- Hollow (only proper submodule of Z_4 *is* Z_2 and $Z_2 = \{0,1\}$ is μ-small in Z.

3) Every simple module is R-Ann-μ-Hollow module (Z_3 as Z-module).

4) Z_6 as Z-module is not R-Ann-μ-Hollow module since $\{0,3\}$ and $\{0,2,4\}$ are not μ-small in Z_6.

An R-Ann-μ-small submodule of an R-module M need not be small submodule.

For example, consider the module Z as Z-module, for every n>1, claim that nZ is Z-Ann-μ-small submodule of Z, let Z = nZ + mZ, where mZ is submodule of Z . Since Z has a nonzero divisors, then Ann mZ={r∈ Z;r.mZ=0}=0.Thus nZ is Z-Ann-μ-small submodule of Z. But {0} is the only small submodule of Z. Therefore Z as Z-module is R-Ann-μ-Hollow module.

A small submodule of an R-module M need not be R-Ann-μ-small submodule.

For example, consider Z_4 as Z-module.{0} and {0,2} are small submodule of Z_4. But Z_4={0}+Z_4 and Z_4 = {0,2}+Z_4 and Ann Z_4={n∈Z: n. Z_4= 0}= 4Z ≠ 0. Thus each of {0} and {0,2} is not Z-Ann-small sub modules of Z_4 is not R-Ann-μ-small submodule.

Results:

Proposition1. Let f: M→M′ be a homomorphism and let M' is R-Ann-μ-Hollow module such that for all N≤M such that kerf is μ-small in M then M is R-Ann-μ- Hollow module.

Proof: Let N⪇M with M = N+K, where K≤M.To show Ann K = 0. f(N) + f(K) = f(M) = M' (f is epimorphism),if f(N) = M' = f(M) then,

f^{-1}(f(N)) = M. N + kerf = M, since kerf is μ-small of M then N = M(contradiction).Therefore N ≠ M.since f(N) is R-Ann-μ-small submodule of f(M), then f(N) ≠ M' = f(M).Thus M' is M is R-Ann-μ- Hollow module. But Ann f(K) = 0 and Ann K≤ Ann(f(K)) = 0, Then N is R-Ann-μ-small submodule of M .Thus M is R-Ann-μ- Hollow module.

Remark: Let f: M→M′ be a homomorphism from R-module M to M'. The inverse of R-Ann-μ-small submodule of M' need not be is R-Ann-μ-small submodule in M.

Consider $\pi: Z \to \frac{Z}{3Z} \cong Z_3$ the natural epimorphism since 0 is R-Ann-μ-small submodule in Z_3 , but $f^{-1}(0)$ = 3Z is not R-Ann-μ-small in

$f^{-1}(Z_3)$ = Z.

Proposition 2. If $\frac{M}{k}$ is R-Ann-μ- Hollow module then M is R-Ann-μ- Hollow module for all K proper submodule of M.

Proof: Let $\frac{M}{K}$ is R-Ann-μ- Hollow module and let N⫋M such that M = N + L, L is a submodule of M then Ann L= 0,if every proper submodule K of M such that $\frac{M}{k}$ is μ-hollow and K is a submodule of M then M is μ-hollow[1]. $\frac{M}{K}=\frac{N+L}{K}=\frac{N+K}{K}+\frac{L+K}{K}. Since\ \frac{N+K}{K}\lneqq\frac{M}{K}, then\ \frac{N+K}{K}\ is$ R − Ann − μ − small submodule $\frac{M}{K}$, $then\ ann\ \frac{L+K}{K}=0$.Thus Ann(L) is submodule of Ann $\frac{L+K}{K}$=0, therefore Ann L= 0 , and M is μ-hollow. Therefore M is R-Ann-μ- hollow module.

Proposition 3. Let M be a faithful μ-hollow R-module then every proper small submodule of M is R-Ann-μ- hollow module.

Proof: Let M be a faithful μ-hollow R-module, N be proper small submodule of M.

M = M + U, since N is small in M then M = U, Ann (M)= Ann(U) so Ann (U) = 0. Thus N is R-a-small submodule of M, such that $\frac{M}{N}$ is cosingular. Thus M is R-Ann-μ- hollow module.

Theorem 1.: Let M be an R-module, then M is R-Ann-μ-hollow module if and only if every proper submodule A of M (Ann (A) = 0) such that $\frac{M}{A}$ is cosingular is small in M.

Proof: ⇒ Let A be a proper submodule of M, such that $\frac{M}{A}$ is cosingular and Ann(A) = 0.

To show that A is small in M (A≪M), assume that there exists B⊂ M such that M = A + B, since M is R-Annμ-hollow module then $B\ll_{\mu}M$ and we have $\frac{M}{A}$ is cosingular then M = A. which is a contradiction. Thus A is small in M (A≪M).

⇐ To show that M is R-Ann-μ- hollow module, let A be a proper submodule of M assume that A is not μ-small in M ,then there exists a proper submodule B of M such that $\frac{M}{B}$ is cosingular and M=A+B thus B≪M then M=A (since A is a proper submodule of M)which is a contradiction ,thus M is R-Ann-μ- hollow module.

Proposition 4. Let M be a R-module and K be R-Ann-μ-small submodule in M. if Rad(M) is μ-small submodule of M and Z(M) is finitely generated then K+ Rad(M) + Z(M) is R-Ann-μ-small submodule of M.

Proof: Let Z(M) = $Rz_1 + Rz_2 + Rz_3 + \ldots + Rz_n$ where $z_i \in$ Z(M), ∀ I = 1, 2 ,3….n

Let K+ Rad(M) + Z(M) + X = M where X is a submodule of M .since Rad(M) is μ-small submodule of M, then K + Z(M) + X = M. but K is R-Ann-μ-small submodule in M, therefore Ann(Z(M) + X) = Ann($Rz_1 + Rz_2 + Rz_3 + \ldots + Rz_n$ + X) = 0.

So $(\bigcap_{i=1}^{n} Ann\ Rz_i\ ) \cap AnnX = 0.$ Since $z_i \in$ Z(M), then ann $z_i \leq^{e}$R, so Ann(X) = 0 and $\frac{M}{X}$ is cosingular. Thus K+ Rad(M) + Z(M) is R-Ann-μ-small submodule of M.

Proposition 5. Let R be an integral domain, let M be a faithful R-module then every proper submodule N of M ; with Ann N ≠ 0 is R-Ann-μ-small.

Proof: Let M = N + K, then 0 = Ann(M) = Ann(N+K) = Ann(N) ∩ Ann(K). Since Ann ≠ 0 and R is an integral domain, then ann N$\leq^{e}$ R. therefore Ann K = 0, hence K<<M where $\frac{M}{K}$ is cosingular, thus N is R-Ann-μ-small.

Proposition 6. Let R be an integral domain and M be a faithful and torsion μ-small module then every finitely generated proper submodule N of M is R-Ann-μ-small.

Proof: Let N= $Rx_1 + Rx_2 + Rx_3 + \ldots + Rx_n$ be a finitely generated proper submodule of M and M = N + K ,then 0 = Ann(M) = Ann(N + K) = Ann(N) ∩ Ann(K) = (Ann($Rx_1 + Rx_2 + Rx_3 + \ldots + Rx_n$)) ∩ Ann(K) =($\bigcap_{i=1}^{n} ann\ Rx_i) \cap Ann(K)$.). Since M is torsion, then

Ann (Rx_i) ≠0 ∀ I = 1,2,3,….,n. But R is integral domain then Rx_i is essential in R, for all i. Hence $\bigcup_{i=1}^{n} Ann$ Rx_i = Ann(K) = 0 thus, N is R-Ann-small in M, but M is μ-small module, thus M is cosingular implies $\frac{M}{K}$ is cosingular. thus N is R-Ann-μ-small.

Corollary 1. Let R be an integral domain and let M be a projective small R-module then every proper small submodule of M is R-Ann-μ-small submodule of M.

Corollary 2. Let M be a small R-module and let $K \ll_{\mu} N \ll_{\mu} L \ll_{\mu} M$ such that $\frac{L}{N}$ is R-Ann-μ-small submodule of $\frac{M}{N}$ then $\frac{L}{K}$ is R-Ann-μ-small submodule of $\frac{M}{K}$.

Proof: Let f: $\frac{M}{K} \rightarrow \frac{M}{N}$ be the map defined by f(x + N) = x + N, ∀ x ∈ M. easily to show f is an epimorphism.since $\frac{L}{N}$ is R-Ann-μ-small submodule of $\frac{M}{N}$, therefore $\frac{L}{K} = f^{-1}(\frac{L}{N})$ is R-Ann-μ-small submodule of $\frac{M}{K}$.Thus $\frac{L}{K}$ is R-Ann-μ-small submodule of $\frac{M}{K}$.

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