Uniqueness of Bounded Variation Solutions for Measure Functional Differential Equations with Infinite Delay

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Abstract—**By using Henstock-Kurzweil integral, the uniqueness theorems of bounded variation solutions for measure functional differential equations with infinite delay are established. This result generalizes theorem concerning uniqueness in Lebesgue integral setting to a Henstock -Kurzweil integral setting.**

***Index Terms*—measure functional differential equations with infinite delay; Henstock-Kurzweil integral; bounded variation solution; uniqueness.**

I. INTRODUCTION

The Henstock-Kurzweil integral encompasses the Riemann and Lebesgue integrals [1]. the integral was introduced by Henstock and Kurzweil independently in 1957-1958 and was proved useful in the study of ordinary differential equations (see [2]).

Measure functional differential equations with infinite delay have the form

$$\begin{cases} x(t)=x(t_0)+\int_{t_0}^{t} f(x_s,s)dg(s), \\ x(t_0)=\phi, \end{cases} \quad (1)$$

which have been introduced in the paper [3] by slavík, where x is an unknown function with values in R^n and the symbol x_s denotes the function $x_s(\tau)=x(s+\tau)$ defined on $[-r,0], r\geq 0$ being a fixed number corresponding to the length of the delay. The integral on the right-hand side of (1) is the Kurzweil-Stieltjes integral with respect to a nondecreasing function g. we consider that the integrands f is Henstock-Kurzweil integrable and ϕ is a regulated function.

Let $G([a,b],R^n)$ be the space of regulated functions $x:[a,b]\to R^n$, that is, the lateral limits

$$x(t+)=\lim_{\rho\to 0_+} x(t+\rho), t\in[a,b)$$

and

$$x(t-)=\lim_{\rho\to 0_-} x(t+\rho), t\in(a,b],$$

exist and are finite. $G([a,b],R^n)$ which is a Banach space when endowed with the norm $\|\phi\|=\sup_{a\leq t\leq b}\|\phi(t)\|$ for all $\|\phi\|\in G([a,b],R^n)$. Also, any function in $G([a,b],R^n)$ is the uniform limit of step functions.

Define

$$G^-([a,b],R^n)=\{u\in G([a,b],R^n): u \text{ is left continuous at every } t\in(a,b]\}.$$

In $G^-([a,b],R^n)$, we consider the norm induced by $G^-([a,b],R^n)$. We denote by $BV([a,b],R^n)$ the space of functions $x:[a,b]\to R^n$ which are of bounded variation. In $BV([a,b],R^n)$, we consider the variation norm given by $\|x\|_{BV}=\|x(a)\|+Var_a^b x$, where $Var_a^b x$ Stands for the variation of x in the interval $[a,b]$. Then $(BV([a,b],R^n),\|\cdot\|_{BV})$ is a Banach space and $BV([a,b],R^n)\subset G([a,b],R^n)$. When $x\in BV([a,b],R^n)$ is also left continuous, we write $x\in BV^-([a,b],R^n)$.

It is clear that for a function $x\in G^-([-\infty,t_0+\sigma],R^n)$, we have $x_t\in G^-([-r,0],R^n)$ for all $t\in[t_0,t_0+\sigma]$.

Let $x\in G^-([-\infty,t_0+\sigma],R^n)$ with the following property: if $x=x(t), t\in(-\infty,t_0+\sigma]$, is an element of G_1 and $\bar{t}\in(-\infty,t_0+\sigma]$, then $\bar{x}$ given by

$$\bar{x}(t)=\begin{cases} x(t), & -\infty\leq t\leq \bar{t}, \\ x(\bar{t}+), & \bar{t}<t\leq t_0+\sigma \end{cases}$$

Also belongs to G_1.

Let $H_1\in G^-([-r,0],R^n)$ be such that

$$\{x_t \mid t\in[t_0,t_0+\sigma], x\in G_1\}\subset H_1.$$

The existence of bounded variation solutions for measure functional differential equation with infinite delay were obtained in [4]. On this basis, the uniqueness of bounded variation solutions for measure functional differential equation

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with infinite delay is discussed in this paper.

II. PRELIMINARIES

Let $[a,b]$ be a compact interval in R and $\|\cdot\|$ be a norm in R^n.

Let $\delta(\xi)$ be a positive function on $[a,b]$, i.e. $\delta(\xi):[a,b]\to R^+$. We say $D=\{([t_{i-1},t_i],\xi_i)\}_{i=1}^m$ is $\delta-$fine of $[a,b]$ if $\xi_i\in[t_{i-1},t_i]\subset(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i))$ for all $i=1,2,\cdots,n$.

Definition.1[1]

A function $u:[a,b]\to R^n$ is said to be Henstock-Kurzweil integrable on $[a,b]$ if there exists an $I\in R^n$ such that for every $\varepsilon>0$, there exists $\delta(\xi):[a,b]\to R^+$ such that for every $\delta-$fine partition $D=\{([t_{i-1},t_i],\xi_i)\}_{i=1}^m$, we have

$$\left\|\sum_{i=1}^m u(\xi_i)(t_i-t_{i-1})-I\right\|<\varepsilon.$$

We denote the Henstock-Kurzweil integral (also write as H-K integral) I by (H-K) $\int_a^b u(s)ds$.

Definition.2[4]

Assume $f(x_t,t):H_1\times[t_0,t_0+\sigma]\to R^n$ satisfy the following conditions:

(A) Then exists a positive function $\delta(\tau):[t_0,t_0+\sigma]\to R^+$ such that for every $[u,v]$ satisfy $\tau\in[u,v]\subset(\tau-\delta(\tau),\tau+\delta(\tau))\subset[t_0,t_0+\sigma]$ and $x\in G_1$, we have

$$\|f(x_\tau,\tau)(g(v)-g(u))\|<|h(v)-h(u)|.$$

(B) For every $[u,v]$ satisfy $\tau\in[u,v]\subset(\tau-\delta(\tau),\ \tau+\delta(\tau))\subset[t_0,t_0+\sigma]$ and $x,y\in G_1$, we have

$$\|f(x_\tau,\tau)-f(y_\tau,\tau)\|(g(v)-g(u))<\omega(\|x_\tau-y_\tau\|)|h(v)-h(u)|,$$

where $h:[t_0,t_0+\sigma]\to R$ is a nondecreasing function and continuous the left. $\omega:[0,\infty]\to R$ is a continuous and increasing function with $\omega(0)=0,\omega(r)>0$ for $r>0$.

Definition.3[4]

Let $\Omega\subset H_1\times[t_0,t_0+\sigma]$ be open. Assume that function $f:\Omega\to R^n$ is a Caratheodory function and belongs to the class $W(\Omega,,h,\omega)$ if f satisfys the condition (A), (B).

Lemma.1[5]

Assume that $\psi:[a,b]\to[0,+\infty)$ is a bounded function on $[a,b]$, $h:[a,b]\to[0,+\infty)$ is a nondecreasing function and continuous from the left on the interval $[a,b]$, $\omega:[0,+\infty]\to R$ is a continuous and increasing function with $\omega(0)=0,\omega(r)>0$ for $r>0$. Assume

$$F(u)=\int_{u_0}^u\frac{1}{\omega(r)}dr, \qquad (2)$$

for $u>0$, where $u_0>0$. $F:[0,\infty]\to R$ is a increasing function, $F(u_0)=0$ and

$$\lim_{u\to0_+}F(u)=\alpha\ge-\infty,\quad \lim_{u\to+\infty}F(u)=\beta\le+\infty.$$

Assume the inequality

$$\psi(\xi)\le k+\int_a^\xi\omega(\psi(t))dh(t)$$

holds for $\xi\in[a,b]$, where k is a constant and $k>0$.

If $F(k)+h(b)-h(a)<\beta$, then the inequality

$$\psi(\xi)\le F^{-1}[F(k)+h(\xi)-h(a)] \qquad (3)$$

holds for $\xi\in[a,b]$, where $F^{-1}:[\alpha,\beta]\to R$ is the inverse function to F form (2).

III. PRIME RESULT

Definition.1

A bounded variation solution $x:[-\infty,t_0+\eta]\to R^n$ of (1) is said to be locally unique for increasing values of t if for any solution $y:[-\infty,t_0+\alpha]\to R^n,\alpha>0$ of (1) with $y_{t_0}=x_{t_0}=\phi$ there exists $\eta_1>0$ such that $x(t)=y(t)$ for $t\in[-\infty,t_0+\eta]\cap[-\infty,t_0+\alpha]\cap[-\infty,t_0+\eta_1]$.

Theorem.1

Assume that $f\in W(\Omega,h,\omega)$, where h is a function and continuous from the left, ω: $[0,+\infty)\to R$ is a continuous and increasing function with $\omega(0)=0$, $\omega(r)>0$ for $r>0$, and for $u>0$, we have

$$\lim_{v\to0+}\int_v^u\frac{1}{\omega(r)}dr=+\infty. \qquad (4)$$

Then every bounded variation solution $x=x(t)$ with $x_{t_0}=\phi$ of (1) is locally unique for increasing values of t, $(t_0,\phi)\in\Omega$.

Proof Assume that $x,y:[-\infty,t_0+\eta]\to R^n$ are two bounded variation solution of (1) with $x_{t_0}=y_{t_0}=\phi$, where $\eta>0$. Then

$$\left\|x(t)-y(t)\right\|=\left\|\int_{t_0}^{t}\left[f(x_s,s)-f(y_s,s)\right]dg(s)\right\|,$$
$$t_0\le t\le t_0+\eta. \qquad (5)$$

For arbitrary $\varepsilon>0$ there exists a positive function $\delta(\tau):[t_0,t]\to R^+$, such that for any $\delta(\tau)$-fine partition $D=\{([t_{i-1},t_i],\tau_i)\}_{i=1}^{m}$. By Definition 1 and condition (B), we have

$$\left\|\int_{t_0}^{t}\left[f(x_s,s)-f(y_s,s)\right]dg(s)\right\|\le$$
$$\left\|\int_{t_0}^{t}\left[f(x_s,s)-f(y_s,s)\right]dg(s)-\sum_{i=1}^{m}\left[f(x_{\tau_i},\tau_i)-f(y_{\tau_i},\tau_i)\right]\left(g(t_i)-g(t_{i-1})\right)\right\|+$$
$$\left\|\sum_{i=1}^{m}\left[f(x_{\tau_i},\tau_i)-f(y_{\tau_i},\tau_i)\right]\left(g(t_i)-g(t_{i-1})\right)\right\|<$$
$$\frac{\varepsilon}{2}+\sum_{i=1}^{m}\omega\left(\left\|x_{\tau_i}-y_{\tau_i}\right\|\right)\left(h(t_i)-h(t_{i-1})\right). \qquad (6)$$

Owing to

$$\sum_{i=1}^{m}\omega\left(\left\|x_{\tau_i}-y_{\tau_i}\right\|\right)\left(h(t_i)-h(t_{i-1})\right)\le$$
$$\left\|\sum_{i=1}^{m}\omega\left(\left\|x_{\tau_i}-y_{\tau_i}\right\|\right)\left(h(t_i)-h(t_{i-1})\right)-\int_{t_0}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)\right\|+$$
$$\int_{t_0}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)<$$
$$\frac{\varepsilon}{2}+\int_{t_0}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s). \qquad (7)$$

By (5)-(7), we get

$$\left\|x(t)-y(t)\right\|=\left\|\int_{t_0}^{t}\left[f(x_s,s)-f(y_s,s)\right]dg(s)\right\|$$
$$<\varepsilon+\int_{t_0}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)$$
$$<\int_{t_0}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)$$
$$=\int_{t_0}^{t_0+\delta}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)+\int_{t_0+\delta}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s),$$

Where $0<\delta<t-t_0$.

According to [5], we obtain

$$\int_{t_0+\delta}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)=$$
$$\omega\left(\left\|x_{t_0}-y_{t_0}\right\|\right)\left[h(t_0+)-h(t_0)\right]+$$
$$\lim_{t_1\to t_0^+}\int_{t_1}^{t_0+\delta}\omega\left(\left\|x_s-y_s\right\|\right)dh(s)\le$$
$$\sup_{s\in(t_0,t_0+\delta]}\omega\left(\left\|x_s-y_s\right\|\right)\left(h(t_0+\delta)-h(t_0)\right)=A(\delta),$$

becasure of $h(t_0+)$ exists, then $\lim_{\delta\to 0^+}A(\delta)=0$. Thus

$$\left\|x(t)-y(t)\right\|\le A(\delta)+\int_{t_0+\delta}^{t}\omega\left(\left\|x_s-y_s\right\|\right)dh(s),$$

where $t\in[t_0+\delta,t_0+\eta]$.

For given $u_0>0$, set $F(u)=\int_{u_0}^{u}\frac{1}{\omega(r)}dr$. Then the imequality

$$\left\|x(t)-y(t)\right\|\le F^{-1}\left(FA(\delta)+h(t)-h(t_0+\delta)\right)$$

holds, where

$$t\in[t_0+\delta,t_0+\eta], F\left(A(\delta)\right)+h(t)-h(t_0+\delta)<\beta,$$
$$\beta=\lim_{u\to+\infty}F(u)\le+\infty.$$

Obviously, we have

$$F\left(A(\delta)\right)+h(t_0+\eta)-h(t_0+\delta)\le$$
$$F\left(A(\delta)\right)+h(t_0+\eta)-h(t_0+),$$

and for

$$\lim_{\delta\to 0^+}A(\delta)=0, \qquad \lim_{u\to 0^+}F(u)=-\infty.$$

Hence

$$\lim_{\delta\to 0^+}F\left(A(\delta)\right)+h(t_0+\eta)-h(t_0+)=-\infty.$$

Therefore there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$ the inequality $F\left(A(\delta)\right)+h(t_0+\eta)-h(t_0+)<\beta$ holds, and we get

$$F\left(\left\|x_t-y_t\right\|\right)-A(\delta)\le h(t)-h(t_0+\delta)$$
$$\le h(t)-h(t_0+).$$

According the definition of function F, for $t\in[t_0+\delta,t_0+\eta]$ and $\delta\in(0,\delta_0)$,we have

$$\int_{A(\delta)}^{\left\|x(t)-y(t)\right\|}\frac{1}{\omega(r)}dr\le h(t_0+\eta)-h(t_0+).$$

If $\left\|x(t^*)-y(t^*)\right\|=k>0$ for some $t^*\in(t_0,t_0+\eta]$, then for $\delta\in(0,\delta_0)$ such that $\delta<t^*-t_0$, we have

$$\int_{A(\delta)}^{k}\frac{1}{\omega(r)}dr\le h(t_0+\eta)-h(t_0+)<+\infty,$$

thus

$$\lim_{\delta\to 0^+}\int_{A(\delta)}^{k}\frac{1}{\omega(r)}dr\le h(t_0+\eta)-h(t_0+)<\infty,$$

this is in contradiction with the function ω, so $\left\|x(t)-y(t)\right\|=0$ for all $t\in[t_0,t_0+\eta]$. Therefore, the theorem is proved.

Corollary.1

If $f \in W(\Omega, h, \omega)$, where $\omega(r) = Lr$, $r \geq 0$, $L \geq 0$, then the bounded variation solution of (1) which satisfies $(t_0, \phi) \in \Omega$ is locally unique for increasing values of t.

Proof For $u > 0$, obviously

$$\lim_{v \to 0^+} \int_v^u \frac{1}{\omega(r)} dr = \lim_{v \to 0^+} \frac{1}{L} \ln \frac{u}{v} = +\infty,$$

the conditions of theorem 3.1 are satisfied, then the corollary holds.

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