Exponential Inequalities for a WOD Sequence

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Abstract — **In this paper, we give exponential inequalities for the sequence of WOD random variables.**

Keywords— **WOD random variables; exponential inequalities.**

I. INTRODUCTION

Since Wang [2] introduced the concept of WOD random variable, many scholars have shown great interest in it and achieved many meaningful results.See,for example, Shen [3] established the Bernstein type inequality for WOD random variables and gave some applications, Wang and Cheng [4] presented some basic renewal theorems for a random walk with widely dependent increments and gave some applications. He [5] provided the asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables.

II. DEFINITION OF A WOD SEQUENCE

Definition 1[1] For the random variables $\{X_n, n\geq 1\}$, if there exists a finite real sequence $\{g_U(n), n\geq 1\}$ satisfying for each $n\geq 1$ and for all $x_i\in(-\infty,\infty), 1\leq i\leq n$

$$P(X_1>x_1, X_2>x_2\cdots, X_n>x_n)\leq g_U(n)\prod_{i=1}^{n}P(X_i>x_i),$$

then we say that the $\{X_n, n\geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite real sequence $\{g_L(n), n\geq 1\}$ satisfying for each $n\geq 1$ and for all $x_i\in(-\infty,\infty), 1\leq i\leq n$

$$P(X_1\leq x_1, X_2\leq x_2\cdots, X_n\leq x_n)\leq g_L(n)\prod_{i=1}^{n}P(X_i\leq x_i).$$

then we say that the $\{X_n, n\geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n\geq 1\}$ are widely orthant dependent (WOD, in short), $g_U(n), g_L(n), n\geq 1$ are called dominating coefficients.

Lemma 1[1] Let $\{X_n, n\geq 1\}$ be a sequence of WOD random variables

(1) If $\{f_n(\cdot), n\geq 1\}$ are all nondecreasing(or all nonincreasing), then $\{f_n(X_n), n\geq 1\}$ are still WOD;

(2) For each $n\geq 1$ and any $s\in R$,

$$\mathrm{E}\exp\left\{s\sum_{i=1}^{n}X_i\right\}\leq g(n)\prod_{i=1}^{n}\mathrm{E}\exp\{sX_i\},$$

where $g_n=\max\{g_U(n), g_L(n)\}$.

Let $\{X_n, n\geq 1\}$ be a sequence of random variables and $\{c_n, n\geq 1\}$ be a sequence of positive number. Define $1\leq i\leq n$, $n\geq 1$,

$$X_{1,i,n}=-c_nI(X_i<-c_n)+X_iI(-c_n\leq X_i\leq c_n)+c_nI(X_i>c_n),$$
$$X_{2,i,n}=(X_i-c_n)I(X_i>c_n),$$
$$X_{3,i,n}=(X_i+c_n)I(X_i<-c_n). \quad (1)$$

It is easy to check that $X_{1,i,n}+X_{2,i,n}+X_{3,i,n}=X_i$ for $1\leq i\leq n$, $n\geq 1$ and $X_{1,1,n}, X_{1,2,n},\cdots, X_{1,n,n}$ are bounded by c_n for each fixed $n\geq 1$. If $\{X_n, n\geq 1\}$ are WOD random variables, then $\{X_{1,i,n}, 1\leq i\leq n\}$, $\{X_{2,i,n}, 1\leq i\leq n\}$, $\{X_{3,i,n}, 1\leq i\leq n\}$ are also WOD random variables for each fixed $n\geq 1$ by Lemma 1.

Lemma 2[1] Let $p\geq 1$ and $\{X_n, n\geq 1\}$ be a sequence of WOD random variables with $E|X_n|^p<\infty$ for each $n\geq 1$. Assume further that $EX_n=0$ for each $n\geq 1$ when $p\geq 2$. Then there exist positive constant $C_1(p)$ and $C_2(p)$ depending only on p such that

$$E\left|\sum_{i=1}^{n}X_i\right|^p\leq[C_1(p)+C_2(p)g(n)]\sum_{i=1}^{n}E|X_i|^p \quad 1\leq p\leq 2,$$

and

$$E\left|\sum_{i=1}^{n}X_i\right|^p\leq C_1(p)\sum_{i=1}^{n}E|X_i|^p+C_2(p)g(n)\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2} \quad p\geq 2.$$

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III. MAIN RESULTS

Theorem 1 Let $\{X_n, n \ge 1\}$ be a sequence of WOD random variables with $EX_n = 0$ for each $n \ge 1$, if there exist a sequence of positive numbers $\{c_n, n \ge 1\}$ such that $|X_i| \le c_i$ for each $i \ge 1$, then for any $t > 0, n \ge 1$,

$$E\exp\left\{t\sum_{i=1}^{n} X_i\right\} \le g(n)\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{tc_i} EX_i^{\ 2}\right\}.$$

Proof. It is easy to check that for all $x \in R$, $e^x \le 1 + x + \frac{1}{2}x^2 e^{|x|}$. Thus, by $EX_i = 0$, and $|X_i| \le c_i$ for each $i \ge 1$, we have

$$Ee^{tX_i} \le 1 + tEX_i + \frac{1}{2}t^2 E\left[X_i^{\ 2} e^{t|X_i|}\right] = 1 + \frac{1}{2}t^2 E\left[X_i^{\ 2} e^{t|X_i|}\right]$$
$$\le 1 + \frac{1}{2}t^2 e^{tc_i} EX_i^{\ 2} \le \exp\left\{\frac{1}{2}t^2 e^{tc_i} EX_i^{\ 2}\right\}.$$

for any $t > 0$, by Lemma 1,we can see that

$$E\exp\left\{t\sum_{i=1}^{n} X_i\right\} \le g(n)\prod_{i=1}^{n} Ee^{tX_i}$$
$$\le g(n)\exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{tc_i} EX_i^{\ 2}\right\}.$$

Corollary 1 Let $\{X_n, n \ge 1\}$ be a sequence of WOD random variables and $\{X_{1,i,n}, 1 \le i \le n, n \ge 1\}$ be defined by (1). Then for any $t > 0$ and $n \ge 1$,

$$E\exp\left\{t\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right)\right\}$$
$$\le g(n)\exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n} EX_i^{\ 2}\right\}.$$

Proof. It is easily seen that $\{X_{1,i,n} - EX_{1,i,n}, 1 \le i \le n, n \ge 1\}$ are still WOD random variables with $|X_{1,i,n} - EX_{1,i,n}| \le 2c_n$ for each $1 \le i \le n, n \ge 1$,by Theorem1,we have

$$E\exp\left\{t\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right)\right\}$$
$$\le g(n)\exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n} E\left(X_{1,i,n} - EX_{1,i,n}\right)^2\right\}$$
$$\le g(n)\exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n} EX_i^{\ 2}\right\}.$$

Theorem 2 Let $\{X_n, n \ge 1\}$ be a sequence of WOD random variables and $\{X_{1,i,n}, 1 \le i \le n, n \ge 1\}$ be defined by (1). Define $B_n^{\ 2} = \sum_{i=1}^{n} EX_i^{\ 2}, n \ge 1$. Then for any $\varepsilon > 0$ such that $\varepsilon \le \frac{eB_n^{\ 2}}{2c_n}$ and $n \ge 1$,

$$P\left(\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right) \ge \varepsilon\right) \le g(n)\exp\left\{-\frac{\varepsilon^2}{2eB_n^{\ 2}}\right\},$$
$$P\left(\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right) \le -\varepsilon\right) \le g(n)\exp\left\{-\frac{\varepsilon^2}{2eB_n^{\ 2}}\right\},$$
$$P\left(\left|\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right)\right| \ge \varepsilon\right) \le 2g(n)\exp\left\{-\frac{\varepsilon^2}{2eB_n^{\ 2}}\right\}.$$

Proof. By Markov's inequality and Corollary1, we have that for $t > 0$

$$P\left(\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right) \ge \varepsilon\right)$$
$$\le e^{-t\varepsilon} E\exp\left\{t\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right)\right\}$$
$$\le e^{-t\varepsilon} g(n)\exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n} EX_i^{\ 2}\right\}$$
$$= g(n)\exp\left\{-t\varepsilon + \frac{t^2}{2}e^{2tc_n}B_n^{\ 2}\right\},$$

Taking $t = \frac{\varepsilon}{eB_n^{\ 2}}$,and noting that $2tc_n \le 1$,so

$$g(n)\exp\left\{-t\varepsilon + \frac{t^2}{2}e^{2tc_n}B_n^{\ 2}\right\} \le g(n)\exp\left\{-\frac{\varepsilon^2}{2eB_n^{\ 2}}\right\}.$$

since $\{-X_{1,i,n}, n \ge 1\}$ is a sequence of WOD random variables,so

$$P\left(\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right) \le -\varepsilon\right)$$
$$= P\left(\sum_{i=1}^{n}\left(-X_{1,i,n} - E\left(-X_{1,i,n}\right)\right) \ge \varepsilon\right)$$
$$\le g(n)\exp\left\{-\frac{\varepsilon^2}{2eB_n^{\ 2}}\right\}$$

then

$$P\left(\left|\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right)\right| \ge \varepsilon\right) \le 2g(n)\exp\left\{-\frac{\varepsilon^2}{2eB_n^{\ 2}}\right\}.$$

Corollary 2 Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables and $\{X_{1,i,n}, 1 \leq i \leq n, n \geq 1\}$ be defined by (1). Then for any $\varepsilon > 0$ such that $\varepsilon \leq \frac{eEX_1^2}{2c_n}$,

$$P\left(\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right) \geq n\varepsilon\right) \leq g(n)\exp\left\{-\frac{n\varepsilon^2}{2eEX_1^2}\right\},$$

$$P\left(\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right) \leq -n\varepsilon\right) \leq g(n)\exp\left\{-\frac{n\varepsilon^2}{2eEX_1^2}\right\},$$

$$P\left(\left|\sum_{i=1}^{n}\left(X_{1,i,n} - EX_{1,i,n}\right)\right| \geq n\varepsilon\right) \leq 2g(n)\exp\left\{-\frac{n\varepsilon^2}{2eEX_1^2}\right\}.$$

Theorem 3 Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables and $\{X_{q,i,n}, 1 \leq i \leq n, n \geq 1\}$ $q = 2,3$, be defined by (1). Assume that there exists a $\delta > 0$ statisfying $\sup_{|t|\leq\delta} Ee^{tX_1} \leq M_\delta < \infty$, where M_δ is a positive constant depending only on δ. Then for any $\varepsilon > 0$ and $t \in (0, \delta]$,

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{q,i,n} - EX_{q,i,n})\right| \geq \varepsilon\right) \leq \frac{[C_1(p)+C_2(p)g(n)]2M_\delta e^{-tc_n}}{\varepsilon^2 nt^2}.$$

Proof. For $q = 2$, by Markov's inequality and Lemma2, we can see that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{2,i,n} - EX_{2,i,n})\right| \geq \varepsilon\right) \leq \frac{E\left|\sum_{i=1}^{n}(X_{2,i,n} - EX_{2,i,n})\right|^2}{\varepsilon^2 n^2}$$
$$\leq \frac{[C_1(p)+C_2(p)g(n)]nE(X_{2,1,n} - EX_{2,1,n})^2}{\varepsilon^2 n^2}$$
$$\leq \frac{[C_1(p)+C_2(p)g(n)]EX^2_{2,1,n}}{\varepsilon^2 n}.$$

Therefore, it remains only to estimate $EX^2_{2,1,n}$. Here, we will adopt the method in[6,Lemma4].By that,it is easy to find that

$$EX^2_{2,1,n} \leq \frac{2M_\delta}{t^2}e^{-tc_n},$$

so

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{2,i,n} - EX_{2,i,n})\right| \geq \varepsilon\right) \leq \frac{[C_1(p)+C_2(p)g(n)]2M_\delta e^{-tc_n}}{\varepsilon^2 nt^2}.$$

For $q = 3$,the proof is similar to the case for $q = 2$ and is omitted.

Corollary 3 Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$.Let $\{X_{q,i,n}, 1 \leq i \leq n, n \geq 1\}$ $q = 2,3$, be defined by (1), Then for any $\varepsilon > 0$,

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{q,i,n} - EX_{q,i,n})\right| \geq \varepsilon\right) \leq \frac{[C_1(p)+C_2(p)g(n)]2Ee^{\delta|X_1|}}{\varepsilon^2 n\delta^2}e^{-\delta c_n}.$$

Proof It is easily seen that $\sup_{|t|\leq\delta} Ee^{tX_1} \leq Ee^{\delta|X_1|} = M_\delta < \infty$,which implies the desired results immediately from Theorem3.

Theorem 4 Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed WOD random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$, and $\{c_n, n \geq 1\}$ be a sequence of positive numbers such that

$$0 < c_n \leq \left(\frac{enEX_1^2}{8\delta}\right)^{1/3} \text{ for some } n \geq n_0,$$

where n_0 is a positive integer. Define $\varepsilon_n = \sqrt{2\delta eEX_1^2 c_n / n}$. Then for $n \geq n_0$,

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i - EX_i)\right| \geq 3\varepsilon_n\right) \leq 2\left(g(n) + \frac{[C_1(p)+C_2(p)g(n)]Ee^{\delta|X_1|}}{\delta^3 eEX_1^2 c_n}\right)e^{-\delta c_n}.$$

Proof It is easy to check that $2\varepsilon_n c_n \leq eEX_1^2$ and $n\varepsilon_n^2 / 2eEX_1^2 = \delta c_n$. It follows from Corollary2 and Corollary3 that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_i - EX_i)\right| \geq 3\varepsilon_n\right) \leq P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| \geq \varepsilon_n\right)$$
$$+ P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{2,i,n} - EX_{2,i,n})\right| \geq \varepsilon_n\right)$$
$$+ P\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{3,i,n} - EX_{3,i,n})\right| \geq \varepsilon_n\right)$$

$$\leq 2g(n)\exp\left\{-\frac{n\varepsilon_n^{\ 2}}{2eEX_1^{\ 2}}\right\}$$

$$+\frac{4[C_1(p)+C_2(p)g(n)]Ee^{\delta|X_1|}}{\varepsilon_n^{\ 2}n\delta^2}e^{-\delta\varepsilon_n}$$

$$=2\left(g(n)+\frac{[C_1(p)+C_2(p)g(n)]Ee^{\delta|X_1|}}{\delta^3 eEX_1^{\ 2}c_n}\right)e^{-\delta\varepsilon_n}.$$

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IV REFERENCES

[1] Wang X , Xu C , Hu T C , et al. On complete convergence for widely orthant dependent random variables and its applications in nonparametric regression models[J]. TEST, 2014, 23(3):607-629.
[2] Wang K , Wang Y , Gao Q . Uniform Asymptotics for the Finite-Time Ruin Probability of a Dependent Risk Model with a Constant Interest Rate[J]. methodology & computing in applied probability, 2013, 15(1):109-124.
[3] Shen, Aiting. Bernstein-Type Inequality for Widely Dependent Sequence and Its Application to Nonparametric Regression Models[J]. Abstract & Applied Analysis, 2013, 2013:1-9.
[4] Wang Y , Cheng D . Basic renewal theorems for random walks with widely dependent increments[J]. journal of mathematical analysis & applications, 2011, 384(2):
597-606.
[5] He W , Cheng D , Wang Y . Asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables[J]. Statistics & Probability Letters, 2013, 83(1):331-338.
[6] Oliveira P E . An exponential inequality for associated variables[J]. Statistics & Probability Letters, 2005, 73(2):
189-197.

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