The balancedness of N -complexes based on cotorsion pairs

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Abstract — **Let Φ be a abelian ctegory, and (*A*,*B*) be a complete hereditary cotorsion pair on Φ. We define relative cohomology groups based on (*A*,*B*) in the category of *N*-complexes on Φ. Especially we are devoted to consider the balancedness of the relative cohomology functors.**

Keywords—***N*-complex; cotorsion pair; relative cohomology group; balancedness**

*MSC(2010): **81E10; 18G25; 18G35***

I. Introduction

The notion of *N*-complexes was introduced by Mayer [15] in his study of simplicial complexes. In 1996, Kapranov [10] and Dubois-Violette [5] gave an abstract framemark of homological theory of *N*-complexes. Since then the notion of *N*-complexes has concerned by many authors, for example [7,9,10,20], and some important applications to theoretical physics, quantum theories and representation theories of algebras have been found. In 2015, Yang and Ding [20] provied an effective construction of left and right triangles, and showed that both the homotopy category and the derived category of *N*-complexes are pretiangulated categories. Later Iyama, Kato and Miyachi [9] futher showed that the homotopy category and derived category of *N*-complexes are triangulated categories. Recently, Luo and Zhu [14] described how complete hereditary cotorsion pairs gave rise to model structures on the derived category of *N*-complexes.

The relative cohomology theory in various ablian categories was studied by many authors, for example [1,2,8,11,17]. Furthermore, the balancedness of the relative cohomology functors is one of the most important studies in the relative cohomology theory. Let R is an arbitrary ring. In 2004, Holm [8] defined relative cohomology groups in module catgeory and showed that if R-module M, with a proper Gorenstein projec- tive resolution, has finite Gorenstein projective dimen- sion and R-module N, with a proper Gorenstein injecti- ve resolution, has finite Gorenstein injective dimension, then there exist isomorphisms $Ext^n_{GP}(M,N)=Ext^n_{GI}(M,N)$ for $n\in Z$ (see [8], theorem 3.6). In complex category, Liu [11] defined relative cohomology groups and the balancedness was already considerd by showed that if X is a complexes with finite Gorenstein projective dimension and Y is a complexes with finite Gorenstein injective dimension, then for each $n\in Z$ there exists an isomorphism (*): $Ext^n_{GP}(X,Y)=Ext^n_{GI}(X,Y)$, where X has a strict Gorenstein projective precover and Y has a strict Gorenstein injective preenvelope. Recently, Lu and Di [13] extended the (*) to the category of *N*-complexes.

Cotorsion pairs (or cotorsion theories) were inv-Ented by Salace [16] and its importance in homological algebra has been showed by its use in the proof of the flat cover conjecture in [3]. Liu and Xie [12] considered relative cohomology theories based on cotorsion pairs in the setting of unbounded complexes of modules over an associative ring R. In [12], suppose (A,L) and (L,B) be two complete hereditary cotorsion pairs in the R-Mod, then by [6] and [18], $(dg\widetilde{A},\widetilde{L})$ and $(\widetilde{L},dg\widetilde{B})$ be two complete hereditary cotosion pairs in the category of R-complexes. For any complex X, Y and any $n\in Z$, Liu and Xie defined the nth relative cohomology groups $Ext^n_{dg\widetilde{A}}(X,Y)$ as $H^nHom(A,Y)$ and $Ext^n_{dg\widetilde{B}}(X,Y)$ as $H^nHom(X,B)$, where $A\to X$ is a special $dg\widetilde{A}$ -precover of X and $Y\to B$ is a

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special $dg\widetilde{B}$ -preenvelope of Y. Of course, Liu and Xie also discussed the balancedness that there are an isomophism $Ext^n_{dg\widetilde{A}}(X,Y) = Ext^n_{dg\widetilde{B}}(X,Y)$.

Motrivated by the above works of authors, we define in this paper relative cohomology groups based on cotorsion pair (A, B) in the category of *N*-complexes on Φ. Since Yang and Cao [19], for a cotorsion pair (χ, γ) in Φ, defined cotorsion pairs $(\widetilde{\chi}_N, dg\widetilde{\gamma}_N)$ and $(dg\widetilde{\chi}_N, \widetilde{\gamma}_N)$ in the ctegory of *N*-complexes on Φ. So we suppose (A, L) be a complete hereditary cotorsion pair in Φ. Then, by Yang and Cao [19], $(dg\widetilde{A}_N, \widetilde{L}_N)$ is a complete here- ditary cotorsion pair in the category of *N*-complexes on Φ. For any *N*-complexes X and Y any $n \in \mathrm{Z}$, we define the *n*th relative cohomology group $Ext^n_{dg\widetilde{A}_N}(X,Y)$ as $H^t_n Hom_{\Phi}(A, Y)$, where $A \to Y$ is a special $dg\widetilde{A}_N$ -precover of X. If (L, B) is a complete hereditary cotorsion piar in Φ, then the *n*th relative cohomology group $Ext^n_{dg\widetilde{B}_N}(X,Y)$ is defined by a dual argument. For relative cohomology groups we show that there exists an isomorphism $Ext^n_{dg\widetilde{A}_N}(X,Y) = Ext^n_{dg\widetilde{B}_N}(X,Y)$.

II. PRELIMINARIES

By an *N*-comlex $X (N \geq 2)$ we mean a sequence of objiects in Φ

$$\cdots \xrightarrow{d} X_{n+1} \xrightarrow{d} X_n \xrightarrow{d} X_{N-1} \xrightarrow{d} \cdots$$

satisfying $d^N = 0$, that is, composing any *N*-consecative morphisms gives 0. So a 2-complex is a clain complex in the usual sense. Let C nd D be *N*-complexes. We denote by $Hom_{\Phi}(C, D)$ the sequence of abelian groups with $Hom_{\Phi}(C,D)_n = \prod_{t \in Z} Hom_{\Phi}(C_t, D_{n+t})$ and such that if $f \in Hom_{\Phi}(C,D)_n$ then $(d_n(f))_m = d_D f_m - (q)^n f_{m+1} d_C$, where q is a *N*th root of unity, $q^N = 1$ and $q \neq 1$. Then $Hom_{\Phi}(C, D)$ is an *N*-complex of abelian groups. f is called a chain map of degree n if $d_n(f) = 0$. A chain map of degr-ee 0 is called a morphism. We use $Hom_{C_N(\Phi)}(C, D)$ to denote the abelian group of morphisms from C to D and $Ext^i_{C_N(\Phi)}(C, D)$ for $i \geq 0$ to denote the groups we get from the right derived functor of Hom (see [10]).

In this way, we get a category of *N*-complexes, deno-ted by $C_N(\Phi)$, whose objects are *N*-complexes and whose morphisms are chain maps of degree 0. It is an abelian category. For an *N*-complexes X, there are *N*-1 choices for homology. For $t = 1, \cdots, N$, we define $Z^t_n(X) = Ker(d_{n-(t-1)} \cdots d_{n-1} d_n)$ and $B^t_n(X) = \mathrm{Im}(d_{n+1} d_{n+2} \cdots d_{n+t})$. In particular, we have $Z^1_n(X) = Ker d_n$, $Z^N_n(X) = X_n$, $B^1_n(X) = \mathrm{Im}\, d_{n+1}$ and $B^N_n(X) = 0$. Set $C^t_n(X) = X_n / B^t_n(X)$. Define $H^t_n(X) = Z^t_n(X)/B^{N-t}_n(X)$ the amplitude homology objects of X for all t. We say that X is *N*-exact, or just exact, if $H^t_n(X) = 0$ for al integers n and t. Two morphisms $f, g : X \to Y$ of N -complexes are called homotopic if there exists a collection of morphis-ms $\{s_n : X_n \to Y_{n+N-1}\}$ such that

$$g_n - f_n = d^{N-1} s_n + d^{N-2} s_{n-1} d + \cdots + s_{n-(N-1)} d^{N-1}$$

$$= \sum_{i=0}^{N-1} d^{(N-1)-i} s_{n-i} d^i, \qquad \forall \quad n.$$

If f and g are homotopic, then we write $f \sim g$. We call a chain map f null homotopic if $f \sim 0$.

If Φ is an abelian category and ***F*** a class of objects of Φ, then an ***F***-precover of an object X of Φ is a morphism $\phi : F \to X$, where $F \in$ ***F*** and such that $Hom(G, F) \to Hom(G, X)$ is surjective for all $G \in$ ***F***. The dual notions are that of an ***F***-preenvelope. An ***F***-precover $\phi : F \to X$ is said to be a special ***F***-precover if $Ext^1(G, Ker\phi) = 0$ for all $G \in$ ***F***. An

F-preenvelope $\varphi : X \to F$ is special if $Ext^1(Co\ker\varphi, G) = 0$ for all $G \in \mathcal{F}$. Let H a class of objects in Φ. We will denote the class of objects X satisfying $Ext^1(H', X) = 0$ respectively, $Ext^1(X, H') = 0$ for every $H' \in H$ by $H^{\perp}$ Respectively, ${}^{\perp}H$. A pair of classes of objects (A, B) is said to be a cotorsion pair if $A^{\perp} = B$ and ${}^{\perp}B = A$. A cotorsion pair (A, B) is said to be complete if for any X there are exact sequence

$0 \to X \to B_1 \to A_1 \to 0, 0 \to B_2 \to A_2 \to X \to 0$

with $A_1, A_2 \in A$ and $B_1, B_2 \in B$. A cotorsion pair (A, B) is said to be hereditary if whenever

$0 \to A_2 \to A_1 \to A_3 \to 0$ is exact with $A_1, A_3 \in A$ then A_2 is also in A, or equivalently, if

$0 \to B_2 \to B_1 \to B_3 \to 0$ is exact with $B_1, B_2 \in B$ then B_3 is also in B.

III. Relative Cohomology Group

The following definition appeared in [19, Definition 3.1].

Definition 3.1 Let (A, B) be a cotorsion pair in Φ and X be an objective $C_N(\Phi)$.

(1) X is called an A *N*-complex if it is *N*-exact and $Z_n^t(X) \in A$ for all n and $t = 1, \cdots, N-1$.

(2) X is called a B *N*-complex if it is *N*-exact and for all n and $t = 1, \cdots, N-1$.

(3) X is called a dgA *N*-complex if $X_n \in A$ for all n and every chain map $f : X \to Y$ is null homotopic where Y is a B *N*-complex.

(4) X is called a dgB *N*-complex if $X_n \in B$ for all n and every chain map $f : Y \to X$ is null homotopic where X is an A *N*-complex.

We denote the class of A *N*-complex by $\widetilde{A}_N$ and the class of dgA *N*-complex by $dg\widetilde{A}_N$. Similarly, the class of B *N*-complex by $\widetilde{B}_N$ and the class of dgB *N*-complex by $dg\widetilde{B}_N$. If follows from [19, Theorem 3.7] that if (A, B) is a complete hereditary cotorsion pair in Φ, then $(dg\widetilde{A}_N, \widetilde{B}_N)$ and $(\widetilde{A}_N, dg\widetilde{B}_N)$ are complete hereditary cotorsion pair in $C_N(\Phi)$.

In the following (A, L) will always stand for a complete hereditary cotorsion pair in Φ. Then $(dg\widetilde{A}_N, \widetilde{L}_N)$ is a complete hereditary cotorsion pairs in $C_N(\Phi)$. For any *N*-complex X there is an exact sequence of *N*-complexes

$$0 \to L \to A \to X \to 0$$

with $A \in dg\widetilde{A}_N$ and $L \in \widetilde{L}_N$. It is easy to see that $A \to X$ is a special $dg\widetilde{A}_N$-precover of X.

Lemma 3.2 Let (A, L) be a complete hereditary cotorsion pair in Φ, and X and X' be an *N*-complexes. Then, for each morphism $\alpha : X \to X'$ there exists a unique, up to homotopy, morphism $\beta : A \to A'$ such that the diagram

$$\begin{array}{ccccccc} 0 \to L \to & A & \xrightarrow{\phi} & X \to 0 \\ & \beta\downarrow & & \alpha\downarrow \\ 0 \to L' \to & A' & \xrightarrow{\phi'} & X' \to 0 \end{array}$$

commutes, where $A, A' \in dg\widetilde{A}_N$ and $L, L' \in \widetilde{L}_N$ are such that the rows are exact.

Proof. Clearly there exists a morphism $\beta : A \to A'$ such that the diagram commutes since $A' \to X'$ is a $dg\widetilde{A}_N$-precover of X'. Suppose that $\beta' : A \to A'$ is another one such that the diagram commutes. Then $\phi'(\beta - \beta') = 0$ and so $\beta - \beta' : A \to L'$. By the completeness of cotorsion pair $(dg\widetilde{A}_N, \widetilde{L}_N)$, there is an exact sequence of *N*-complexes

$$0 \to A \xrightarrow{\tau} L_1 \to A_1 \to 0 \qquad (**)$$

with $A_1 \in dg\widetilde{A}_N$ and $L_1 \in \widetilde{L}_N$. Note that $Ext^1(A_1, L') = 0$ and, so the sequence

$$Hom_{\Phi}(L_1, L') \to Hom_{\Phi}(A, L') \to 0$$

is exact. Hence there exists a morphism $\gamma : L_1 \to L'$ such that $\beta - \beta' = \gamma\tau$. Form (**) it follows $L_1 \in dg\widetilde{A}_N$ since A and A' belong to $dg\widetilde{A}_N$. Note that $L' \in \widetilde{L}_N$. Thus the complex $Hom_\Phi(L_1, L')$ is exact, which implies that γ is homotopic to zero and therefore $\beta - \beta'$ is homotopic to zero.

According to lemma 3.2 we have the following definition.

Definition 3.3 Let X and Y be *N*-complexes. For each $n \in Z$, the *n*th relative cohomology group $Ext^n_{dg\widetilde{A}_N}(X,Y)$, based on a cotorsion pair (A,B), is defined by the equality

$$Ext^n_{dg\widetilde{A}_N}(X,Y) = H^t_n Hom_\Phi(A,Y),$$

where $A \to X$ is a special $dg\widetilde{A}_N$-precover of X and $t = 1,\cdots,N-1$.

Proposition 3.4 For each $n \in Z$, the assignment $(X,Y) \mapsto Ext^n_{dg\widetilde{A}_N}(X,Y)$ defines a functor

$$C_N(\Phi) \times C_N(\Phi) \to Z - Mod$$

Proof It follows from lemma 3.2

Dual argument 3.5 Let (L,B) be a complete hereditary cotorsion pair in Φ. Then $(\widetilde{L}_N, dg\widetilde{B}_N)$ is a complete hereditary cotorsion pair in $C_N(\Phi)$. For any *N*-complex Y there is an exact sequence of *N*-complex

$$0 \to Y \to B \to H \to 0$$

With $B \in dg\widetilde{B}_N$ and $H \in \widetilde{L}_N$. It is easy to see that $Y \to B$ is a special $dg\widetilde{B}_N$-preenvelope of Y.For any *N*-complex and any $n \in Z, t = 1,\cdots,N-1.$ the *n*th relative cohomology group $Ext^n_{dg\widetilde{B}_N}(X,Y)$ is defined by the equality

$$Ext^n_{dg\widetilde{B}_N}(X,Y) = H^t_n Hom_\Phi(X,B).$$

By the following lemma, the *n*th relative cohomology group is well defined.

Lemma 3.6 Let (L,B) be a complete hereditary cotorsion pair in Φ, Y and Y' and be an *N*-complexes. Then, for each morphism $\alpha : Y \to Y'$, there exists a unique, up to homotopy, morphism $\beta : B \to B'$ such that the diagram

$$\begin{array}{ccccccc} 0 \to & Y & \longrightarrow & B \to & H \to 0 \\ & \alpha\downarrow & & \beta\downarrow & \\ 0 \to & Y' & \longrightarrow & B' \to & H' \to 0 \end{array}$$

commutes, where $B, B' \in dg\widetilde{B}_N$ and $H, H' \in \widetilde{L}_N$ are such that the rows are exact.

IV. THE BALANCE RESULT

Now let (A,L) and (L,B) are two complete hereditary cotorsion pairs in Φ.It would be intersting to know if one has balancedness in relative cohomolgy. The aim of this section is to show that the relative cohomology functor we discussed is balanced.

The following definition from [20, Defintion 3.1]

Definition 4.1 Let X and Y be *N*-complexes. A morphism $f : X \to Y$ induces homomorphisms $H^t_n(f) : H^t_n(X) \to H^t_n(Y)$ for all $n \in Z$ and $t = 1,\cdots,N-1$. We say that f is a quasi-isomorphism if each $H^t_n(f)$ is an isomorphism.

Lemma 4.2 Let X and Y be *N*-complexes. Then

$$H^1_n Hom_\Phi(X,Y) \cong Hom_{K_N(\Phi)}(X, \theta^{-n}Y).$$

In particular, $Hom_\Phi(X,Y)$ is exact if and only if $Hom_{K_N(\Phi)}(X,Y)$, i.e. $H^t_n Hom_\Phi(X,Y) = 0$ is equivalent to $Hom_{K_N(\Phi)}(X, \theta^{-n}Y) = 0$ for $t = 1,\cdots,N-1.$

Remark The above lemma 4.2 from [13, lemma 4.4], where the $K_N(\Phi)$ denote the homotopy category of *N*-complexes, and the θ is a shift functor (see [9]).

Theorem 4.3 Let (A,L) and (L,B) be complete reditary cotorsion pairs in Φ. Then, for any N-complex X and Y and for all $n \in Z$, $t = 1,\cdots,N-1.$ there exists an isomophism

$$Ext^n_{dg\widetilde{A}_N}(X,Y) \cong Ext^n_{dg\widetilde{B}_N}(X,Y).$$

Proof Since (X,L) is a complete hereditary cotorson pair in Φ, we know $\left(dg\widetilde{A}_N, \widetilde{L}_N\right)$ is a complete hereditary cotorsion pair in $C_N(\Phi)$ .Thus there exists an exact sequence $0 \to L \to A \to X \to 0$ with $A \in dg\widetilde{A}_N$ and $L \in \widetilde{L}_N$. Similarly since (L, B) is a complete hereditary cotosion pair in Φ, we know $\left(\widetilde{L}_N, dg\widetilde{B}_N\right)$ is a complete hereditary cotorsion pair in $C_N(\Phi)$. Thus there exists an exact sequence

$$0 \to Y \to B \xrightarrow{\tau} H \to 0$$

With $B \in dg\widetilde{B}_N$ and $H \in \widetilde{L}_N$. Next we show that $H_n^t Hom_\Phi(A,Y) \cong H_n^t Hom_\Phi(X,B)$, it is just to prove that $H_n^1 Hom_\Phi(A,Y) \cong H_n^1 Hom_\Phi(X,B)$ by lemma 4.2. Note that for each $m \in Z$, we have

$$Hom_\Phi(A,\tau)_m = \prod_{i\in Z} Hom_\Phi(A_i, \tau_{i+m}).$$

We clain that

$$Hom_\Phi(A_i, \tau_{i+m}) : Hom_\Phi(A_i, B_{i+m}) \to Hom_\Phi(A_i, H_{i+m})$$

is an epimorphism for any $m \in Z$. In fact, suppose that $f \in Hom_\Phi(A_i, H_{i+m})$. Consider the exact sequence $0 \to K \to P \to H_{i+m} \to 0$, since (L,B) is a complete hereditary cotorsion pair, it follows that $K \in L$, which implie $Ext_\Phi^1(A_i, K) = 0$. Thus there exists $g : A_i \to P$ such that the following diagram commutes

$$\begin{array}{ccccccc} & & A_i \to & & A_i & & \\ & & \downarrow g & & \downarrow f & & \\ 0 \to & K \to & P \to & & H_{i+m} & \to 0 & \\ & & \downarrow h & & \downarrow = & & \\ 0 \to & Y_{i+m} \to & B_{i+m} \to & & H_{i+m} & \to 0 & \end{array}$$

This means that

$$Hom_\Phi(A_i, \tau_{i+m}) : Hom_\Phi(A_i, B_{i+m}) \to Hom_\Phi(A_i, H_{i+m})$$

is an epimorphism for any $m \in Z$. Hence, the homorphism of complexes

$$Hom_\Phi(A,\tau)_m : \prod_{i\in Z} Hom_\Phi(A_i, B_{i+m}) \to \prod_{i\in Z} Hom_\Phi(A_i, H_{i+m})$$

is an epimorphism for any $m \in Z$. Thus the sequence of N-complexes

$$0 \to Hom(A,Y) \to Hom(A,B) \to Hom(A,H) \to 0$$

is exact.

Note that $A \in dg\widetilde{A}_N$ and $H \in \widetilde{L}_N$. Thus $Hom_\Phi(A,H)$ is an exact N-complex. Hence the morphism $Hom_\Phi(A,Y) \to Hom_\Phi(A,B)$ is a qusi-isomorphism. Similar discussion will yield that the morphism $Hom_\Phi(X,B) \to Hom_\Phi(A,B)$ is quasi-siomorphism. Thus the desired conclusio $H_n^t Hom_\Phi(A,Y) \cong H_n^t Hom_\Phi(X,B)$ follows.

Corollary 4.4 Let (A,L) and (L,B) be complete hereditary cotorsion pairs in Φ and X and Y be N-complexes. If A is a special $dg\widetilde{A}_N$-precover of X and B is a special $dg\widetilde{B}_N$-preenvelope of Y then, for any $n \in Z$ and $t = 1, \cdots, N-1$, there exists an isomorphism $Ext_{dg\widetilde{A}_N}^n(X,Y) \cong H_n^t Hom_\Phi(A,B)$.

Let (A,L) and (L,B) be complete hereditary cotorsion pairs in Φ. Then (A,L,B) is called a cotorsion triple. This notion was introduced recently by Chen [4]. The cotorsion triple (A,L,B) is said to be hereditary (resp. complete) provided that both of the two cotorsion pairs (A,L) and (L,B) are hereditary (resp. complete). For any N-comlexes X and Y and for any $n \in Z$, we define the nth relative cohomology group based on a complete hereditary cotorsion triple (A,L,B) as

$$Ext_{(A,L,B)}^n(X,Y) = Ext_{dg\widetilde{A}_N}^n(X,Y) = Ext_{dg\widetilde{B}_N}^n(X,Y).$$

By lemma 4.3, the nth relative cohomology group $Ext_{(A,L,B)}^n(X,Y)$ can be computed by a special $dg\widetilde{A}_N$-precover of the first variable $dg\widetilde{B}_N$-preenvelope of the second variable.

V. References

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