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Bochner integral for $\mathcal{S}^*(M)$-valued function

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Abstract— **Let $\mathcal{S}^*(M)$ be the generalized functional space of a discrete-time normal martingale M. In this paper, we define a Bochner integration for $\mathcal{S}^*(M)$-valued functions. What's more, we examine fundamental properties of this integration.**

Index Terms— **Fock transform; Discrete-time normal martingale; Bochner integral.**

I. INTRODUCTION

Bochner integral, which according to the definition of Lebesgue integral, is a commonly used integral of vector-valued function. In 1932, S.Bochner established Bochner integral, which is a direct generalization of Lebesgue integral in the case of vector-valued function. Bochner integral is widely used in many branches of mathematics, such as probability theory, operator theory, stochastic process, Banach space geometry theory and vector-valued measure theory. In 2015, Wang et al. considered a class of discrete-time normal martingale and assumed that it has chaotic representation. The author constructed the testing functional space and generalized functional space on M in discrete-time normal martingale by the orthonormal basis of

the square integrable functional space $\mathcal{L}^2(M)$ of M. if $\mathcal{L}^2(M)$ is equated with its dual space, we can obtain a Gel'fand triple $\mathcal{S}(M) \subset \mathcal{L}^2(M) \subset \mathcal{S}^*(M)$

It is worth noting that the generalized functional space $\mathcal{S}^*(M)$ has many good properties, while $\mathcal{L}^2(M)$ doesn't have. Therefore, it is very meaningful to discuss the properties of generalized functional space $\mathcal{S}^*(M)$ in discrete-time normal martingale.

II. PRELIMINARIES

In this section, we briefly recall some notions and results for discrete-time normal martingale.

Let N be the set of all nonnegative integers and Γ the finite power set of N, namely,

$$\Gamma = \{\sigma \mid \sigma \subset N \text{ and } \#\sigma < \infty\},$$

where $\#\sigma$ denotes the cardinality of σ as a set. It is not hard to check that Γ is countable as an infinite set. Additionally, we assume that $(\Omega, \mathcal{F}, P)$ is a given probability space with $\mathbb{E}$ denoting the expectation with respect to P. We denote by $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ the usual Hilbert space of square integrable $\|\cdot\|$ to means its inner product and norm, respectively. By convention, $\langle\cdot,\cdot\rangle$ is conjugate-linear in its first argument and linear in its second argument.

Definition 2.1.[1] A (real-valued) stochastic process $M = (M_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, P)$ is called a discrete-time normal martingale if it is square integrable and satisfies

(i) $\mathbb{E}[M_0 \mid \mathcal{F}_{-1}] = 0$ and $\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] = M_{n-1}$ for $n \geq 1$,

(ii) $\mathbb{E}[M_0^2 \mid \mathcal{F}_{-1}] = 1$ and $\mathbb{E}[M_n^2 \mid \mathcal{F}_{n-1}] = M_{n-1}^2 + 1$ for $n \geq 1$.

where $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(M_k; 0 \leq k \leq n)$ for $n \in \mathbb{N}$.

Now let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale on $(\Omega, \mathcal{F}, P)$. We given some necessary notions concerning M. First we construct from M a process $Z = (Z_n)_{n\in\mathbb{N}}$ as

$$Z_0 = M_0,\ Z_0 = M_n - M_{n-1},\ n \geq 1. \qquad (1)$$

It can be verified that Z admits the following properties:

$$\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] = 0, \mathbb{E}[Z_n^2 \mid \mathcal{F}_{n-1}] = 1, n \in \mathbb{N}.$$

Thus, it can be viewed as a discrete-time noise (see[1]).

Definition 2.2.[1] The process Z defined by (1) is called the discrete-time normal noise associated with M.

Lemma 2.1.[2] Let $Z = (Z_n)_{n\in\mathbb{N}}$ be the discrete-time normal noise associated with M. Define $Z_\emptyset = 1$, where $\emptyset$ denotes the empty set, and

$$Z_\sigma = \prod_{i\in\sigma} Z_i,\ \sigma \in \Gamma,\ \sigma \neq \emptyset. \qquad (2)$$

Then $\{Z_\sigma \mid \sigma \in \Gamma\}$ form a countable orthonormal system in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$.

Let $\mathcal{F}_\infty = \sigma(M_n;\ n \in \mathbb{N})$, the σ-field over Ω generated by M. In the literature, $\mathcal{F}_\infty$-measurable functions on Ω are also known as functional of M. Thus elements of $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$ can be called square integrable functional of M.

Definition 2.3.[2] The discrete-time normal martingale M is said to have the chaotic representation property if the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ defined by (2) is total in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$.

Lemma 2.2.[5] Let $\sigma \to \lambda_\sigma$ be the $\mathbb{N}$-valued function on Γ given by

$$\lambda_\sigma = \begin{cases} \prod_{k\in\sigma}(k+1), & \sigma \neq \emptyset,\ \sigma \in \Gamma; \\ 1, & \sigma = \emptyset,\ \sigma \in \Gamma. \end{cases} \qquad (3)$$

Then, for $p > 1$, the positive term series $\sum_{\sigma\in\Gamma} \lambda_\sigma^{-p}$ converges and, moreover,

$$\sum_{\sigma\in\Gamma} \lambda_\sigma^{-p} \leq \exp[\textstyle\sum_{k=1}^{\infty} k^{-p}] < \infty.$$

Using the $\mathbb{N}$-valued function defined by (3), we can construct a chain of Hilbert spaces consisting of functionals of M as follows. For $p \geq 0$, we define a norm $\|\cdot\|_p$ on $\mathcal{L}^2(M)$ through

$$\|\xi\|_p^2 = \sum_{\sigma\in\Gamma} \lambda_\sigma^{2p} |\langle Z_\sigma, \xi\rangle|^2,\ \xi \in \mathcal{L}^2(M)$$

and put

$$\mathcal{S}_p(M) = \{\xi \in \mathcal{L}^2(M) \mid \|\xi\|_p < \infty\}$$

Lemma 2.3.[3] For each $p \geq 0$, one has $\{Z_\sigma \mid \sigma \in \Gamma\} \subset \mathcal{S}_p(M)$ and moreover the system $\{\lambda_\sigma^{-p} Z_\sigma \mid \sigma \in \Gamma\}$ forms an orthonormal basis for $\mathcal{S}_p(M)$.

It is easy to see that $\lambda_\sigma \geq 1$, for all $\sigma \in \Gamma$. This implies that $\|\cdot\|_p \leq \|\cdot\|_q$ and $\mathcal{S}_q(M) \subset \mathcal{S}_p(M)$ whenever $0 \leq p \leq q$. Thus, we actually get a chain of Hilbert spaces of functional of M:

$$\cdots \subset \mathcal{S}_{p+1}(M) \subset \mathcal{S}_p(M) \subset \cdots \subset \mathcal{S}_1(M) \subset \mathcal{S}_0(M) = \mathcal{L}^2(M).$$

We now put

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$$\mathcal{S}(\mathrm{M}) = \cap_{p=0}^{\infty} \mathcal{S}_p(\mathrm{M})$$

and endow it with the topology by the norm sequence $\{\|\cdot\|_p\}_{p\geq 0}$.

Lemma 2.4.[3] The space $\mathcal{S}(\mathrm{M})$ is a nuclear space; namely, for any $p \geq 0$, there exists $q > p$ such that the inclusion mapping $i_{pq} : \mathcal{S}_q(\mathrm{M}) \to \mathcal{S}_p(\mathrm{M})$ defined by $i_{pq}(\xi) = \xi$ is a Hilbert-Schmidt operator.

Lemma 2.5.[3] Let $\mathcal{S}^*(\mathrm{M})$ be the dual of $\mathcal{S}(\mathrm{M})$ and endow it with the strong topology. Then,

$$\mathcal{S}^*(\mathrm{M}) = \cup_{p=0}^{\infty} \mathcal{S}_p^*(\mathrm{M})$$

and moreover the inductive limit topology on $\mathcal{S}^*(\mathrm{M})$ given by space sequence $\{\mathcal{S}_p^*(\mathrm{M})\}_{p\geq 0}$ coincides with the strong topology.We mention that, by identifying $\mathcal{L}^2(\mathrm{M})$ with its dual, one comes to a Gel'fand triple:

$$\mathcal{S}(\mathrm{M}) \subset \mathcal{L}^2(\mathrm{M}) \subset \mathcal{S}^*(\mathrm{M})$$

which we refer to as the Gel'fand triple associated with M.

Lemma 2.6.[3] The system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is contained in $\mathcal{S}(\mathrm{M})$ and moreover it serves as a basis in $\mathcal{S}(\mathrm{M})$ in the sense that

$$\xi = \sum_{\sigma\in\Gamma} \langle Z_\sigma, \xi\rangle Z_\sigma, \quad \xi \in \mathcal{S}(\mathrm{M})$$

where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathcal{L}^2(\mathrm{M})$ and the series converges in the topology of $\mathcal{S}(\mathrm{M})$.

Definition 2.4.[3] Elements of $\mathcal{S}^*(\mathrm{M})$ are called generalized functional of M, while elements of $\mathcal{S}(\mathrm{M})$ are called testing functional of M.

Denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the canonical bilinear form on $\mathcal{S}^*(\mathrm{M}) \times \mathcal{S}(\mathrm{M})$; namely,

$$\langle\langle \Phi, \xi\rangle\rangle = \Phi(\xi), \qquad \Phi \in \mathcal{S}^*(\mathrm{M}), \xi \in \mathcal{S}(\mathrm{M})$$

where $\Phi(\xi)$ means Φ acting on ξ as usual. Note that $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathcal{L}^2(\mathrm{M})$, which is different from $\langle\langle\cdot,\cdot\rangle\rangle$.

Definition 2.5.[3] For $\Phi \in \mathcal{S}^*(\mathrm{M})$, its Fock transform is the function $\hat{\Phi}$ on Γ given by

$$\hat{\Phi}(\sigma) = \langle\langle \Phi, Z_\sigma\rangle\rangle, \quad \sigma \in \Gamma, \qquad (4)$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is the canonical bilinear form.

Lemma 2.7.[3] Let $\Phi \in \mathcal{S}_p^*(\mathrm{M})$, where $p \geq 0$. Then the norm of Φ in $\mathcal{S}_p^*(\mathrm{M})$ satisfies

$$\|\Phi\|_{-p}^2 = \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2p} |\hat{\Phi}(\sigma)|^2.$$

III. MAIN RESULTS

In this section, we assume that (E, ϵ, v) is a given finite measure space and $\mathcal{S}^*(\mathrm{M})$ is the generalized functional space of a discrete-time normal martingale M. we mainly give the definition of Bochner integral of $\mathcal{S}^*(\mathrm{M})$-valued functions and examine fundamental properties of this integration.

Definition 3.1. The mapping $\Phi : E \to \mathcal{S}^*(\mathrm{M})$ is called a simple mapping, if Φ can be expressed in the following form

$$\Phi = \sum_{i=1}^{n} \Phi_i \mathbb{I}_{A_i}$$

where $\Phi_i \in \mathcal{S}^*(\mathrm{M})$, A_i, i = 1, 2, ⋯ ,n, $n \geq 1$,is a measurable subset of pairwise disjoint in ϵ, and $\cup_{i=1}^{n} A_i = E$.

Definition 3.2. For simple mapping $\Phi = \sum_{i=1}^{n} \Phi_i \mathbb{I}_{A_i}$, we can define Bochner integral

$$\int_A \Phi(e)\,dv = \sum_{i=1}^{n} \Phi_i(e) v(A_i)$$

By Definition 3.2, for $q > p + \frac{1}{2}$, we have

$$\begin{aligned}
&\left\|\int_A \Phi(e)\,dv\right\|_{-q}^2 \\
&= \left\|\sum_{i=1}^{n} \Phi_i(e) v(A_i)\right\|_{-q}^2 \\
&\leq \sum_{i=1}^{n} \|\Phi_i(e) v(A_i)\|_{-q}^2 \\
&= \sum_{i=1}^{n} \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2q} \left|\widehat{\Phi_i(e) v(A_i)}(\sigma)\right|^2 \\
&\leq \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)} \sum_{i=1}^{n} \|\Phi_i(e)\|_{-p}^2 v(A_i) \\
&= \sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)} \int_A \|\Phi(e)\|_{-p}^2\,dv,
\end{aligned}$$

namely,

$$\left\|\int_A \Phi(e)\,dv\right\|_{-q} \leq C \int_A \|\Phi(e)\|_{-p}\,dv. \qquad (5)$$

where $C = \left\{\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)}\right\}^{\frac{1}{2}}$.

Definition 3.3. A mapping $\Phi(e)$ is called strong measurable. If $\exists p > 0$, and a sequence of simple mapping $\Phi_n(e)$, $n \geq 1$ such that

$$\|\Phi(e) - \Phi_n(e)\|_{-p} \to 0, \ (n \to \infty), \ v\text{-a.s.} \qquad (6)$$

Definition 3.4. Strong measurable mapping $\Phi : E \to \mathcal{S}^*(\mathrm{M})$ is called Bochner integral. If $\exists p > 0$, and a sequence of simple mapping $\Phi_n(e)$, $n \geq 1$ such that

$$\lim_n \int_E \|\Phi_n(e) - \Phi(e)\|_{-p}\,dv = 0. \qquad (7)$$

In this case, $\int_E \Phi_n(e)\,dv$ strongly converges to an unique element in $\mathcal{S}_p^*(\mathrm{M})$, which denoted by $\int_E \Phi(e)\,dv$, and be called the Bochner integral of $\Phi(e)$.

Next we need to add to the reasonableness of the above definition.

First of all, according to equation (6), we know that the integral in equation (7) can be defined.

Secondly, we prove that $\{\int_A \Phi_n(e)\,dv\}$ is a Cauchy sequence. In fact, by equation (7), for $q > p + \frac{1}{2}$, we have

$$\begin{aligned}
\left\|\int_A \Phi_n(e)dv - \int_A \Phi_m(e)dv\right\|_{-q} &= \left\|\int_A (\Phi_n(e) - \Phi_m(e))dv\right\|_{-q} \leq \int_A \|\Phi_n(e) - \Phi_m(e)\|_{-p}dv \\
&\leq \int_E \|\Phi_n(e) - \Phi(e)\|_{-p}dv + \int_E \|\Phi(e) - \Phi_m(e)\|_{-p}dv \to 0
\end{aligned}$$

Thus, there exists an unique element in $\mathcal{S}_q^*(\mathrm{M})$, which denoted by $\int_A \Phi(e)\,dv$, such that

$$\int_A \Phi_n(e)\,dv \xrightarrow{\mathcal{S}_q^*(\mathrm{M})} \int_A \Phi(e)\,dv.$$

Finally, it is easy to know that $\int_A \Phi(e)\,dv$ is independent of the selection of the sequence of simple mapping $\Phi_n(e)$, $n \geq 1$ satisfied equation (7).

Measurable numerical functions have an important conclusion in Lebesgue integral theory, that is, integrable and absolutely integrable are equivalent. Accordingly, for Bochner integral, the following theorem is also basic. So we can give the following characterization theorem for Bochner integrable mapping.

Theorem 3.1. Strong measurable mapping $\Phi(e)$ is Bochner integrable if and only if $\int_E \|\Phi(e)\|_{-p}\,dv < \infty$.

Proof. The "if" part. It can be seen by the definition of strong measurable mapping $\Phi(e)$ is Bochner integrable, there exists $p > 0$ and the sequence of simple mapping $\Phi_n(e)$, $n \geq 1$ such that $\|\Phi(e) - \Phi_m(e)\|_{-p} \leq \frac{1}{n}$ for each positive integer n.

Since $\|\Phi_n(e)\|_{-p} \leq \|\Phi(e)\|_{-p} + \frac{1}{n}$, v-a.e. and v is finite, then $\int_E \|\Phi_n(e)\|_{-p}\,dv < \infty$.

For each positive integer n, write

$$\Phi_n(e) = \sum_{m=1}^{\infty} \Phi_{n,m} \mathbb{I}_{A_{n,m}},$$

where $A_{n,m}$ is a measurable set of pairwise disjoint in ϵ and $\Phi_{n,m} \in \mathcal{S}_p^*(\mathrm{M})$.

For each n, choose p_n so large that

$$\int_{\cup_{m=p_n+1}^{\infty} A_{n,m}} \|\Phi_n(e)\|_{-p}\,dv < \frac{v(E)}{n},$$

and set $\Psi_n(e) = \sum_{m=1}^{p_n} \Psi_{n,m}(e) \mathbb{I}_{A_{n,m}}$, then $\Psi_n(e)$ is also a sequence of simple mapping and

$$\int_E \|\Phi(e) - \Psi_n(e)\|_{-p} dv$$
$$\leq \int_E \|\Phi(e) - \Phi_n(e)\|_{-p} dv$$
$$+\int_E \|\Phi_n(e) - \Psi_n(e)\|_{-p} dv$$
$$\leq \frac{v(E)}{n} + \frac{v(E)}{n} = \frac{2v(E)}{n}.$$

So $\Phi(e)$ is Bochner integrable. □

Next we can easily prove the properties of the Bochner integral.

Theorem 3.2. Let $\Phi(e)$, $\Psi(e)$ is Bochner integrable.

(1) For $\forall \alpha, \beta \in \mathbb{R}$, we have

$$\int_A [\alpha\Phi(e) + \beta\Psi(e)] dv$$
$$= \alpha\int_A \Phi(e) dv + \beta\int_A \Psi(e) dv.$$

(2) For $q > p + \frac{1}{2}$, such that

$$\left\|\int_A \Phi(e) dv\right\|_{-q} \leq C\int_A \|\Phi(e)\|_{-p} dv,$$

where C=$\left\{\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)}\right\}^{\frac{1}{2}}$.

(3) If $\Phi(e) \doteq \Psi(e)$, then $\int_A \Phi(e) dv = \int_A \Psi(e) dv$.

Proof. 1) Let $\Phi(e)$, $\Psi(e)$ is Bochner integrable, choose the disjoint sets $A_i \cap B_j \in \epsilon$, such that for $\forall \alpha, \beta \in \mathbb{R}$, we have

$$\int_A [\alpha\Phi(e) + \beta\Psi(e)] dv$$
$$= \sum_{i=1}^m \sum_{j=1}^n [\alpha\Phi_i(e) + \beta\Psi_j(e)] v(A_i \cap B_j)$$
$$= \sum_{i=1}^m \alpha\Phi_i(e) v(A_i \cap B_j) + \sum_{j=1}^n \beta\Psi_j(e) v(A_i \cap B_j)$$
$$= \alpha\sum_{i=1}^m \Phi_i(e) v(A_i \cap B_j) + \beta\sum_{j=1}^n \Psi_j(e) v(A_i \cap B_j)$$
$$= \alpha\int_A \Phi(e) dv + \beta\int_A \Psi(e) dv.$$

where $\sum_{i=1}^m v(A_i \cap B_j) = v(A_i)$, $\sum_{j=1}^n v(A_i \cap B_j) = v(B_j)$.

2) According to inequality (5), it can be proved.

3) Since $\Phi(e) \doteq \Psi(e)$, then $\Phi_i(e) = \Psi_i(e)$, i=1,2,⋯,n a.e. Then

$$\int_A \Phi(e) dv = \sum_{i=1}^n \Phi_i(e) v(A_i) = \sum_{i=1}^n \Psi_i(e) v(A_i) = \int_A \Psi(e) dv$$

. □

Theorem 3.3. Let $\Phi(e)$ is Bochner integrable with respect to v, then

1) $\lim_{v(A)\to 0} \int_A \Phi(e) dv = 0$.

2) If $\{A_n\}$ is a sequence of pairwise disjoint in ϵ and $A = \bigcup_{n=1}^\infty A_n$, then

$$\int_A \Phi(e) dv = \sum_{n=1}^\infty \int_{A_n} \Phi(e) dv$$

Proof. 1) Since $\lim_{v(A)\to 0} \int_A \|\Phi(e)\|_{-p} dv = 0$, then according to 2) in Theorem 3.2, it can be proved.

2) First of all, it is noted that series $\sum_{n=1}^\infty \int_{A_n} \Phi(e) dv$ is controlled term-by-term by the convergence series of nonnegative numbers $\sum_{n=1}^\infty \int_{A_n} \|\Phi(e)\|_{-p} dv$ ($\int_E \|\Phi(e)\|_{-p} dv$). Hence $\sum_{n=1}^\infty \int_{A_n} \Phi(e) dv$ is absolutely converges.

In order to verify its limit, according to the finite additivity of Bochner integral, we have

$$\left\|\int_{\bigcup_{n=1}^\infty A_n} \Phi(e) dv - \sum_{n=1}^m \int_{A_n} \Phi(e) dv\right\|_{-p}$$
$$= \left\|\int_{\bigcup_{n=m+1}^\infty A_n} \Phi(e) dv\right\|_{-p}$$

Meanwhile, $\lim_m v(\bigcup_{n=m+1}^\infty A_n) = 0$

From 1), we can get

$$\lim_m \left\|\int_{\bigcup_{n=m+1}^\infty A_n} \Phi(e) dv\right\|_{-p} = 0$$

And then $\int_{\bigcup_{n=1}^\infty A_n} \Phi(e) dv = \sum_{n=1}^\infty \int_{A_n} \Phi(e) dv$. □

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