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The γ-Type Probability Inequalities of Two-Parameter Conditional Demimartingale

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***Abstract*— To study the maximal inequalities for two-parameter conditional demimartingale, on this basis, futher get the γ -type probability inequalities of two-parameter conditional demimartingale.**

***Index Terms*— Maximal inequality; two-parameter conditional demi(sub)martingale; γ -type probability inequalities**

I. INTRODUCTION

Since Newman and Wright [1] introduced the concepts of demimartingale and demi(sub)martingale, many scholars have extended some results of (single parameter) demimartingale and demi(sub)martingale to the case of multi-parameter, and given some probability inequalities and related application results of multi-parameter demi(sub)martingale. For example, Christofides and Hadjikyriakou [2] gave the definitions of multi-parameter demimartingale and multi-parameter demi(sub)martingale, and extended the Chow type maximal inequality of (single parameter) demimartingale to the case of multi-parameter demimartingale, and obtained a Chow type maximal inequality of two-parameter demimartingale; Wang [3] gave the Hajek Renyi type inequality of multi-parameter associated random variable. In reference [4-8], also gave some probability inequalities for two-parameter demimartingales and their related applications.

Inspired by reference [2], this paper first gave the definition of two-parameter conditional demi(sub)martingale, and obtained a kind of maximal inequality of two-parameter conditional demi(sub)martingale. On this basis, we further obtained the γ type probability inequality of two-parameter conditional demimartingale.

Notation and conventions. Throughout this paper, Let $n,m\in N^2, n=(n_1,n_2), m=(m_1,m_2)$ .If $n_i\le m_i, i=1,2$,then $n\le m$ .In particular, if $n_i\le m_i, i=1,2$ at least one of them is strictly less than established, then $n\le m$. I_A represents an indicative function of set A. $\log^+ x=\ln\left(\max(x,1)\right)$

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II. DEFINITION OF TWO-PARAMETER CONDITIONAL DEMI(SUB)MARTINGALE

Definition 1 A collection of two-parameter random variables $\left\{S_n, n\in N^2\right\}$ is called a two-parameter conditional demimartingale if

$$E^{\mathrm{F}}\left\{\left(S_j-S_i\right)f\left(S_k,k\le i\right)\right\}\ge 0\quad a.s..$$

for all $i,j\in N^2$ with $i\le j$ and for any componentwise nondecreasing functions f ,the above conditional expectation meaningful. If, in addition f is required to be nonnegative, then the collection $\left\{S_n, n\in N^2\right\}$ is said to be two-parameter conditional demisubmartingale.

Lemma 1[15] Let $X(\cdot,\cdot,\cdot):\Omega\times R^2\to R$ be $\mathrm{A}\times\mathrm{B}^2$ -masurable and either nonnegative or $P\times\mathbf{u}\times u$ -integrable, where u is Lebesgue measure, and let F be a sub- σ -field of A .Then

$$E^{\mathrm{F}}\int_{R^2}X(\cdot,t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2=\int_{R^2}[E^{\mathrm{F}}X(\cdot,t_1,t_2)]\,\mathrm{d}t_1\mathrm{d}t_2\ a.s..$$

Lemma 2 Let $\left\{S_n, n\in N^2\right\}$ be a two-parameter conditional demi(sub)martingale, and let $g(\cdot)$ be a nondecreasing convex function, $g(S_n)\in L^1$. Then $\{g(S_n), n\in N^2\}$ is a two-parameter conditional demisubmartingale.

Proof. Since $g(x)$ is a nondecreasing convex function,let

$$h(y)=\lim_{x\to y^-}\frac{g(x)-g(y)}{x-y},$$

then $g(x)\ge g(y)+(x-y)h(x)$, and $h(x)$ is a nonnegative nondecreasing function. Let $f(x)$ is a componentwise nondecreasing functions, then

$$\begin{aligned}&E^{\mathrm{F}}\left[\left(g(S_j)-g(S_i)\right)f\left(g(S_k,k\le i)\right)\right]\\&\ge E^{\mathrm{F}}\left[\left(S_j-S_i\right)h(S_i)f\left(g(S_k,k\le i)\right)\right]\\&=E^{\mathrm{F}}\left[\left(S_j-S_i\right)f^*\left(S_k,k\le i\right)\right]a.s..\end{aligned}$$

Where $f^*(S_k,k\le i)=h(S_i)f\left(g(S_k,k\le i)\right)$, and f^* is a componentwise nonnegative nondecreasing function. Because of $\left\{S_n, n\in N^2\right\}$ is a two-parameter conditional demi(sub)martingale, then

$E^{\mathrm{F}}\left[\left(g\left(S_j\right)-g\left(S_i\right)\right) f\left(g\left(S_k, k \leq i\right)\right)\right] \geq 0$ *a.s.*. So $\{g\left(S_n\right), n \in N^2\}$ is a two-parameter conditional demisubmartingale.

III. MAIN RESULTS

Theorem 1 Let $\left\{S_n, n \in N^2\right\}$ be a two-parameter conditional demisubmartingale. Then for any F measurable random variables $\varepsilon \geq 0$ *a.s.*,

$$\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right) \leq \min\left(\begin{array}{l}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right], \\ \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[S_{(i,n_2)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right]\end{array}\right) \quad a.s..$$

Proof. Let

$$A=\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right),$$

$$B_{(1,j)}=\left[S_{(1,j)} \geq \varepsilon\right], 1 \leq j \leq n_2,$$

$$B_{(i,j)}=\left[S_{(r,j)}<\varepsilon, 1 \leq r<i, S_{(i,j)} \geq \varepsilon\right],$$

$$2 \leq i \leq n_1, 1 \leq j \leq n_2.$$

Then

$$\begin{aligned}
&\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right) \\
&\leq \varepsilon \sum_{j=1}^{n_2} \sum_{i=1}^{n_1} P^{\mathrm{F}}\left(B_{(i,j)}\right) \\
&=\sum_{j=1}^{n_2} \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[\varepsilon I_{B(i,j)}\right] \\
&\leq \sum_{j=1}^{n_2} \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B(i,j)}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(1,j)} I_{B_{(1,j)}}\right]+\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(2,j)} I_{B_{(2,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=3}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(1,j)} I_{B_{(1,j)}}\right]+\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(2,j)}\left(I_{B_{(1,j)} \cup B_{(2,j)}}-I_{B_{(1,j)}}\right)\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=3}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(2,j)} I_{B_{(1,j)} \cup B_{(2 j)}}\right]+\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[(S_{(1,j)}-S_{(2,j)}) I_{B_{(1,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=3}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(2,j)} I_{B_{(1,j)} \cup B_{(2 j)}}\right]-\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[(S_{(2,j)}-S_{(1,j)}) I_{B_{(1,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=3}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \quad a.s..
\end{aligned}$$

Since $E^{\mathrm{F}}\left[(S_{(2,j)}-S_{(1,j)}) I_{B_{(1,j)}}\right] \geq 0$ *a.s.*, so we have

$$\begin{aligned}
&\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right) \\
&\leq \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(2,j)} I_{B_{(1,j)} \cup B_{(2 j)}}\right]+\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(3,j)} I_{B_{(3,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(2,j)} I_{B_{(1,j)} \cup B_{(2,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(3,j)} I_{(B_{(1,j)} \cup B_{(2,j)} \cup B_{(3,j)})-(B_{(1,j)} \cup B_{(2,j)})}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(3,j)} I_{B_{(1,j)} \cup B_{(2,j)} \cup B_{(3,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[(S_{(2,j)}-S_{(3,j)}) I_{B_{(1,j)} \cup B_{(2,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \\
&=\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(3,j)} I_{B_{(1,j)} \cup B_{(2,j)} \cup B_{(3,j)}}\right] \\
&\quad-\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[(S_{(3,j)}-S_{(2,j)}) I_{B_{(1,j)} \cup B_{(2,j)}}\right] \\
&\quad+\sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right]
\end{aligned}$$

Additionally $E^{\mathrm{F}}\left[(S_{(3,j)}-S_{(2,j)}) I_{B_{(1,j)} \cup B_{(2,j)}}\right] \geq 0$ *a.s.*, so we have

$$\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right)$$

$$\leq \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(3,j)} I_{B_{(1,j)}\cup B_{(2,j)}\cup B_{(3,j)}}\right] + \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(4,j)} I_{B_{(4,j)}}\right]$$

$$+ \sum_{j=1}^{n_2}\sum_{i=5}^{n_1} E^{\mathrm{F}}\left[S_{(i,j)} I_{B_{(i,j)}}\right] \ a.s..$$

Repeat the above stepts, we get

$$\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right)$$

$$\leq \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1-1,j)} I_{B_{(1,j)}\cup B_{(2,j)}\cup\cdots\cup B_{(n_1-1,j)}}\right]$$

$$+ \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{B_{(n_1,j)}}\right]$$

$$= \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1-1,j)} I_{B_{(1,j)}\cup B_{(2,j)}\cup\cdots\cup B_{(n_1-1,j)}}\right]$$

$$+ \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{(B_{(1,j)}\cup\cdots\cup B_{(n_1,j)})-(B_{(1,j)}\cup\cdots\cup B_{(n_1-1,j)})}\right]$$

$$= \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right]$$

$$- \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[(S_{(n_1,j)} - S_{(n_1-1,j)}) I_{(B_{(1,j)}\cup\cdots\cup B_{(n_1,j)})-(B_{(1,j)}\cup\cdots\cup B_{(n_1-1,j)})}\right]$$

$$\leq \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right] \ a.s..$$

Similarly

$$\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right)$$

$$\leq \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[S_{(i,n_2)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right] \ a.s..$$

Therefore

$$\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq \varepsilon\right)$$

$$\leq \min\left(\begin{array}{l}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right], \\ \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[S_{(i,n_2)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq\varepsilon\right)}\right]\end{array}\right) \ a.s..$$

Corollary 1 Let $\{S_n, n\in N^2\}$ be a two-parameter conditional demisubmartingale,and when $k_1k_2=0,\ S_k=0 a.s.$, where $k=(k_1,k_2)$. Let $g(\cdot)$ be a nondecreasing convex function, $g(S_n)\in L^1, n\in N^2$. Then for any F measurable random variables $\varepsilon\geq 0\ a.s.$, we get

$$\varepsilon P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} g(S_{(i,j)}) \geq \varepsilon\right)$$

$$\leq \min\left(\begin{array}{l}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[g(S_{(n_1,j)}) I_{\left(\max_{(i,j)\leq(n_1,n_2)} g(S_{(i,j)})\geq\varepsilon\right)}\right], \\ \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[g(S_{(i,n_2)}) I_{\left(\max_{(i,j)\leq(n_1,n_2)} g(S_{(i,j)})\geq\varepsilon\right)}\right]\end{array}\right) \ a.s..$$

Proof. From lemma 2, we can see that $\{g(S_n), n\in N^2\}$ is a two-parameter conditional demisubmartingale ,the above results can be obtained from theorem 1.

Theorem 2 Let $\{S_n, n\in N^2\}$ be a nonnegative two-parameter conditional demimartingale.If $S_{(1,1)}=1\, a.s.$. Then

$$\gamma\left(E^{\mathrm{F}}\left[\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right]\right)$$

$$\leq \min\left(\begin{array}{l}\sum_{j=1}^{n_2}\left(E^{\mathrm{F}}\left[S_{(n_1,j)}\ln S_{(n_1,j)}\right] + \ln E^{\mathrm{F}}\left[\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right]\right), \\ \sum_{i=1}^{n_1}\left(E^{\mathrm{F}}\left[S_{(i,n_2)}\ln S_{(i,n_2)}\right] + \ln E^{\mathrm{F}}\left[\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right]\right)\end{array}\right) a.s..$$

Where $\gamma(x) = x-1-\ln x, x>0$.

Proof. Since from theorem 1, we can get that

$$E^{\mathrm{F}}\left[\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right] - 1$$

$$= \int_0^{\infty} P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq t\right) \mathrm{d}t - 1$$

$$= \int_0^{1} P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq t\right) \mathrm{d}t$$

$$+ \int_1^{\infty} P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq t\right) \mathrm{d}t - 1$$

$$= \int_1^{\infty} P^{\mathrm{F}}\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)} \geq t\right) \mathrm{d}t$$

$$\leq \int_1^{\infty} \frac{1}{t} \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} I_{\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\geq t\right)}\right] \mathrm{d}t$$

$$= \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)} \ln\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right)\right] a.s..$$

Similarly

$$E^{\mathrm{F}}\left[\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right] - 1$$

$$\leq \sum_{i=1}^{n_1} E^{\mathrm{F}}\left[S_{(i,n_2)} \ln\left(\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right)\right] a.s..$$

Since γ is non-negative,we have

$$E^{\mathrm{F}}\left[\max_{(i,j)\leq(n_1,n_2)} S_{(i,j)}\right] - 1$$

$$\le \sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(n_1,j)}\left(\ln\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right) + \gamma\left(\frac{\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}}{S_{(n_1,j)}E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]}\right)\right)\right]$$

$$=\sum_{j=1}^{n_2}\left(E^{\mathrm{F}}\left[S_{(n_1,j)}\ln S_{(n_1,j)}\right]+\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]\right) a.s..$$

Then

$$E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]-1-\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]$$

$$\le\sum_{j=1}^{n_2}\left(E^{\mathrm{F}}\left[S_{(n_1,j)}\ln S_{(n_1,j)}\right]+\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]\right) a.s..$$

Similarly

$$E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]-1-\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]$$

$$\le\sum_{i=1}^{n_1}\left(E^{\mathrm{F}}\left[S_{(i,n_2)}\ln S_{(i,n_2)}\right]+\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]\right) a.s..$$

So we have

$$\gamma\left(E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]\right)$$

$$\le \min\left(\begin{array}{l}\sum_{j=1}^{n_2}\left(E^{\mathrm{F}}\left[S_{(n_1,j)}\ln S_{(n_1,j)}\right]+\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]\right),\\ \sum_{i=1}^{n_1}\left(E^{\mathrm{F}}\left[S_{(i,n_2)}\ln S_{(i,n_2)}\right]+\ln E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]\right)\end{array}\right) a.s..$$

Theorem 3 Let $\{S_n, n\in N^2\}$ be a nonnegative two-parameter conditional demimartingale, and for any $(n_1,n_2)\in N^2$, with $S_{(1,1)}=a>0$ *a.s.*,

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(i,j)}\log^+ S_{(i,j)}\right]<\infty \quad a.s.,$$

$$\lim_{(n_1,n_2)\to\infty}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(i,j)}\log^+ S_{(i,j)}\right]=\infty \ a.s.,$$

and $\lim_{(n_1,n_2)\to\infty}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[\mathrm{e}^{-1}\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]<\infty \quad a.s.;$

Then

$$\lim_{(n_1,n_2)\to\infty}\sup\frac{E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]}{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E^{\mathrm{F}}\left[S_{(i,j)}\log^+ S_{(i,j)}\right]}\le\frac{\mathrm{e}}{\mathrm{e}-1} \quad a.s..$$

Proof. We may study the two-parameter conditional demimartingale $\frac{S_{(n_1,n_2)}}{a}$ instead of $S_{(n_1,n_2)}$. Therefore we may assume that $S_{(1,1)}=1$ *a.s.*. Since $\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge S_{(1,1)}=1.$ It can be checked that for $a\log^+ b\le a\log^+ a+b\mathrm{e}^{-1}, a>0, b>0,$ and by Theorem 1, we get that

$$E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]-1$$

$$=\int_0^\infty P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t-1$$

$$=\int_0^1 P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t+\int_1^\infty P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t-1$$

$$=\int_1^\infty P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t$$

$$\le\int_1^\infty\frac{1}{t}\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[S_{(i,n_2)}I_{\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)}\right]\mathrm{d}t$$

$$=\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[S_{(i,n_2)}\log^+\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right)\right]$$

$$\le\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[S_{(i,n_2)}\log^+ S_{(i,n_2)}+\mathrm{e}^{-1}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right)\right]$$

$$=\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[S_{(i,n_2)}\log^+ S_{(i,n_2)}\right]+\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[\mathrm{e}^{-1}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right)\right] a.s.$$

Then

$$E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]-\mathrm{e}^{-1}E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]$$

$$\le\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[S_{(i,n_2)}\log^+ S_{(i,n_2)}\right]+\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[\mathrm{e}^{-1}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right)\right]+1$$

So

$$E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]$$

$$\le\frac{\mathrm{e}}{\mathrm{e}-1}\left(\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[S_{(i,n_2)}\log^+ S_{(i,n_2)}\right]+\sum_{i=1}^{n_1}E^{\mathrm{F}}\left[\mathrm{e}^{-1}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right)\right]+1\right) a.s..$$

Similarly

$$E^{\mathrm{F}}\left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\right]-1$$

$$=\int_0^\infty P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t-1$$

$$=\int_0^1 P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t+\int_1^\infty P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t-1$$

$$=\int_1^\infty P^{\mathrm{F}}\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)}\ge t\right)\mathrm{d}t$$

$$\le \int_1^{\infty} \frac{1}{t} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(n_1,j)} I_{\left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \ge t\right)} \right] \mathrm{d}t$$

$$= \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(n_1,j)} \log^{+} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right]$$

$$\le \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(n_1,j)} \log^{+} S_{(n_1,j)} + \mathrm{e}^{-1} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right]$$

$$= \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(n_1,j)} \log^{+} S_{(n_1,j)} \right] + \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[\mathrm{e}^{-1} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right] \ a.s..$$

Then

$$E^{\mathrm{F}} \left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right] - \mathrm{e}^{-1} E^{\mathrm{F}} \left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right]$$

$$\le \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(n_1,j)} \log^{+} S_{(n_1,j)} \right] + \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[\mathrm{e}^{-1} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right] + 1 \quad a.s..$$

So

$$E^{\mathrm{F}} \left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right] \le \frac{\mathrm{e}}{\mathrm{e}-1} \left(\sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(n_1,j)} \log^{+} S_{(n_1,j)} \right] + \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[\mathrm{e}^{-1} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right] + 1 \right) a.s..$$

Then

$$E^{\mathrm{F}} \left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right] \le \frac{\mathrm{e}}{\mathrm{e}-1} \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(i,j)} \log^{+} S_{(i,j)} \right] + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[\mathrm{e}^{-1} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right] + 1 \right) a.s..$$

So

$$\frac{E^{\mathrm{F}} \left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right]}{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(i,j)} \log^{+} S_{(i,j)} \right]} \le \frac{\mathrm{e}}{\mathrm{e}-1} \left(1 + \frac{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[\mathrm{e}^{-1} \left(\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right) \right]}{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(i,j)} \log^{+} S_{(i,j)} \right]} + \frac{1}{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(i,j)} \log^{+} S_{(i,j)} \right]} \right) a.s..$$

IV. CONCLUSION

$$\lim_{(n_1,n_2)\to\infty} \sup \frac{E^{\mathrm{F}} \left[\max_{(i,j)\le(n_1,n_2)} S_{(i,j)} \right]}{\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E^{\mathrm{F}} \left[S_{(i,j)} \log^{+} S_{(i,j)} \right]} \le \frac{\mathrm{e}}{\mathrm{e}-1} \quad a.s..$$

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