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Some operators acting on functional of discrete-time quantum Bernoulli noises

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***Abstract*— In this paper, we present some commutation relations of operators , which act on functional of discrete-time quantum Bernoulli noises.**

***Index Terms*— quantum Bernoulli noises; com-mut-ation relations; operators**

I. INTRODUCTION

Quantum Bernoulli noises are the family of annihilation and creation operators acting on square integrable Bernoulli functionals, which can describe a two-level quantum system with infinitely many sites. They satisfy a canonical anti-commutation relation in equal-time and also play an active role in buliding a discrete-time quantum stochastic calculus in infinite dimensions. In 2013, Wang and Zhang considered a kind of localization of quantum Bernoulli noises and showed applications of main results to quantum dynamical semigroups and quantum protum probability. Our work devote to discussing some operators acting on functional of discrete-time quantum Bernoulli noises, which play a key role in Bernoulli functionals. Thus, we consider some commutation relations for the operators , which are interesting.

II. PRELIMINARIES

In this section, we briefly recall some notions and results for quantum Bernoulli noises. For details, see [1-2] and references therein.

Let N be the set of all nonnegative integers and Γ the finite power set of N, namely,

$$\Gamma=\{\sigma|\sigma\subset N \text{ and } \#\sigma<\infty\},$$

where $\#\sigma$ denotes the cardinality of σ as a set.

Thoughout, we assume that (Ω,F,P) is a probability space and $Z=(Z_n)_{n\geq 0}$ is an independent sequence of random variables on (Ω,F,P), which satisfies that

$$P\{Z=\theta_n\}=p_n,\quad P\{Z=-1/\theta_n\}=q_n,\quad n\geq 0$$

with $\theta_n=\sqrt{q_n/p_n}$, $q_n=1-p_n$ and $0\leq p_n\leq 1$. And, moreover, $F=\sigma(Z_n,n\geq 0)$, the σ-filed generated by $Z=(Z_n)_{n\geq 0}$. And Z is actually a discrete-time Bernoulli noise.

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Let $L^2(Z)$ be the space of square integrable complex-valued random variables on (Ω,F,P).

We denote by $\langle\cdot,\cdot\rangle$ the inner product of $L^2(Z)$, and by $\|\cdot,\cdot\|$ the corresponding norm. It is known that Z has the orthonormal basis $\{Z_\sigma|\sigma\in\Gamma\}$, where $Z_\varnothing=1$ and

$$Z_\sigma=\prod_{i\in\sigma}Z_i,\sigma\in\Gamma,\sigma\neq\varnothing,$$

which shows that $L^2(Z)$ is an infinite dimensional space.

For $k\geq 0$, we use E_k to denote the conditional exceptation given σ-filed F_k, namely

$$E_k=E[\cdot\mid F_k],$$

where F_k is the σ-filed generated by $(Z_j)_{0\leq j\leq k}$. It is known that E_k is a projection operator on $L^2(Z)$ and its range is $L^2(\Omega,F_k,P)$, which is a 2^{k+1}-dimensional of $L^2(Z)$.

Lemma 1.[1] For $k\geq 0$, there exists a bounded operator ∂_k on $L^2(Z)$ such that

$$\partial_k Z_\sigma=1_\sigma(k)Z_{\sigma\setminus k},\quad \sigma\in\Gamma, \tag{1}$$

where $\sigma\setminus k=\sigma\setminus\{k\}$ and $1_\sigma(k)$ is the indicator of σ a subset of N.

Lemma 2.[1] For $k\geq 0$, then ∂_k^*, the adjoint operator, has following property:

$$\partial_k^* Z_\sigma=(1-1_\sigma(k))Z_{\sigma\cup k},\quad \sigma\in\Gamma, \tag{2}$$

where $\sigma\cup k=\sigma\cup\{k\}$.

Lemma 3. [2] For $k\geq 0$, we call $l_k=\partial_k E_k$ the local annihilation and its adjoint operator $l_k^*=\partial_k^* E_k$ the local creation operator.

Lemma 4.[2] For $k\geq 0$, we have

$$l_k^* l_k=(\partial_k^*\partial_k)E_k. \tag{3}$$

III. MAIN RESULTS

Definition 1. A diagonal operator C on functionals of Z is defined as

$$Cx = \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle Z_\sigma, x\rangle Z_\sigma,\ x \in Dom\, C,\ p \ge 0,$$

with

$$Dom\ C = \left\{x \in L^2(z) \middle| \sum_{\sigma\in\Gamma} \lambda_\sigma^{2p} \left|\langle Z_\sigma, x\rangle\right|^2 < \infty\right\},$$

where

$$\lambda_\sigma = \begin{cases} \prod_{\sigma\in\Gamma}(k+1), \sigma \ne \emptyset, \sigma\in\Gamma; \\ 1, \quad\quad\quad\quad , \sigma = \emptyset, \sigma \in \Gamma. \end{cases}$$

Clearly, $Dom\, C$ contains the canonical orthonormal basis of $\{Z_\sigma | \sigma\in\Gamma\}$, which means the C is a densely defined operator in $L^2(Z)$. $L^2(Z)$ has an orthonormal basis of $\{Z_\sigma | \sigma\in\Gamma\}$. Thus, for each $n \ge 0$, we put

$$H_n = span\{Z_\sigma | \sigma\in\Gamma, \sigma \subset [0,n]\} = span\{Z_\sigma | \sigma \in \Gamma_{n]}\}.$$

Clearly, for each $n \ge 0$, $H_n \subset L^2(Z)$ and the dimension of H_n is 2^{n+1}, which means that H_n is a closed subspace of $L^2(Z)$ and $\{Z_\sigma | \sigma\in\Gamma_{n]}\}$ is an orthonormal basis of H_n.

Definition 2. For $n \ge 0$, P_n is the projection operator from $L^2(Z)$ onto H_n, namely, for all $x \in L^2(Z)$, $P_n x = \sum_{\sigma\in\Gamma_{n]}} \langle Z_\sigma, x\rangle Z_\sigma$.

Theorem 1. Let $p \ge 0$ be a nonnegative real number. Then, for all $n \ge 0$, CP_n makes sense, and moreover $CP_n = P_n C$ on $Dom\, C$.

Proof. Let $n \ge 0$. Then, $H_n \subset L^2(Z) \subset Dom\, C$, which together with the fact that H_n is just the range of P_n, impiles that CP_n makes sense. Now, for $x \in Dom\, C$, it follows from the definitions of C and P_n that

$$CP_n x = \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle Z_\sigma, P_n x\rangle Z_\sigma = \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle P_n Z_\sigma, x\rangle Z_\sigma = \sum_{\sigma\in\Gamma_{n]}} \lambda_\sigma^p \langle Z_\sigma, x\rangle Z_\sigma,$$

and

$$P_n Cx = P_n \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle Z_\sigma, x\rangle Z_\sigma = \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle Z_\sigma, x\rangle P_n Z_\sigma = \sum_{\sigma\in\Gamma_{n]}} \lambda_\sigma^p \langle Z_\sigma, x\rangle Z_\sigma,$$

which givens $CP_n x = P_n Cx$. □

Theorem 2. Let $p \ge 0$ be a nonnegative real number. Then, for all $k \ge 0$, both $l_k^* l_k C$ and $Cl_k^* l_k$ make sense on $L^2(Z)$, and moreover it holds on $L^2(Z)$ that $Cl_k^* l_k = l_k^* l_k C$.

Proof. Let $k \ge 0$. It is easy to see that $Dom\, l_k^* l_k = L^2(Z)$, which together with the fact $L^2(Z) \subset Dom\, C$, implies that $Cl_k^* l_k$ makes sense on $L^2(Z)$. Similarly, $l_k^* l_k C$ also makes sense on $L^2(Z)$. To complete the proof, it suffices to show that $Cl_k^* l_k Z_\sigma = l_k^* l_k CZ_\sigma$ holds for all $\sigma\in\Gamma$, in fact, for all $\sigma\in\Gamma$, by (1), (2) and definition of C, we have

$$Cl_k^* l_k Z_\sigma = C\left[\left(\partial_k^* \partial_k\right) E_k Z_\sigma\right] = C1_{\max\sigma=k} Z_\sigma = 1_{\max\sigma=k} CZ_\sigma = 1_{\max\sigma=k} \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle Z_\sigma, Z_\sigma\rangle Z_\sigma = 1_{\max\sigma=k} \sum_{\sigma\in\Gamma} \lambda_\sigma^p Z_\sigma,$$

and

$$l_k^* l_k CZ_\sigma = l_k^* l_k \sum_{\sigma\in\Gamma} \lambda_\sigma^p \langle Z_\sigma, Z_\sigma\rangle Z_\sigma = l_k^* l_k \sum_{\sigma\in\Gamma} \lambda_\sigma^p Z_\sigma = \sum_{\sigma\in\Gamma} \lambda_\sigma^p l_k^* l_k Z_\sigma = \sum_{\sigma\in\Gamma} \lambda_\sigma^p \left(\partial_k^* \partial_k\right) E_k Z_\sigma = 1_{\max\sigma=k} \sum_{\sigma\in\Gamma} \lambda_\sigma^p Z_\sigma.$$

which givens $Cl_k^* l_k Z_\sigma = l_k^* l_k CZ_\sigma$. □

Theorem 3. For all $k, n \ge 0$, $l_k^* l_k P_n$ makes sense on $L^2(Z)$, and moreover it holds on $L^2(Z)$ that $P_n l_k^* l_k = l_k^* l_k P_n$.

Proof. Let $k \ge 0$. It is easy to see that $Dom\, l_k^* l_k = L^2(Z)$. Obviously, $l_k^* l_k P_n$ makes sense. Now, we prove $P_n l_k^* l_k = l_k^* l_k P_n$. In fact, it suffices to show that $P_n l_k^* l_k Z_\sigma = l_k^* l_k P_n Z_\sigma$ holds for all $\sigma\in\Gamma$, by (1), (2) and definition of P_n, we have

$$P_n l_k^* l_k Z_\sigma = P_n\left[\left(\partial_k^* \partial_k\right) E_k Z_\sigma\right] = P_n 1_{\max\sigma=k} Z_\sigma = 1_{\max\sigma=k} P_n Z_\sigma, \sigma\in\Gamma_{n]},$$

and

$$l_k^* l_k P_n Z_\sigma = l_k^* l_k Z_\sigma = \left(\partial_k^* \partial_k\right) E_k Z_\sigma, \sigma\in\Gamma_{n]},$$

which givens $P_n l_k^* l_k Z_\sigma = l_k^* l_k P_n Z_\sigma$. □

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