Some operators acting on functional of discrete-time quantum Bernoulli noises

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Abstract—In this paper, we present some commutation relations of operators, which act on functional of discrete-time quantum Bernoulli noises.

Index Terms—quantum Bernoulli noises; commutation relations; operators

I. INTRODUCTION

Quantum Bernoulli noises are the family of annihilation and creation operators acting on square integrable Bernoulli functionals, which can describe a two-level quantum system with infinitely many sites. They satisfy a canonical anti-commutation relation in equal-time and also play an active role in building a discrete-time quantum stochastic calculus in infinite dimensions. In 2013, Wang and Zhang considered a kind of localization of quantum Bernoulli noises and showed applications of main results to quantum dynamical semigroups and quantum probability. Our work devote to discussing some operators acting on functional of discrete-time quantum Bernoulli noises, which play a key role in Bernoulli functionals. Thus, we consider some commutation relations for the operators, which are interesting.

II. PRELIMINARIES

In this section, we briefly recall some notions and results for quantum Bernoulli noises. For details, see [1]-[2] and references therein.

Let $N$ be the set of all nonnegative integers and $\Gamma$ the finite power set of $N$, namely,

$$\Gamma = \{\sigma | \sigma \subset N \text{ and } \# \sigma < \infty\},$$

where $\# \sigma$ denotes the cardinality of $\sigma$ as a set.

Thoughout, we assume that $(\Omega, F, \mathcal{P})$ is a probability space and $Z = (Z_n)_{n \geq 0}$ is an independent sequence of random variables on $(\Omega, F, \mathcal{P})$, which satisfies that

$$P[Z = q_n] = p_n, \quad P[Z = -1/\theta] = q_n, \quad n \geq 0$$

with $\theta = \sqrt{q_n/p_n}$, $q_n = 1 - p_n$ and $0 \leq p_n \leq 1$. And, moreover, $F = \sigma(Z_n, n \geq 0)$, the $\sigma$-filed generated by $Z = (Z_n)_{n \geq 0}$. And $Z$ is actually a discrete-time Bernoulli noise.

Let $L^2(Z)$ be the space of square integrable complex-valued random variables on $(\Omega, F, \mathcal{P})$.

We denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(Z)$, and by $\| \cdot \|$ the corresponding norm. It is known that $Z$ has the orthonormal basis

$$\{Z_\sigma | \sigma \in \Gamma\},$$

where $Z_\sigma = 1$ and

$$Z_\sigma = \prod_{i \in \sigma} Z_i, \sigma \in \Gamma, \sigma \neq \emptyset,$$

which shows that $L^2(Z)$ is an infinite dimensional space.

For $k \geq 0$, we use $E_k$ to denote the conditional expectation given $\sigma$-filed $F_k$, namely

$$E_k = E[Z_k | F_k],$$

where $F_k$ is the $\sigma$-filed generated by $(Z_i)_{0 \leq i \leq k}$. It is known that $E_k$ is a projection operator on $L^2(Z)$ and its range is $L^2(\Omega, F_k, \mathcal{P})$, which is a $2^{k+1}$-dimensional of $L^2(Z)$.

Lemma 1.[1] For $k \geq 0$, there exists a bounded operator $\partial_k$ on $L^2(Z)$ such that

$$\partial_k Z_\sigma = 1_{\sigma \cup \{k\}} Z_{\sigma \cup \{k\}}, \quad \sigma \in \Gamma,$$

where $\sigma \cup \{k\} = \sigma \cup \{k\}$ and $1_\sigma(k)$ is the indicator of $\sigma$ a subset of $N$.

Lemma 2.[1] For $k \geq 0$, then $\partial_k^* \partial_k$, the adjoint operator, has following property:

$$\partial_k^* Z_\sigma = (1 - 1_{\sigma \cup \{k\}}) Z_{\sigma \cup \{k\}}, \quad \sigma \in \Gamma.$$ (2)

where $\sigma \cup \{k\} = \sigma \cup \{k\}$.

Lemma 3. [2] For $k \geq 0$, we call $l_k = \partial_k E_k$ the local annihilation and its adjoint operator $l_k^* = \partial_k^* E_k$ the local creation operator.

Lemma 4.[2] For $k \geq 0$, we have

$$l_k^* l_k = (\partial_k^* \partial_k) E_k.$$ (3)
III. MAIN RESULTS

Definition 1. A diagonal operator $C$ on functionals of $Z$ is defined as
\[ Cx = \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle Z_{\sigma}, x \rangle Z_{\sigma}, \quad x \in \text{Dom } C, \quad p \geq 0, \]
with
\[ \text{Dom } C = \left\{ x \in L^2(Z) \mid \left( \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \right) \left( Z_{\sigma}, x \right)^2 < \infty \right\}, \]
where
\[ \lambda_{\sigma} = \prod_{\alpha = 1}^{n_k} (k + 1), \quad \sigma \neq \emptyset, \sigma \in \Gamma; \]
\[ \lambda_{\sigma} = 1, \quad \sigma = \emptyset, \sigma \in \Gamma. \]

Clearly, $\text{Dom } C$ contains the canonical orthonormal basis of $\{ Z_{\sigma} \mid \sigma \in \Gamma \}$, which means the $C$ is a densely defined operator in $L^2(Z)$. $L^2(Z)$ has an orthonormal basis of $\{ Z_{\sigma} \mid \sigma \in \Gamma \}$. Thus, for each $n \geq 0$, we put
\[ H_n = \text{span} \left\{ Z_{\sigma} \mid \sigma \in \Gamma, \sigma \in [0,n] \right\}. \]

Clearly, for each $n \geq 0$, $H_n \subset L^2(Z)$ and the dimension of $H_n$ is $2^{n+1}$, which means that $H_n$ is a closed subspace of $L^2(Z)$ and $\{ Z_{\sigma} \mid \sigma \in \Gamma_{n} \}$ is an orthonormal basis of $H_n$.

Definition 2. For $n \geq 0$, $P_n$ is the projection operator from $L^2(Z)$ onto $H_n$, namely, for all $x \in L^2(Z)$, $P_n x = \sum_{\sigma \in \Gamma_{n}} \langle Z_{\sigma}, x \rangle Z_{\sigma}.$

Theorem 1. Let $p \geq 0$ be a nonnegative real number. Then, for all $n \geq 0$, $C P_n$ makes sense, and moreover $C P_n = P_n C$ on $\text{Dom } C$.

Proof. Let $n \geq 0$. Then, $H_n \subset L^2(Z) \subset \text{Dom } C$, which together with the fact that $H_n$ is just the range of $P_n$, implies that $C P_n$ makes sense. Now, for $x \in \text{Dom } C$, it follows from the definitions of $C$ and $P_n$ that
\[ C P_n x = \sum_{\sigma \in \Gamma_{n}} \lambda_{\sigma}^p \langle Z_{\sigma}, x \rangle Z_{\sigma} = \sum_{\sigma \in \Gamma_{n}} \lambda_{\sigma}^p \langle P_n Z_{\sigma}, x \rangle Z_{\sigma}, \]
and
\[ P_n C x = P_n \sum_{\sigma \in \Gamma_{n}} \lambda_{\sigma}^p \langle Z_{\sigma}, x \rangle Z_{\sigma} = \sum_{\sigma \in \Gamma_{n}} \lambda_{\sigma}^p \langle Z_{\sigma}, x \rangle P_n Z_{\sigma}, \]
which gives $C P_n x = P_n C x.$

Theorem 2. Let $p \geq 0$ be a nonnegative real number. Then, for all $k \geq 0$, both $l^*_k C$ and $C l^*_k$, make sense on $L^2(Z)$, and moreover it holds on $L^2(Z)$ that $C l^*_k = l^*_k C$.

Proof. Let $k \geq 0$. It is easy to see that $\text{Dom } l^*_k = L^2(Z)$, which together with the fact $L^2(Z) \subset \text{Dom } C$, implies that $C l^*_k$ makes sense on $L^2(Z)$. Similarly, $l^*_k C$ also makes sense on $L^2(Z)$. To complete the proof, it suffices to show that $C l^*_k Z_{\sigma} = l^*_k C Z_{\sigma}$ holds for all $\sigma \in \Gamma$, in fact, for all $\sigma \in \Gamma$, by (1), (2) and definition of $C$, we have
\[ C l^*_k Z_{\sigma} = C \left( \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle Z_{\sigma}, \sigma \rangle \right) = C \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle Z_{\sigma}, \sigma \rangle Z_{\sigma}, \]
and
\[ l^*_k C Z_{\sigma} = l^*_k \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle Z_{\sigma}, \sigma \rangle \]

which gives $C l^*_k Z_{\sigma} = l^*_k C Z_{\sigma}$.

Theorem 3. For all $k, n \geq 0$, $l^*_k l^*_n P_n$ makes sense on $L^2(Z)$, and moreover it holds on $L^2(Z)$ that $P_n l^*_k l^*_n = l^*_n l^*_k P_n$.

Proof. Let $k \geq 0$. It is easy to see that $\text{Dom } l^*_k = L^2(Z)$.

Obviously, $l^*_k l^*_n P_n$ makes sense. Now, we prove $P_n l^*_k = l^*_k P_n$. In fact, it suffices to show that $P_n l^*_k Z_{\sigma} = l^*_k P_n Z_{\sigma}$ holds for all $\sigma \in \Gamma$, by (1), (2) and definition of $P_n$, we have
\[ P_n l^*_k Z_{\sigma} = P_n \left( \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle Z_{\sigma}, \sigma \rangle \right) = P_n \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle Z_{\sigma}, \sigma \rangle Z_{\sigma}, \]
and
\[ l^*_k P_n Z_{\sigma} = l^*_k \sum_{\sigma \in \Gamma} \lambda_{\sigma}^p \langle \sigma, Z_{\sigma} \rangle \]
which gives $P_n l^*_k Z_{\sigma} = l^*_k P_n Z_{\sigma}$.

ACKNOWLEDGEMENT

This work is supported by National Natural Science Foundation of China (Grant No. 11461061).

REFERENCE


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