Computing the Subgroup Commutativity Degree, Normality Degrees and Cyclicity Degrees of Dicyclic Group T_{4n}

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***Abstract*— In this paper we want computed and study *subgroup commutative degree, normality degree* and *cyclicity degree* of Dicyclic group T_{4n}. It is clear that the subgroups H and K of a group G we can say that H permutes with K if $HK = KH$ and the number of subgroups of the Dicyclic group T_{4n} be $\tau(2n)+\sigma(n)$.**

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***Index Terms*—Subgroup, Dicyclic group, Subgroup Commutative Degree, Cyclicity Degree**

I. INTRODUCTION

Let G be a finite group. the subgroups H and K of G, their product $HK=\{hk \mid h \in H;\ k \in K\}$ is a subgroup in G if and only if $HK = KH$. The subgroup commutative degree is define by $sd(G)=\frac{|\{(H;K)\in Sub(G)\times Sub(G)|HK=KH\}|}{Sub(G)^2}$ It is introduced by M. Tărnăuceanu in [8, Section 2.2.3] and [5], the difficult work in field group theory is computing the subgroup commutative degree $sd(G)$, since it must the counting of subgroups of G. Dicyclic group T_{4n} defined in [1], $T_{4n}=\langle a,b \mid a^{2n}=b^4=e;\ b^2=a^n;\ b^{-1}ab=a^{-1}\rangle$. In [4, Chapter 2], Shelash and Ashrafi could counted the number of subgroups of the dicylic group T_{4n} and studied the structure description of subgroups of group T_{4n} are $\langle a^i\rangle$ for $i|2n$ and $\langle a^i, a^j b\rangle$ where $i \mid n$ and $1 \le j \le i$. For this we refer [5,6,7].

In [2], D.E.Otera, F.G.Russo defined the permutability degrees of finite groups. In [1, Theorem], Let $n = 2rm$ where $m=\prod_{i=1}^{s} p_i^{\alpha_i}$, p_i is a odd prime number for any i and $r \ge 0$. Then the number of all subgroups, normal subgroups and characteristic subgroups of T_{4n} can be computed by the following formulas:

$$Sub(T4n) = \tau(2n)+\sigma(n);$$

$$NSub(T4n)=\begin{cases}\tau(2n)+3 & \text{if } 2\mid n\\ \tau(2n)+1 & \text{if } 2\nmid n\end{cases};$$

$$CSub(T4n) = \tau(2n)+1.$$

M. Tărnăuceanu in [8, Theorem 9] computed the subgroup commutativity degree $sd(D2n)$ and some of finite groups. In [3], A. Stefanos computed the $sd(G)$ where G is simple Suzuki groups. we known the $\tau(n)$ is the number of all divisors of n and $\sigma(n)$ is the number summation of all divisors of n. in this paper the our goal is compute the subgroup commutativity degree of dicyclic group T_{4n}.

Let H is a subgroup of G, we can say that H is a subnormal subgroup if there exist series of subgroups such that satisfy the following:

$$1 \trianglelefteq H \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_i \trianglelefteq G$$

we denoted for the subnormal subgroup of group G by $Sbn(G)$. In [6, Chapter six] computed all subnormal subgroup of dicyclic group $T4n$ it is equal to

$$Sbn(T_{4n})=\begin{cases}\tau(2n)+1 & if\ r=0\\ \tau(2n)+\sigma(2^r) & if\ r\ge 1\end{cases}$$

it easy see that the define of subnormality degree of G be

$$Sbndeg(G)=\frac{Sbn(G)}{Sub(G)},$$

where $Sub(G)$, $Sbn(G)$ and $Sbndeg(G)$ the number of subgroups, subnormal subgroup and subnormality degree respectively.

II. MAIN RESULTS

In the section we will compute the subgroup commutativity degree of the Dicyclic group T_{4n}.

Theorem(2.1). The subgroup commutativity degree $sd(T_{4n})$ of the dicyclic group T_{4n} be equal to :

$$sd(T_{4n})=\frac{\tau(2n)^2+2\tau(2n)\sigma(n)+g(n)}{(\tau(2n)+\sigma(n))^2}$$

where $n = 2rm$, m is odd number.

Proof:

From [4, Theorem 2.2.11], it is clear that the number of normal subgroups of Dicyclic group is given by $\tau(2n)+1$ if n is odd number and $\tau(2n)+4$ if n is even number. Suppose that $\Psi(H)=\{K \mid HK = KH; K \in Sub(T_{4n})\}$ the set of subgroups are permutable with subgroup H.

$$\sum_{H\in Sub(T_{4n})}|\Psi(H)|=\sum_{i|2n}|\Psi(H_c^i)|+\sum_{i|n}\sum_{1\le j\le i}|\Psi(H_j^i)|$$

Where $\Psi(H_c^i)$ be the number of cyclic and normal subgroup. then

$$\sum_{H\in Sub(T_{4n})}|\Psi(H)| = \tau(2n)(\tau(2n)+\sigma(n)) + \sum_{i|n}\sum_{1\le j\le i}|\Psi(H_j^i)|$$

and

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$$\sum_{i|n}\sum_{1\le j\le i}|\Psi(H_j^i)| = \sum_{i|n}\sum_{1\le j\le i}(\tau(2n)+x_j^i)$$
$$= \tau(2n)\sigma(n) + \sum_{i|n}\sum_{1\le j\le i} x_j^i$$

where Tărnăuceanu explain that

$$g(n) = \sum_{i|n}\sum_{1\le j\le i} x_j^i = [(r-1)2^{r+3}+9]g(m)$$

Where $g(m) = \prod_{i=1}^{s}\frac{(2\alpha_i+1)p^{\alpha_i+2}+(2\alpha_i+3)p^{\alpha_i+1}+p_i+1}{(p_i-1)^2}$

a) ***The Normality Degree, Cyclicty Degree and Subnormality Degree****:*

In this section we will counting the normality degree, cyclicty degree and subnormality degree. In [1] Shelash and Ashrefi computed the number of subgroups, cyclic subgroups and subnormal subgroups of the group T_{4n}.

Corollary(2.2) *. The normality degree of the Dicyclic group T_{4n} is given by :*

$$ndeg(T_{4n}) = \begin{cases} \dfrac{\tau(2n)+1}{\tau(2n)+\sigma(n)} & if \ 2\nmid n \\ \dfrac{\tau(2n)+3}{\tau(2n)+\sigma(n)} & if \ 2|n \end{cases}$$

Proof:

Direct From [4, Theorem 2.2.11].

Now we want compute the $ndeg(T_{4n}) \le \frac{1}{2}$, it is clear that if n is odd number. Then the $ndeg(T_{4n})$ be

$$\frac{\tau(2n)+1}{\tau(2n)+\sigma(n)} \le \frac{1}{2} ;$$
$$2\tau(2n)+2 \le \tau(2n)+\sigma(n);$$
$$\tau(2n)+2 \le \sigma(n)$$

for each odd number but $n \neq 1,3$ and iIf n is even number . Then $\tau(2n) + 6 \le \sigma(n)$ but $n \neq 2,4$.

Corollary(2.3). *The following holds:*

a) If $n = 5,6$. Then $ndeg(T_{4n}) = \frac{1}{2}$;
b) If $n \ge 7$. Then $ndeg(T_{4n}) > \frac{1}{2}$.

Proof:

Suppose n is odd number then $\tau(2n)+2 = \sigma(n)$, Let $n = p$ is odd prime number then $\tau(2p) + 2 = 6$ and $\sigma(p) = p + 1$, thus $p = 5$. If n is even number. Then $\tau(2n) + 6 = \sigma(n)$, supposethat $n = pq$, where p, q are prime numbers. Let $p = 2$; $q = 3$, then its true. Conversely, it is clear.

***Proposition*(2.4).** The number of cyclic subgroups of the Dicyclic group T_{4n} is given by:

$$CySub(T_{4n}) = \tau(2n) + n$$

Proof:

Clear that, the first type of the subgroups is isomorphic to $C_{\frac{2n}{d}}$ where $d|2n$, then it has $\tau(2n)$ of cyclic subgroups. The second type of the subgroups has cyclic subgroups only when $i = n$, since $\langle a^n, a^j b\rangle \sim C4$, where $1 \le j \le n$.

Now, we can compute the cyclicty degree of the dicyclic group T_{4n} by the theorem in the following:

Corollary(2.5) The cyclicity degree of the T_{4n} is given by the following :

$$cydeg(T4n) = \frac{\tau(2n) + n}{\tau(2n) + \sigma(n)}$$

Proof.

Direct from [4, Theorem 2.2.11] & Proposition 2.4.

Proposition 2.7. The following holds:

a) $n = p$ is an odd prime number if and only if $cydeg(T_{4n}) = \frac{Sub(T_{4n})-1}{Sub(T_{4n})}$;
b) $n = 2p$, p is an odd prime number if and only if $cydeg(T_{4n}) = \frac{2}{3}$.

Proof:

For part (a):

Suppose that $n = p$ is odd prime number, then $cydeg(T_{4n}) = \frac{\tau(2p)+p}{\tau(2p)+\sigma p} = \frac{4+p}{5+p} = \frac{Sub(T_{4p})-1}{Sub(T_{4p})}$

Conversely, let

$$cydeg(T_{4n}) = \frac{Sub(T_{4n})-1}{Sub(T_{4n})} = \frac{\tau(2n)+\sigma(n)-1}{\tau(2n)+\sigma(n)}$$

Since $\sigma(p) = p+1$, thus $cydeg(T_{4n}) = \frac{Sub(T_{4n})-1}{Sub(T_{4n})}$ is true.

For part (b):

Suppose that $n = 2p$, then $cydeg(T_{4n}) = \frac{\tau(2p)+2p}{\tau(2p)+\sigma(2p)} = \frac{6+2p}{9+3p} = \frac{2(3+p)}{3(3+p)} = \frac{2}{3}$.

Conversely, let $\frac{\tau(2n)+n}{\tau(2n)+\sigma(n)} = \frac{2}{3}$, we can obtain on the $3\tau(2n) + 3n = 2\tau(2n) + 2\sigma(n)$ it is equal to $\tau(2n) + 3n = 2\sigma(n)$, it is true when $n = 2p$ checked this by GAP program.

Corollary 2.8. If $n = 2^3 5$ or $n = 6p$, where p is odd prime number. Then $cydeg(T_{4n}) = \frac{1}{2}$.

Proof:

Suppose that $n = 6p$ and p is odd prime number, then

$$cydeg(T_{4n}) = \frac{\tau(2^3 3p)+6p}{\tau(2^3 3p)+\sigma(23p)} = \frac{12+6p}{24+12p} = \frac{1}{2}.$$

If $n = 2^3 5$, then $cydeg(T_{4n}) = \frac{1}{2}$.

The number of subnormal subgroup of Dicyclic group T_{4n} it was computed in [4] by Shelash and Ashrafi.

In the following proposition we will find the relation between the number of subnormal subgroups and the number of subgroups.

Proposition (2.9). The following hold:

a) $Sbndeg(T_{4n}) = \frac{\tau(2n)+1}{\tau(2n)+\sigma(n)}$ if $r = 0$;
b) $Sbndeg(T_{4n}) = \frac{\tau(2n)+\sigma(2^r)}{\tau(2n)+\sigma(n)}$ if $r \ge 1$;
c) $Sbndeg(T_{4n}) = \frac{1}{2}$ if $n = 5,6$.

Proof:

Direct from definition.

III. Conclusion

In this paper we studied the subgroup commutative degree of Dicyclic group and extension this work to computing normality degree, cyclicty degree and subnormality degree subgroup, we test all of the results by GAP program.

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