Computing the Subgroup Commutativity Degree, Normality Degrees and Cyclicity Degrees of Dicyclic Group $T_{4n}$

Hayder Shelash, Ahmed M. AL-obaidi, Muayad G. Mohsin

Abstract—In this paper we want computed and study subgroup commutativity degree, normality degree and cyclicity degree of Dicyclic group $T_{4n}$. It is clear that the subgroups $H$ and $K$ of a group $G$ we can say that $H$ permutes with $K$ if $HK = KH$ and the number of subgroups of the Dicyclic group $T_{4n}$ be $\tau(2n) + \sigma(n)$.


Index Terms—Subgroup, Dicyclic group, Subgroup Commutative Degree, Cyclicity Degree

I. INTRODUCTION

Let $G$ be a finite group, the subgroups $H$ and $K$ of $G$, their product $HK = \{hk \mid h \in H; k \in K\}$ is a subgroup in $G$ if and only if $HK = KH$. The subgroup commutative degree is defined by $sd(G) = \frac{\sum_{(H,K) \in Sub(G) \times Sub(G),HK=KH}}{Sub(G)^2}$ It is introduced by M. Târnăuceanu in [8, Section 2.2.3] and [5], the difficult work in field group theory is computing the subgroup commutative degree $sd(G)$, since it must the counting of subgroups of $G$. Dicyclic group $T_{4n}$ defined in [1]. $T_{4n} = \{a, b \mid a^{2n} = b^2 = e; b^2 = a^n; b^{-1}ab = a^{-1}\}$. In [4, Chapter 2], Shelash and Ashrafi could counted the number of subgroups of the dicyclic group $T_{4n}$ and studied the structure description of subgroups of group $T_{4n}$ are $(a^i)$ for $i \mid 2n$ and $(a^i, a^j b)$ where $i \mid n$ and $1 \leq j \leq i$. For this refer [5,6,7].

In [2], D.E.Otera, F.G.Russo defined the permutability degrees of finite groups. In [1, Theorem], Let $n = 2rm$ where $m = \prod_{i=1} p_i^e_i$, $p_i$ is a odd prime number for any $i$ and $r \geq 0$. Then the number of all subgroups, normal subgroups and characteristic subgroups of $T_{4n}$ can be computed by the following formulas:

$Sub(T_{4n}) = \tau(2n) + \sigma(n)$;
$NSub(T_{4n}) = \begin{cases} \tau(2n) + 3 & \text{if } 2 \mid n \\ \tau(2n) + 1 & \text{if } 2 \nmid n \end{cases}$;
$CSub(T_{4n}) = \tau(2n) + 1$.

M. Târnăuceanu in [8, Theorem 9] computed the subgroup commutativity degree $sd(D_{2n})$ and some of finite groups. In [3], A. Stefanos computed the $sd(G)$ where $G$ is simple Suzuki groups. we known the $\tau(n)$ is the number of all divisors of $n$ and $\sigma(n)$ is the number summation of all divisors of $n$. in this paper the our goal is compute the subgroup commutativity degree of dicyclic group $T_{4n}$.

Let $H$ is a subgroup of $G$, we can say that $H$ is a subnormal subgroup if there exist series of subgroups such that satisfy the following:

$1 \subseteq H \subseteq H_2 \subseteq \ldots \subseteq H_i \subseteq G$

we denoted for the subnormal subgroup of group $G$ by $Sbnm(G)$. In [6, Chapter six] computed all subnormal subgroup of dicyclic group $T_{4n}$ it is equal to

$Sbn(T_{4n}) = \begin{cases} \tau(2n) + 1 & \text{if } r = 0 \\ \tau(2n) + \sigma(2^r) & \text{if } r \geq 1 \end{cases}$

it easy see that the define of subnormality degree of $G$ be

$Sbnmdeg(G) = \frac{Sbn(G)}{Sub(G)}$

where $Sub(G)$, $Sbnm(G)$ and $Sbnmdeg(G)$ the number of subgroups, subnormal subgroup and subnormality degree respectively.

II. MAIN RESULTS

In the section we will compute the subgroup commutativity degree of the Dicyclic group $T_{4n}$.

Theorem 2.1. The subgroup commutativity degree $sd(T_{4n})$ of the dicyclic group $T_{4n}$ be equal to:

$sd(T_{4n}) = \frac{\tau(2n)^2 + 2\tau(2n)\sigma(n) + g(n)}{(\tau(2n) + \sigma(n))^2}$

where $n = 2rm$, $m$ is odd number.

Proof:

From [4, Theorem 2.2.11], it is clear that the number of normal subgroups of Dicyclic group is given by $\tau(2n) + 1$ if $n$ is odd number and $\tau(2n) + 4$ if $n$ is even number. Suppose that $\Psi(H) = \{K \mid HK = KH; K \in Sub(T_{4n})\}$ the set of subgroups are permutable with subgroup $H$.

$\sum_{H \in Sub(T_{4n})} |\Psi(H)| = \sum_{i|n} \sum_{1 \leq j \leq d} |\Psi(H_{ij})|$

Where $\Psi(H_{ij})$ be the number of cyclic and normal subgroup, then

$\sum_{H \in Sub(T_{4n})} |\Psi(H)| = \tau(2n)\tau(2n) + \sigma(n) + \sum_{i|n} \sum_{1 \leq j \leq d} |\Psi(H_{ij})|$

and

Hayder Shelash, His Department Mathematics, University of Kufa Faculty of Computer Sciences and Mathematics, Najaf, Iraq

Ahmed M. AL-OBaidi, His Department Mathematics, University of Kufa Faculty of Computer Sciences and Mathematics, Najaf, Iraq

Muayad G. Mohsin, His Department Mathematics, University of Kufa Faculty of Computer Sciences and Mathematics, Najaf, Iraq


\[ \sum_{i \in \mathbb{Z}^+} \sum_{j \in \mathbb{Z}^+} |\Psi(H_j)| = \sum_{i \in \mathbb{Z}^+} \sum_{j \in \mathbb{Z}^+} (\tau(2n) + x_j) \]

where Tănăaseanu explain that

\[ g(n) = \sum_{i \in \mathbb{Z}^+} \sum_{j \in \mathbb{Z}^+} x_j = [(r-1)2^{r+3} + 9]g(m) \]

\[ g(n) = \prod_{i=1}^{\infty} \left( \frac{2a_i+1)p^{i+2}+r(2a_i+3)2^{i+1}+p+1}{(p-1)^2} \right) \]

\[ \text{Proof:} \]

Suppose n is odd number then

\[ \text{ndeg}(T_{4n}) = \begin{cases} 
\frac{\tau(2n)+1}{\tau(2n)+\sigma(n)} & \text{if } 2 \n \frac{\tau(2n)+3}{\tau(2n)+\sigma(n)} & \text{if } 2|n
\end{cases} \]

\[ \text{Corollary(2.2)} \text{. The normality degree of the Dicyclic group } T_{4n} \text{ is given by :}
\]

\[ \text{ndeg}(T_{4n}) = \begin{cases} 
\frac{\tau(2n)+1}{\tau(2n)+\sigma(n)} & \text{if } 2 \n \frac{\tau(2n)+3}{\tau(2n)+\sigma(n)} & \text{if } 2|n
\end{cases} \]

\[ \text{Corollary(2.3)} \text{. The following holds:}
\]

a) If \( n = 5, 6 \), then \( \text{ndeg}(T_{4n}) = \frac{1}{2} \).

b) If \( n \geq 7 \), then \( \text{ndeg}(T_{4n}) > \frac{1}{2} \).

\[ \text{Corollary(2.5)} \text{. The cyclicity degree of the } T_{4n} \text{ is given by the following :}
\]

\[ \text{cydeg}(T_{4n}) = \frac{\tau(2n) + n}{\tau(2n) + \sigma(n)} \]

\[ \text{Proof:} \]

Direct from [4, Theorem 2.2.11] & Proposition 2.4.

\[ \text{Proposition 2.7.} \text{ The following holds:}
\]

a) \( n = p \) is an odd prime number if and only if

\[ \text{cydeg}(T_{3n}) = \frac{\tau(p)+1}{\tau(p)+\sigma(p)} = \frac{p+1}{p+1} = 1 \]

Conversely, let

\[ \text{cydeg}(T_{3n}) = \frac{\tau(p)+1}{\tau(p)+\sigma(p)} = \frac{p+1}{p+1} = 1 \]

Since \( p = p = 1 \), thus \( \text{cydeg}(T_{3n}) = \frac{\tau(p)+1}{\tau(p)+\sigma(p)} = 1 \) is true.

b) \( n = 6p \), then \( \text{cydeg}(T_n) = \frac{\tau(2p)+2p}{\tau(2p)+\sigma(p)} = \frac{2p+2p}{2p+2p} = 1 \)

Conversely, let

\[ \text{cydeg}(T_{3n}) = \frac{\tau(p)+1}{\tau(p)+\sigma(p)} = \frac{p+1}{p+1} = 1 \]

\[ \text{Corollary 2.8. If } n = 2^35 \text{ or } n = 6p, \text{ then cydeg}(T_n) = \frac{1}{2} \]

\[ \text{Proof:} \]

Suppose that \( n = 6p \) and \( p \) is odd prime number, then

\[ \text{cydeg}(T_{3n}) = \frac{\tau(2^3p)+6p}{\tau(2^3p)+\sigma(2^3p)} = \frac{12+6p}{24+12p} = \frac{1}{2} \]

If \( n = 2^35 \), then \( \text{cydeg}(T_{3n}) = \frac{1}{2} \).

The number of normal subgroup of Dicyclic group \( T_{4n} \) it was computed in [4] by Shelaš and Ashrafi.

In the following proposition we will find the relation between the number of subnormal subgroups and the number of subgroups.

\[ \text{Proposition (2.9).} \text{ The following hold:}
\]

a) \( \text{Sbndeg}(T_{4n}) = \frac{\tau(2n)+1}{\tau(2n)+\sigma(n)} \) if \( r = 0 \);

b) \( \text{Sbndeg}(T_{4n}) = \frac{\tau(2n)+\sigma(2^r)}{\tau(2n)+\sigma(n)} \) if \( r \geq 1 \);

c) \( \text{Sbndeg}(T_{4n}) = \frac{1}{2} \) if \( n = 5, 6 \).

Proof:

Direct from definition.

\[ \text{III. Conclusion} \]

In this paper we studied the subgroup commutativity degree of Dicyclic group and extension this work to computing normality degree, cyclicity degree and subnormality degree subgroup, we test all of the results by GAP program.
REFERENCES


Author Profile

Haider B. Shelah received the B.S. degree in mathematics from Faculty Education in Al-Mustansiriya University in 2004, M.S. degrees in representation theory from faculty comp sciences and math, university of Kufa in 2011 and Ph.D degree in Algebra from university of Kashan in 2018.

AHMEDMAL-OBAIDI received a B.Sc. degree in Math. from University of BAGHDAD, IRAQ in 1998 and is currently pursuing a M.Sc. degree in Math. from University of TIKRIT, IRAQ.

Muayad G. Mohsin received his B.Sc. from collage of Science, University of Mosul. He has completed the study of M.Sc. degrees in Mathematics at 2012 in the Faculty of Computer Science and Mathematics at University of Kufa, Iraq. He is working now, at the same Faculty which gave him his M.Sc., as academic member staff.