Asymptotic estimates for finite-time ruin probability of a bidimensional risk model based on entrance process

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Abstract—Consider a bidimensional risk model based on entrance process with constant force of interest in which the claim size from the same business are heavy-tailed and pairwise strong quasi-asymptotically independent, the two counting processes of different business satisfy a certain dependence structure. A precise asymptotic formula for the finite-time ruin probability is obtained.

Index Terms—Bidimensional risk model; ruin probability; independence; asymptotic formula MSC(2010): 60B12, 91B30

I. INTRODUCTION

We know that the literature [1] put forward into a new model (LIG model) based on an entrance process and discussed asymptotic normality of the risk process. Furthermore, some scholars got some conclusions through the study of the LIG model. [2–4] investigated the one-dimensional risk model based on entrance processes. Recently, more attention has been paid to multi-dimensional risk models, especially bidimensional ones. [5] discussed the precise large deviations based on the entry process risk model in the independent case of multi-risk. [6] studied the ruin probability of a bidimensional risk model based on entrance processes with constant interest rate.

In this paper, we investigate finite-time ruin probability of a bidimensional risk model based on entrance processes, in which an insurance company operates two kinds of business. Suppose that the initial fund for the \(i-\)th class is \(x_i\) and \(S_j^i\) is entry time of the \(j-\)th policy with \(0 < S_1^i < S_2^i < \cdots\) and \(S_j^i = \sum_{k=1}^{j} \theta_k^i\), \(i = 1, 2\). \((N_1(t), N_2(t))\) is a bidimensional renewal counting process. Here, \(N_i(t) = \sup\{j \geq 0: S_j^i \leq t\}, t \geq 0, i = 1, 2\). For more detail of a bidimensional renewal counting process, we refer the reader to Examples 3.1 and 3.2 of [7]. Denote the mean function by \(\lambda_i(t) = EN_i(t)\) with \(\lambda_i(0) = 0\) and \(\lambda_i(t) < \infty\), and define the set \(\Lambda_i = \{t > 0: \lambda_i(t) > 0\} = \{t > 0: P(S_t^i \leq t)\}. If we set \(t^*_i = \inf\{t > 0: P(S_t^i \leq t)\}\), then it is easy to see that \(\Lambda_i = [t^*_i, \infty]\) if \(P(S_t^i = t^*_i) > 0\); or \(\Lambda_i = (t^*_i, \infty]\) if \(P(S_t^i = t^*_i) = 0, i = 1, 2\). We denote the intersection set by \(\Lambda = \Lambda_1 \cap \Lambda_2\). Let the validity time of the \(j-\)th policy be \(\{C_j^i, j = 1, 2, \cdots\}\) with probability \(P(C_j^i = \alpha_j^i) = p_j^i, \ell = 1, 2, \cdots, K_j^i\), where they are independent and identically distributed. The premium is \(f_i(C_j^i)\) and \(f_i(\cdot)\) is a strictly increasing function. \(D_j^i\) is claim time of the \(j-\)th policy and independent and identically distributed function \(H_j(\cdot)\). \(X_j^i\) is the \(j-\)th claim size and identically distributed function \(F_j(\cdot)\). Suppose that \(X_j^i\) have the same distributions \(X_j^i\). For any time \(t \geq 0\), the surplus process of the insurer can be described as

\[
\begin{align*}
R_i(t) &= \left( x_i e^{\delta t} + \sum_{j=1}^{N_i(t)} f_1(C_j^i) e^{\delta(t-S_j^i)} \right) \\
R_2(t) &= \left( x_2 e^{\delta t} + \sum_{j=1}^{N_2(t)} f_2(C_j^2) e^{\delta(t-S_j^2)} \right) \\
&- \left( \sum_{i=1}^{N_i(t)} X_j^i e^{\delta(t-S_j^i-D_j^i)} I_{[S_j^i+D_j^i \leq s \leq C_j^i]} \right) \\
&- \left( \sum_{j=1}^{N_2(t)} X_j^2 e^{\delta(t-S_j^2-D_j^2)} I_{[S_j^2+D_j^2 \leq s \leq C_j^2]} \right),
\end{align*}
\]

where \(\delta > 0\) denotes the constant force of interest. We further assume that \(\{X_j^1, j \geq 1\}, \{X_j^2, j \geq 1\}\) and \(\{(N_1(t), N_2(t)\), \(i, j \geq 1\)\} are mutually independent.

Define the finite-time ruin probabilities corresponding to risk model (1) as

\[
\Psi(x_1, x_2; t) = P(\tau_{\max}(x_1, x_2) \leq t)
\]

\[
= P\left( \bigcap_{i=1}^{2} R_i(s) < 0, 0 \leq s \leq t \right),
\]

where

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\[
\tau_{\text{max}}(x_1, x_2) = \inf\{t : \max\{R_1(t), R_2(t)\} < 0 | R_i(0) = x_i, i = 1, 2\}.
\]

In the rest of this paper, Section 2 presents our main results after introducing necessary preliminaries, Section 3 gives some lemmas, and Sections 4 gives the proofs of the main results.

II. PRELIMINARIES AND MAIN RESULTS

**Definition 1.** For a distribution \( F \), denote its tail distribution by \( \bar{F} = 1 - F \) and its upper Matuszewska index by

\[
J^*_F = -\lim_{x \to \infty} \log \frac{\bar{F}_*(x)}{\log x},
\]

where

\[
\bar{F}_*(x) = \liminf_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} \text{ for } y > 0,
\]

which can be found in [8]. A distribution \( F \) is said to be dominatedly-varying-tailed, denoted by \( F \in D \), if for every fixed \( y \in (0, 1) \), \( \bar{F}_*(x) < \infty \). Clearly, \( F \in D \). A distribution \( F \) is said to be long-tailed, denoted by \( F \in L \), if for every fixed \( y \in R \), \( \bar{F}(x+y) \bar{F}(x) \) as \( x \to \infty \), that is,

\[
\lim_{x \to \infty} \frac{\bar{F}(x+y)}{\bar{F}(x)} = 1,
\]

which can be found in [9].

**Definition 2.** If real valued random variables \( X_i, i \geq 1 \) with distribution functions \( F_i, i \geq 1 \) satisfy for any \( i \neq j \)

\[
\lim_{\min\{x_i, x_j\} \to \infty} P(X_i > x_i | X_j > x_j) = 0,
\]

Then we say \( X_i, i \geq 1 \) are pairwise strong quasi-asymptotically independent (PSQAI), which can be found in [10].

**Theorem 1.** Consider the bidimensional risk model (1). Suppose that claim sizes, \( \{X_j, j \geq 1\} \) be PSQAI random variables with common distribution \( F_j \in D \cap L \), \( i = 1, 2 \). If \( N_i(t) \) and \( N_j(t) \) are arbitrarily dependent, then for any fixed \( \varepsilon > 0 \) with \( E[N_i(\varepsilon)N_j(\varepsilon)] > 0 \). Then for \( t \in \Lambda \cap [\varepsilon, T] \), we have

\[
\psi(x_1, x_2; t) \leq \int_0^t \int_0^2 \left[ \sum_{j=1}^{K^1} \sum_{i=1}^{s^1(j)} P^i_{j} \int_0^{\theta_j^1(t+u_n)} \bar{F}_j^i(x e^{\theta_j^1(t+u_n)} \nu_2^i) dH_j^i(y) \right] E[N_i(\varepsilon)N_j(\varepsilon)] d\varepsilon.
\]

\[ (2) \]

III. SOME LEMMAS

The following lemma is an immediate corollary of Theorems 3.2 and 3.4 of [11].

**Lemma 1.** Let \( \{N_i(t), t \geq 0\} \) and \( \{N_j(t), t \geq 0\} \) be two renewal counting processes with their inter-arrival time \( \theta_1^1, \theta_2^2, \ldots \), and \( \theta_1^1, \theta_2^2, \ldots \), respectively. For any integer \( i \geq 1 \), denote by \( S_i^1 = \sum_{j=1}^{i} \theta_j^1 \) and \( S_i^2 = \sum_{j=1}^{i} \theta_j^2 \) the corresponding arrival times. If \( \{(\theta_j^1, \theta_j^2); i \geq 1\} \) is a sequence of i.i.d. random vectors, then it holds for any \( u_i \geq 0 \) and \( u_j \geq 0 \)

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(S_i^1 \leq u_i, S_j^2 \leq u_j) = E(N_i(u_i)N_j(u_j)).
\]

The following lemma comes from Theorem 2.1 of [10]

**Lemma 2.** Assume that \( \{X_j, 1 \leq j \leq n\} \) are \( n \) real-valued random variables with functions of distribution \( F_j, 1 \leq j \leq n \). If \( \{X_j, 1 \leq j \leq n\} \) are PSQAI and \( F_j \in D \cap L \) and \( (c_1, \cdots, c_n) \in [a, b]^n \). Then

\[
P\left( \sum_{j=1}^{n} c_j X_j > x \right) \leq \sum_{j=1}^{n} P(c_j X_j > x).
\]

The following lemma comes from proposition 2.2.1 of [9]

**Lemma 3.** If a distribution \( F \in D \), then for any \( \beta > J^*_F \), there exist two positive constant \( C \) and \( D \) such that for all \( x \geq y \geq D \),

\[
\frac{\bar{F}(y)}{\bar{F}(x)} \leq C \frac{x^\beta}{y^\beta}.
\]

**Lemma 4.** Under the conditions of Theorem 1, it holds that uniformly for \( t \in \Lambda \cap [\varepsilon, T] \),

\[ \left( \sum_{i=1}^{N_i(t)} X_i^1 e^{-\delta(S_i^1+D_i^1)} I_{(S_i^1+D_i^1) \leq x_1} \right) \leq \int_0^t \left[ \sum_{i=1}^{N_i(t)} X_i^2 e^{-\delta(S_i^2+D_i^2)} I_{(S_i^2+D_i^2) \leq x_2} \right] d\varepsilon \]

\[ \sum_{j=1}^{N_j(t)} \left[ \int_0^t \left[ \sum_{i=1}^{K^1} \sum_{n=0}^{s^1(j)} P^i_{j} \int_0^{\theta_j^1(t+u_n)} \bar{F}_j^i(x e^{\theta_j^1(t+u_n)} \nu_2^i) dH_j^i(y) \right] E[N_i(\varepsilon)N_j(\varepsilon)] d\varepsilon \right]. \]

\[ (3) \]

**Proof.** For arbitrarily fixed positive integer \( M \), we split the left-hand side of (3) into three parts as

\[
\left( \sum_{m=1}^{M} \sum_{n=m+1}^{M} \sum_{n=M+1}^{M} \sum_{m=M+1}^{M} \sum_{m=M+1}^{M} \sum_{n=M+1}^{M} \right)
\]

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\[ \times P\left( \sum_{i=1}^{m} X_i^1 e^{-\delta(S_i^1+D_i^1)} I_{(S_i^1+D_i^1, \cdot, \cdot, \cdot, C_i^1)} > x_1, \right. \]
\[ \sum_{j=1}^{n} X_j^2 e^{-\delta(S_j^2+D_j^2)} I_{(S_j^2+D_j^2, \cdot, \cdot, \cdot, C_j^2)} > x_2, \]
\[ N_i(t) = m, \quad N_j(t) = n \]
\[ = \sum_{l=1}^{3} I_l(x_1, x_2; t) - I_4(x_1, x_2; t). \] (4)

We first deal with \( I_1(x_1, x_2; t) \). Write \( B_m^i = \{ 0 \leq z_1^i \leq \cdots \leq z_m^i \leq t < z_{m+1}^i \} \), and \( B_m^{i+1} = \{ 0 \leq \cdots \leq z_m^{i+1} \leq t < z_{m+1}^{i+1} \} \). Since the three sequences \( \{ X_i^j, j \geq 1 \}, i = 1, 2, \) and \( \{ (N_i(t), N_j(t)^T, t \geq 0) \} \) are mutually independent, and using Lemma 2, we have that uniformly for \( t \in \Lambda \cap [\varepsilon, T] \)
\[ I_1(x_1, x_2; t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \prod_{i=1}^{m} P\left( \sum_{j=1}^{n} X_j^i e^{-\delta(S_j^i+D_j^i)} I_{(S_j^i+D_j^i, \cdot, \cdot, \cdot, C_j^i)} > x_1, \right. \]
\[ \sum_{j=1}^{n} X_j^2 e^{-\delta(S_j^2+D_j^2)} I_{(S_j^2+D_j^2, \cdot, \cdot, \cdot, C_j^2)} > x_2, \]
\[ N_i(t) = m, \quad N_j(t) = n \]
\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \prod_{i=1}^{m} P\left( \sum_{j=1}^{n} X_j^i e^{-\delta(S_j^i+D_j^i)} I_{(S_j^i+D_j^i, \cdot, \cdot, \cdot, C_j^i)} > x_1, \right. \]
\[ \sum_{j=1}^{n} X_j^2 e^{-\delta(S_j^2+D_j^2)} I_{(S_j^2+D_j^2, \cdot, \cdot, \cdot, C_j^2)} > x_2, \]
\[ \left. N_i(t) = m, \quad N_j(t) = n \right) \]
\[ \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \sum_{j=1}^{m} P(X_j^1 > x_1, X_j^2 > x_2, N_i(t) = m, \right. \]
\[ \left. N_j(t) = n \right) \]
\[ = P(X_1^1 > x_1, X_2^2 > x_2) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mnP(N_i(t) = m, \right. \]
\[ \left. N_j(t) = n \right) \]
\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \prod_{i=1}^{m} P(X_i^1 > x_1, X_j^2 > x_2, N_i(t) = m, \right. \]
\[ \left. N_j(t) = n \right) \]
\[ \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \prod_{i=1}^{m} \sup_{t \in \Lambda [\varepsilon, T]} E(N_i(t)N_j(t)N_i(t)^{N_j(t)^T} > M). \]

We obtain from Hölder’s inequality that
\[ \sup_{t \in \Lambda [\varepsilon, T]} E(N_i(t)N_j(t)N_i(t)^{N_j(t)^T} > M). \]
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Then, by \( F_i \in D, i = 1, 2 \), we have

\[ \lim_{M \to \infty} \sup_{\min \{ \alpha, \beta \} \to \varepsilon} \left( \frac{I_2(x_1, x_2; t)}{E(N_1(t))} \right) = 0. \quad (7) \]

\( K_2(x_1, x_2; t) \) can be dealt with in the same way. Thus, we obtain

\[ \lim_{M \to \infty} \sup_{\min \{ \alpha, \beta \} \to \varepsilon} \left( \frac{J_2(x_1, x_2; t)}{I_1(x_1, x_2; t)} \right) = 0. \quad (8) \]

Substituting (6) and (8) into (5) leads to

\[ \lim_{M \to \infty} \sup_{\min \{ \alpha, \beta \} \to \varepsilon} \left( \frac{I_2(x_1, x_2; t)}{I_1(x_1, x_2; t)} \right) = 0. \quad (9) \]

We next estimate \( I_4(x_1, x_2; t) \). Choose some \( \beta > \max \{ J_{F_1 \varepsilon}^+, J_{F_2 \varepsilon}^+ \} \). According to Lemma 3, uniformly for all \( t \in \Lambda \cap \varepsilon \),

\[ \psi(x_1, x_2; t) \leq CP(X^1 > x_1)CP(X^2 > x_2) \]

\[ \times E(N_1(t)) \beta \gamma N_2(t) \beta \gamma I_{N_2(T) > M} \],

where \( C \) is some positive constant. Similarly to (8) we obtain

\[ \lim_{M \to \infty} \sup_{\min \{ \alpha, \beta \} \to \varepsilon} \left( \frac{I_4(x_1, x_2; t)}{I_1(x_1, x_2; t)} \right) = 0. \quad (10) \]

In the same manner, we can prove

\[ \lim_{M \to \infty} \sup_{\min \{ \alpha, \beta \} \to \varepsilon} \left( \frac{I_4(x_1, x_2; t)}{I_1(x_1, x_2; t)} \right) = 0. \quad (11) \]

Substituting (9), (10) and (11) into (4) the desired relation (3) holds uniformly for all \( t \in \Lambda \cap \varepsilon \).

IV. PROOF OF MAIN RESULTS.

Proof of Theorem 1. We first deal with the asymptotic upper bound of \( \psi(x_1, x_2; t) \). By Lemma 4, we have that uniformly for \( t \in \Lambda \cap \varepsilon \),

\[ \psi(x_1, x_2; t) \leq \frac{1}{\beta \gamma} \sum_{m=1}^{N_1(t)} X^1 \text{e}^{-\delta(t^1)} I_{|s^1_j, d^1_j| > x_1} \]

\[ \leq \frac{1}{\beta \gamma} \sum_{m=1}^{N_1(t)} X^1 \text{e}^{-\delta(t^1)} I_{|s^1_j, d^1_j| > x_1} \]

\[ \times \prod_{j=1}^{N_2(t)} X^2 \text{e}^{-\delta(t^2)} I_{|s^2_j, d^2_j| > x_2} \]

\[ \times dH_i(y) dE[N_1(u_1)N_2(u_2)]. \quad (12) \]

Then we discuss the asymptotic lower bound of \( \psi(x_1, x_2; t) \). For simplicity, write

\[ Z_k(t) = \sum_{i=1}^{N_k(t)} \int_{t^i} f_k(C_k^i) e^{-\delta t^i}, \quad k = 1, 2. \]

For sufficiently large \( N > 0 \), by Lemma 4,
\[ F_i \in L, k = 1, 2 \], we obtain that uniformly for 
\[ t \in \Lambda \cap [\varepsilon, T]. \]
\[ \psi(x_1, x_2; t) = \int_0^\infty \int_0^\infty P_i \sum_{j=1}^{N_i(t)} X_j^i e^{-\delta(S_j^i+D_j^i)} I_{[S_j^i+D_j^i \leq C_i^j]} > x_1 + z_1, \]
\[ \sum_{j=1}^{N_i(t)} X_j^i e^{-\delta(S_j^i+D_j^i)} I_{[S_j^i+D_j^i \leq C_i^j]} > x_2 + z_2, \]
\[ P(Z_i(t) \in dz_1, Z_2(t) \in dz_2) \]
\[ \int_0^\infty \int_0^\infty \int_0^2 \sum_{i=1}^{K'} P_i \int_0^{\lambda_i(t-u_i)} F_i(x_i e^{\delta(u+y)}) \]
\[ dH_i(y) dE[N_i(u_1)N_i(u_2)] \]
\[ \times P(Z_i(t) \in dz_1, Z_2(t) \in dz_2) \]
\[ \int_0^\infty \int_0^\infty \int_0^{2} \sum_{i=1}^{K'} P_i \int_0^{\lambda_i(t-u_i)} F_i(x_i e^{\delta(u+y)}) \]
\[ dE[N_i(u_1)N_i(u_2)] \]
\[ \frac{dH_i(y) dE[N_i(u_1)N_i(u_2)]}{P_i} \]
A combination of (12) and (13) shows that (2) holds uniformly for all \[ t \leq T \]

**REFERENCES**


