International Journal of Engineering and Applied Sciences (IJEAS)
ISSN: 2394-3661, Volume-6, Issue-9, September 2019

Modified Korteweg-De Vries Equation And Schward Derivative

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***Abstract*— The third order PDE which describes the nonlinear shallow water wave equation has been interested since Scott Russel (1844) [1].**

In this work we study this kind of equation (mKDV), through our study we find that even if the (mKDV) equation does not pass Painleve test but by using Schwarzian Derivative techinque, we were able to find analytic solution. Also we support this study by some figures that to describe the behavior of (mKDF) equation.

***Index Terms*—(mKDV) modified Kortewege-de Vrise equation, Painleve's property, Schwarzian Derivative, Resonanse points.**

I. INTRODUCTION

A nonlinear third order PDE (mKDV) equation can be used to describe most phenomena in scientific field and other domain which normally result from spontaneity motion that appears in the daily life such as the waves water [1].

Some authors considers the (mKDV) in the form:

$$\frac{\partial u}{\partial t} - 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

The determination of the Lie symmetry of above equation provides us with the similarity ansatz,

$$u(x,t) = \frac{1}{(3t)^{\frac{1}{3}}}f(s), \qquad s = \frac{x}{(3t)^{\frac{1}{3}}}.$$

The quantity a is called similarity variable. Taking the ansatz $u(x,t)$ yields,

$$\frac{d^3 f}{ds^3} - 6f^2\frac{df}{ds} - \left(f + s\frac{df}{ds}\right) = 0,$$

integration leads to,

$$\frac{d^2 f}{ds^2} - 2f^3 - sf = c.$$

where c is a constant. [7]

The hyper-surface where the singularities lie is known of the singular manifold, so it can define a technique of Painleve for PDE_s [3].

Definition: The painleve technique for PDE_s**,** it is on analytic with definition in mind, it is natural to find the PDE in the form a Laurit-like expansion [3]

$$u(z) = \varphi(z)^{-p}\sum_{j=0}^{\infty} u_j(z)\varphi(z)^j.$$

The leading power P appearing in order of above equation, where power P is positive integer with the expansion coefficient u_j starts analytic function in a neighborhood of the manifold $\varphi = \varphi(t,x) = 0$, [3].

II. PAINLEVE ANALYSIS

In this section we apply Painleve's test in the (mKdV) equation:

$$u_t + \alpha u^2 u_x + u_{xxx} = 0,\ \alpha \in \mathbb{R} \setminus \{0\} \qquad (1)$$

To verify the (mKDV) has Painleve property or not, we use a method for expanding of the non-linear PDE (presently mKDV) about a movable singularly (presently $\varphi(t,x) = 0$)

The series of non-linear PDE is in the form [3].

$$u = \frac{1}{\varphi^p}\sum_{j=0}^{\infty} u_j\,\varphi^j \qquad (2)$$

where u_j and φ are analytic functions.

Some authors to determine equation (1) by using the simplified condition

$\varphi(x,t) = x + \psi(t) = 0$

where ψ is an arbitrary function and φ is a characteristic of equation (1). Then we can take the coefficient in the equation (2) to be function of t only. To find a value of equilibrium p that by substituting (2) into (1) where $u_t(t,x) = \partial u(t,x)/\partial t$, $u_x(t,x) = \partial u(t,x)/\partial x$ and $u_{xxx}(t,x) = \partial^3 u(t,x)/\partial x^3$,and by comparing the lowest powers in the eventual series, we observe $P=1$ in the neighborhood of the singularity manifold (1). By associating the summation, we observe the recursion [1],

$$(j-3)\left[j - \left(\frac{3}{2} \pm \sqrt{\frac{1}{4} - \alpha}\right)\right]\varphi_x^3 u_j =$$
$$-u_{j-3,t} + \alpha\varphi_x u_0 \sum_{i=1}^{j-1} u_{j-i}u_i - (j-3)u_{j-2}\varphi_t$$
$$-\alpha\sum_{k=1}^{j-1}\left[\sum_{i=0}^{k} u_{k-i}u_i\right]u_{j-k}(j-k-1)\varphi_x$$
$$-\alpha\sum_{k=1}^{j-1}\left[\sum_{i=0}^{k} u_{k-i}u_i\right]u_{j-k-1,x} - u_{j-3,xxx}$$
$$-3(j-3)u_{j-2,x}\varphi_{xx} - (j-3)u_{j-2}\varphi_{xxx}$$
$$-3(j-2)(j-3)\left[u_{j-1,x}\varphi_x^2 + u_{j-1}\varphi_x\varphi_{xx}\right]$$
$$-3(j-3)u_{j-2,xx}\varphi_x, \qquad (3)$$

By using the technique of Painleve, and let uj=0 for all j>1. Then the serious solution (2) leads

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$$\boldsymbol{u} = \frac{\boldsymbol{u_0}}{\boldsymbol{\varphi}} \boldsymbol{u_1}. \qquad (4)$$

Now, to find the value of u_j where $j=0,1,2,\ldots$

To find u_0 then at $j = 0$ in the equation (3), we obtain:

$$u_0 = i\sqrt{\frac{6}{\alpha}}\varphi_x, \qquad i = \sqrt{-1}, \qquad (5)$$

To find u_1 ,then at $j = 1$ in the equation (3), we obtain:

$$u_1 = -\frac{i}{2}\sqrt{\frac{6}{\alpha}}\frac{\varphi_{xx}}{\varphi_x}, \qquad (6)$$

To find u_2 ,then at $j = 2$ in the equation (3), we obtain:

$$u_2 = -\frac{i}{\sqrt{6\alpha}\varphi_x}\left[\frac{\varphi_t + \varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2\right], \qquad (7)$$

Now, in equation (3), we have to find all coefficients of u_j and the relation becomes:

We note that the all coefficients of u_j in the equation (7) are $(j - 3)$ and

$$\left[j - \left(\frac{3}{2} \pm \sqrt{\frac{1}{4} - \alpha}\right)\right],$$

then, in the universal of the integer resonance point is $j = 3$.

The other values of resonance depend on the value of α. For example, if $\alpha = -6$, the resonance points will be $j = -1, 3, 4$.

Now, at $j = 3$, and by using the equations (3), (5) ,(6) and (7), we find,

$$-u_{0,t} - u_{0,xxx} + 2\alpha\varphi_x^2 u_1 u_2 - 2\alpha\varphi_x u_0 u_1 u_2$$
$$+\alpha u_0^2 u_{2,x} - 2\alpha\varphi_x u_0 u_1 u_{1,x}$$
$$-\alpha u_{0,x} u_1^2 - 2\alpha u_0 u_2 u_{0,x} = 0,$$

but, $u_j=0$ for all j > 1, we get.

$$u_{0,t} - u_{0,xxx} + 2u_0 u_1 u_{1,x} + \alpha u_{0,x} u_1^2 = 0, \qquad (8)$$

Inconsistent at the resonance point $j = 3$, this leads that the (mKDV1) does not satisfy the Painleve's test.

Now, at $j = 4$ in the equation (8), we get,

$$-u_{1,t} - u_{1,xxx} - \varphi_t\varphi_2 - 3\varphi_x u_{2,xx} - 3\varphi_{xx} u_{2,x} - \varphi_{xxx} u_2$$
$$-6\varphi_x^2 u_{3,x} - 6\varphi_x\varphi_{xx} u_3 + \alpha\varphi_x^3 \sum_{i=1}^{3} u_{4-i} u_i$$
$$-\alpha\varphi_x \sum_{k=1}^{3}\left[\sum_{i=0}^{k} u_{k-i} u_i\right](3-k) u_{4-k}$$
$$+\alpha \sum_{k=0}^{3}\left[\sum_{i=0}^{k} u_{k-i} u_i\right] u_{3-k,x} = 0, \qquad (9)$$

By realizing the equation (5) into the equation (9), and $u_j = 0$ for all j > 1,

we get,

$$u_{1,t} + \alpha u_1^2 u_{1,x} + u_{1,xxx} = 0,$$

Then u_1 is also a solution of the (mKdV) equation (1).

III. Analytic Solution:

In this section, we follow the project to derive analytic solution with the transmutation,

$$T : \varphi \to \frac{a\varphi + b}{c\varphi + d}, \qquad ad \neq bc,$$

The Schwartzian derivative [2].

$$S(\varphi) = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2, \qquad (10)$$

The dimension of velocity [2],

$$C(\varphi) = -\frac{\varphi_t}{\varphi_x}, \qquad (11)$$

The compatibility of C and S depicted by:

$$S_t + C_{xxx} + 2C_x S + CS_x = 0.$$

By comparing the equations (10) and (11) with the equation (6), and, $u_j=0$ for all j>1

we find:

$$C = S.$$

By substituting $S = C$ into the equation (12), we get:

$$S_t + 3SS_x + S_{xxx} = 0, \qquad (13)$$

This is (mKdV1) like equation.

IV. Schwarzian Derivative:

Solution for a constant ***S***.

The functions of a constant $S = \pm 2\lambda^2$ hence λ is a constant, are solutions of the (mKDV1) like equation (13).

Lemma [1]: Let τ_1 and τ_2 be two linearly independent solutions of the equation,

$$\frac{d^2\tau}{dz^2} + f(z) = 0, \qquad (14)$$

which are defined and holomorphic on some simply connected domain D in complex plane, then $w(z) = \tau_1(z)/\tau_2(z)$ satisfies the equation [1] [2],

$$\{w, z\} = 2f(z), \qquad (11)$$

(8)

Conversely, if $w(z)$ is a solı 6) at all points of D, then one can find two linearly holomorphic independent solutions τ_1 and τ_2 of (15) such that $w(z) = \tau_1(z)/\tau_2(z)$ in some neighborhood of $z_0 \in D$. [1].

Lemma [2]: The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, the form:

$$\left\{\frac{aw+b}{cw+d}; z\right\} = \{w; z\}, \qquad ad \neq bc,$$

where a, b, c and d are constants [1] [2].

Step 1:

For $S=-2\lambda^2$, we get,

$S=\{\varphi,x\}=-2\lambda^2$. Where $f(x)=-\lambda^2$ in (15),

and two linearly independent solutions are:

$$\Psi_1=E(t)e^{\lambda x}+F(t)e^{-\lambda x},$$

$$\Psi_2=G(t)e^{\lambda x}+H(t)e^{-\lambda x},$$

Therefore by Lemma [1] and Lemma [2], obtain:

$$\varphi(t,x)=\frac{E(t)e^{\lambda x}+F(t)e^{-\lambda x}}{G(t)e^{\lambda x}+H(t)e^{-\lambda x}},EH\neq FG \qquad (16)$$

By using the equations (10) and (11),
then:

$$C=S=-\frac{\varphi_t}{\varphi_x}=-2\lambda^2. \qquad (17)$$

Now, to find the differential equation of coefficients E(t), F(t), G(t) and H(t),

we derive $\varphi(t,x)$ in the equation (16), to get $\varphi_t(t,x)$ and $\varphi_x(t,x)$, and substituting them into the equation (17), we obtain:

$$\varphi_x=\frac{2\lambda\left(H(t)E(t)-G(t)F(t)\right)}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2},$$

and

$$\varphi_t=\frac{\left(G(t)E'(t)-E(t)G'(t)\right)}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2}e^{2\lambda x}+\frac{\left(H(t)F'(t)-F(t)H'(t)\right)}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2}e^{-2\lambda x}+\frac{\left(G(t)F'(t)-F(t)G'(t)\right)}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2}+\frac{\left(H(t)E'(t)-E(t)H'(t)\right)}{\left(G(t)e^{\lambda x}+H(t)e^{-\lambda x}\right)^2},$$

Then, the equation (17) becomes:

$$C=\frac{\left(G(t)E'(t)-E(t)G'(t)\right)}{2\lambda\left(H(t)E(t)-G(t)F(t)\right)}e^{2\lambda x}+\frac{\left(H(t)F'(t)-F(t)H'(t)\right)}{2\lambda\left(H(t)E(t)-G(t)F(t)\right)}e^{-2\lambda x}+\frac{\left(G(t)F'(t)-F(t)G'(t)\right)}{2\lambda\left(H(t)E(t)-G(t)F(t)\right)}+\frac{\left(H(t)E'(t)-E(t)H'(t)\right)}{2\lambda\left(H(t)E(t)-G(t)F(t)\right)}=2\lambda^2,$$

Then,

$$\left(G(t)E'(t)-E(t)G'(t)\right)e^{2\lambda x}+\left(H(t)F'(t)-F(t)H'(t)\right)e^{-2\lambda x}+G(t)F'(t)-F(t)G'(t)+\left(H(t)E'(t)-E(t)H'(t)\right)=4\lambda^3\left(H(t)E(t)-G(t)F(t)\right),$$

This lets us to a system of nonlinear ODE in all coefficients $E(t)$, $F(t)$,$G(t)$ a[illegible]hen:
(17)

$(i)\quad GE'-EG'=0$

$(ii)\quad HF'-FH'=0$

$(iii)\quad \left(GF'-FG'\right)+\left(HE'-EH'\right)=4i\lambda^3\left(HE-GF\right)$

particular solutions of (*i*) and (*ii*) are:
$E(t)=AG(t)$ and $F(t)=BH(t)$
where A and B are real arbitrary constants.

By using (*i*), (*ii*) and (*iii*), we have:

$$B\left(G(t)H'(t)-H(t)G'(t)\right)+A\left(H(t)G'(t)-G(t)H'(t)\right)=4\lambda^3H(t)G(t)\left(A-B\right),$$

then;

$$\frac{H'(t)}{H(t)}-\frac{G'(t)}{G(t)}=-4\lambda^3,$$

By integrating, we get:

$$\frac{H(t)}{G(t)}=\exp\left(-4\lambda^3t\right)$$

Then, equation (16), leads:

$$\varphi(t,x)=\frac{AG(t)\exp(\lambda x)+BG(t)\exp\left(-4\lambda^3t-\lambda x\right)}{G(t)\exp(\lambda x)+G(t)\exp\left(-4\lambda^3t-\lambda x\right)},$$

This leads to

$$\varphi(t,x)=\frac{Ae^{\lambda\xi_1}+Be^{-\lambda\xi_1}}{e^{\lambda\xi_1}+e^{-\lambda\xi_1}},\quad \xi_1=x+2\lambda^2$$

$$=\frac{(A+B)\cosh\lambda\xi_1+(A-B)\sinh\lambda\xi_1}{2\cosh\lambda\xi_1},$$

Then:

$$\varphi(t,x)=K_1+K_2\tanh\lambda\xi_1, \qquad (18)$$

Where K_1 and K_2 are arbitrary constants, such that $K_1=(A+B)/2$ and $K_2=(A-B)/2$.
For $K_1=0$, and by substituting the equation (18) into the equation (4), we get:

$$u_1=-i\sqrt{\frac{6}{\alpha}}\frac{-K_2\lambda^2\operatorname{sec}h^2\lambda\xi_1\tanh\lambda\xi_1}{K_2\lambda\operatorname{sec}h^2\lambda\xi_1},$$

Then: $$u_1 = i\lambda\sqrt{\frac{6}{\alpha}}\tanh\lambda\xi_1,\ \ \xi_1 = x + 2\lambda^2 t.$$

Where $u_1(x,t)$ is the first solution for (mKDV) equation (1).

Now, by the equations (4), (5), (6) and (18), we have:

$$u = \frac{i\sqrt{\frac{6}{\alpha}}K_2\lambda\sec h^2\lambda\xi_1}{K_2\lambda\tanh\lambda\xi_1} + u_1,$$

Then

$$u = i\lambda\sqrt{\frac{6}{\alpha}}\coth\lambda\xi_1\ ,\ \ \xi_1 = x + 2\lambda^2 t.$$

Where $u(t, x)$ is the second solution for (mKDV) equation (1).

Step 2:

For $S=2\lambda^2$,we have:

$S = \{\varphi, x\} = 2\lambda^2$. Where $f(x) = \lambda^2$ in (15), and linearly independent solutions are

$$\Psi_3 = E(t)e^{\lambda ix} + F(t)e^{-\lambda ix},$$

$$\Psi_4 = G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}$$

Therefore, Lemma [1] and Lemma [2] obtain:

$$\varphi(t,x) = \frac{E(t)e^{\lambda ix} + F(t)e^{-\lambda ix}}{G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}}\ , \qquad (19)$$

$$EH \neq FG,$$

By using the equations (11) and $S=C$, then:

$$C = S = -\frac{\varphi_t}{\varphi_x} = 2\lambda^2. \qquad (20)$$

Now, to find coefficients of the differential equation, E(t), F(t), G(t) and H(t), we derive $\varphi(t,x)$ in the equation (19), to get $\varphi_t(t,x)$ and $\varphi_x(t,x)$ and by substituting them into the equation(20), then:

$$C = \frac{(G(t)E'(t) - E(t)G'(t))}{2\lambda(H(t)E(t) - G(t)F(t))}e^{2\lambda ix}$$
$$+ \frac{(H(t)F'(t) - F(t)H'(t))}{2i\lambda(H(t)E(t) - G(t)F(t))}e^{-2\lambda ix}$$
$$+ \frac{(G(t)F'(t) - F(t)G'(t))}{2i\lambda(H(t)E(t) - G(t)F(t))}$$
$$+ \frac{(H(t)E'(t) - E(t)H'(t))}{2i\lambda(H(t)E(t) - G(t)F(t))} = -2\lambda^2.$$

This takes us to a system of nonlinear ODE$_s$ in all coefficients $E(t)$, $F(t)$,$G(t)$ and $H(t)$, then:

$(i)\ \ GE' - EG' = 0$

$(ii)\ \ HF' - FH' = 0$

$(iii)\ \ (GF' - FG') + (HE' - EH') = -4i\lambda^3(HE - GF)$

particular solutions of (i) and (ii) are:

$E(t) = MG(t)$ and $F(t) = NH(t)$

where M and N are real arbitrary constants.

By substituting these into (iii), we get:

$$\frac{H(t)}{G(t)} = \exp\left(4i\lambda^3 t\right),$$

Then the equation (19), leads:

$$\varphi(t,x) = \frac{MG(t)\exp(\lambda ix) + NG(t)\exp\left(4\lambda^3 it - \lambda ix\right)}{G(t)\exp(\lambda ix) + G(t)\exp\left(4\lambda^3 it - \lambda ix\right)}$$

which leads to:

$$\varphi(t,x) = \frac{Me^{\lambda i\xi_2} + Ne^{-\lambda i\xi_2}}{e^{\lambda i\xi_2} + e^{-\lambda i\xi_2}},\ \ \xi_2 = x - 2\lambda^2 t,$$
$$= \frac{(M+N)\cos\lambda\xi_2 + (M-N)\sin\lambda\xi_2}{2\cos\lambda\xi_2}.$$

Then:

$$\varphi(t,x) = K_3 + K_4\tan\lambda\xi_2, \qquad (21)$$

where K_3 and K_4 are arbitrary constants, such that $K_3 = (M+N)/2$ and $K_4 = (M-N)/2$

For $K_3=0$,by substituting the equation (21) into the equation (4), we get:

$$\hat{u}_1 = -i\sqrt{\frac{6}{\alpha}}\frac{K_4\lambda^2\sec^2\lambda\xi_2\tan\lambda\xi_2}{K_4\lambda\sec^2\lambda\xi_2},$$

Then:

$$\hat{u}_1 = -i\lambda\sqrt{\frac{6}{\alpha}}\tan\lambda\xi_2,\ \ \xi_2 = x - 2\lambda^2 t.$$

Where $\hat{u}_1(x,t)$ is the third solution for (mKDV) equation (1).

Now, by the equations (4), (5), (6) and (21), we get:

$$\hat{u} = \frac{i\sqrt{\frac{6}{\alpha}}K_4\lambda\sec^2\lambda\xi_2}{K_4\lambda\sec^2\lambda\xi_2} + \hat{u}_1,$$

Then:

$$\hat{u} = i\lambda\sqrt{\frac{6}{\alpha}}\cot\lambda\xi_2\ ,\xi_2 = x - 2\lambda^2 t.$$

Where $\hat{u}(x,t)$ is the fourth solution for (mKDV) equation (1).

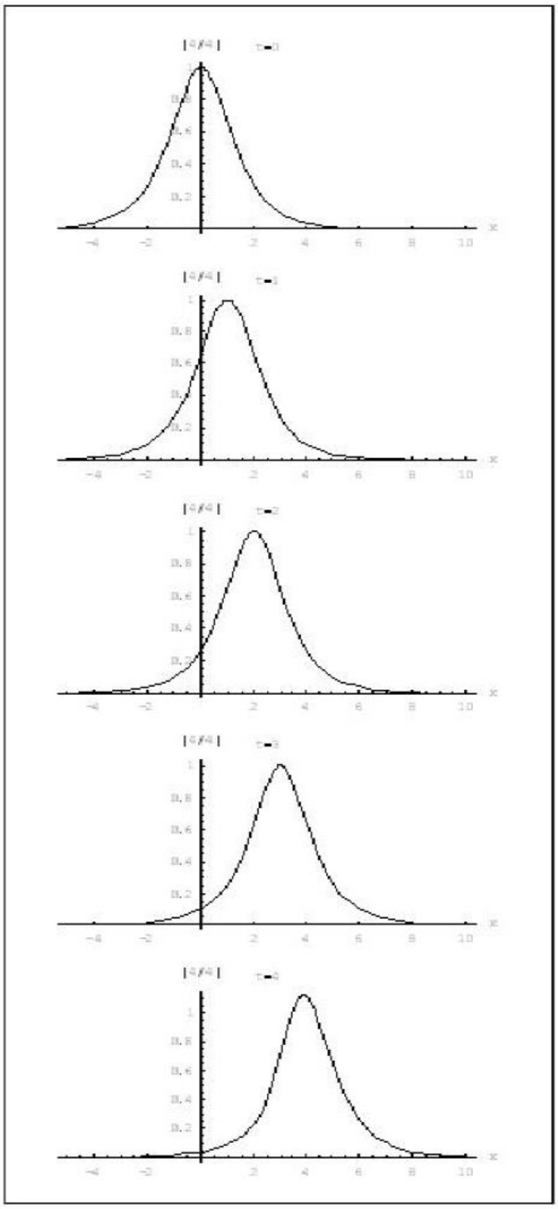

Fig. Shows the behavior of the solutions for mKdV equation at different times.

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