International Journal of Engineering and Applied Sciences (IJEAS)
ISSN: 2394-3661, Volume-6, Issue-9, September 2019

A Unitary Quantum Walk Constructed Directly from Quantum Bernoulli Noises

Yin Zhang, Caishi Wang, Jia Shi, Lixia Zhang

***Abstract*— In this paper, we present a new model of quantum walk, which is constructed directly from quantum Bernoulli noises. We examine its basic properties, and two representation results are obtained of the walk.**

***Index Terms*— Unitary representation; Quantum walk; Quantum Bernoulli noise MSC(2014):—, 81S25; 60G50; 81P68**

I. INTRODUCTION

As quantum analogs of classical random walks, quantum walks were introduced almost three decade ago [12,13], and they have found numerous applications in quantum information processing and communication science [15].

Recently, by using quantum walks with finite number degrees of freedom, Konno has introduced a new time-series model and shown its interesting properties [16].

Quantum Bernoulli noises (QBN) are the family of annihilation and creation operators acting on Bernoulli functionals, which satisfy an anti-commutation relation (ACR) in equal time, and can provide an approach to the effects of environment on an open quantum system. In 2016, Wang and Ye introduced a quantum walk model in terms of quantum Bernoulli noises [14], and showed its interesting properties, one of which says that at some special initial states this walk has the same limit probability distribution as the classical random walk .

The quantum walk introduced by Wang and Ye actually uses the following operator pairs $L_k = \frac{1}{2}\left(\partial_k^* + \partial_k + I\right)$ an $R_k = \frac{1}{2}\left(\partial_k^* + \partial_k - I\right)$ as its coin operators, where ∂_k and ∂_k^* are annihilation and creation operators acting on Bernoulli functionals, which form quantum Bernoulli noises. In this paper, we present a new model of quantum walk, which is constructed directly from quantum Bernoulli noises. We examine its basic properties, and two representation results are obtained of the walk.

Notation and conventions. Throughout this paper, Z always denotes the set of all integers, while N means the Set of all nonnegative integers. We denote by Γ the finite power set of N, namely

$$\Gamma = \{\sigma \mid \sigma \subset N \text{ and } \#\sigma < \infty\}, \qquad (1.1)$$

where $\#\sigma$ means the cardinality of σ. Unless otherwise stated, letters like j, k and n stand for nonnegative integers, namely elements of N.

II. QUANTUM BERNOULLI NOISES

In this section, we briefly recall quantum Bernoulli noises (see [8] for details).

Let $\Omega = \{-1,1\}^N$ be the set of all mapping ω: $N \mapsto \{-1,1\}$, and $(\zeta_n)_{n\geq 0}$ the sequences of canonical projections on Ω given by

$$\zeta_n(\omega) = \omega(n), \qquad \omega \in \Omega. \qquad (2.1)$$

Let $\Im$ be the σ-field on Ω generated by the sequences $(\zeta_n)_{n\geq 0}$, and $(p_n)_{n\geq 0}$ a given sequences of positives numbers with the property that $0 < p_n < 1$ for all $n \geq 0$. Then there exists a unique probability measure P on $\Im$ such that

$$P \circ (\zeta_{n1}, \zeta_{n2}, \ldots \zeta_{nk})^{-1}\{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)\} = \prod_{j=1}^{k} p_j^{\frac{1+\varepsilon_j}{2}} (1-p_j)^{\frac{1-\varepsilon_j}{2}} \qquad (2.2)$$

for $n_j \in N, \varepsilon_j \in \{-1,1\}$ $(1 \leq j \leq k)$ with $n_i \neq n_j$ when $i \neq j$ and $k \in N$ with $k \geq 1$. Thus one has a probability measure space $(\Omega, \Im, P)$, which is referred to as the Bernoulli space and random variables on it are known as Bernoulli functionals.

Let $Z = (Z_n)_{n\geq 0}$ be the sequences of Bernoulli functionals defined by

$$Z_n = \frac{\zeta_n + q_n + p_n}{2\sqrt{q_n p_n}}, \qquad n \geq 0, \qquad (2.3)$$

where $q_n = 1 - p_n$. Clearly $Z = (Z_n)_{n\geq 0}$ is an independent sequence of random variables on the probability measure space $(\Omega, \Im, P)$.

Let H be the space of square integrable complex-valued Bernoulli functionals, namely

$$H = L^2(\Omega, \Im, P), \qquad (2.4)$$

we denote by $\langle \cdot, \cdot \rangle$ the usual inner product of the space H, and by $\|\cdot\|$ the corresponding norm. It is known that Z has the chaotic representation property, which means that the family $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms an orthonormal basis (ONB) of H, where $Z_\varnothing = 1$ and

Yin Zhang, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, China, Mobile No. 18368918324
Caishi Wang, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China.
Jia Shi, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China.
Lixia Zhang, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China.

$$Z_\sigma = \prod_{j\in\sigma} Z_j, \qquad \sigma\in\Gamma, \sigma\neq\emptyset. \qquad (2.5)$$

In the following, we call $\{Z_\sigma \mid \sigma\in\Gamma\}$ the canonical ONB of H. Clearly H is infinitely dimensional since $\{Z_\sigma \mid \sigma\in\Gamma\}$ is countably infinite.

Lemma 2.1. [8] For each $k\in N$, there exists a bounded operator ∂_k on H such that

$$\partial_k Z_\sigma = 1_\sigma(k) Z_{\sigma\setminus k}, \qquad \sigma\in\Gamma, \qquad (2.6)$$

where $\sigma\setminus k = \sigma\setminus\{k\}$ and $\mathbf{1}_\sigma(k)$ the indicator of σ as a subset of N.

Lemma 2.2. [8] Let $k\in N$. Then ∂_k^*, the adjoint of operator ∂_k, has following property:

$$\partial_k^* Z_\sigma = [1 - 1_\sigma(k)] Z_{\sigma\cup k}, \qquad \sigma\in\Gamma, \qquad (2.7)$$

where $\sigma\cup k = \sigma\cup\{k\}$.

Remark 2.1. The family $\{\partial_k, \partial_k^*\}_{k\geq 0}$ is called quantum Bernoulli noises, while ∂_k and ∂_k^* are known as the annihilation and creation operators, respectively.

Lemma 2.3. [8] Let $k, l\in N$. Then, it holds true that

$$\partial_k\partial_l = \partial_l\partial_k, \quad \partial_k^*\partial_l^* = \partial_l^*\partial_k^*, \quad \partial_k^*\partial_l = \partial_l\partial_k^*, (k\neq l) \qquad (2.8)$$

and

$$\partial_n\partial_n = \partial_n^*\partial_n^* = 0 \;\; and \;\; \partial_n\partial_n^* + \partial_n^*\partial_n = I. \qquad (2.9)$$

Here I is the identity operator on H.

III. DEFINITION OF THE QUANTUM WALK

In this section, we define our quantum walk and examine its basic properties.

Let $l^2(Z,H)$ be the space of square summable functions defined on Z and valued in H, namely

$$l^2(Z,H) = \{\Phi : Z\to H \mid \sum_{x=-\infty}^{\infty} \|\Phi(x)\|^2 < \infty\}. \qquad (3.1)$$

Then $l^2(Z,H)$ remains a complex Hilbert space, whose inner product $\langle\cdot,\cdot\rangle_{l^2(Z,H)}$ is given by

$$\langle\Phi,\Psi\rangle_{l^2(Z,H)} = \sum_{x=-\infty}^{\infty} \langle\Phi(x),\Psi(x)\rangle, \quad \Phi,\Psi\in l^2(Z,H), \qquad (3.2)$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of H as indicated in section II. By convention, we denote by $\|\cdot\|_{l^2(Z,H)}$ the norm induced by $\langle\cdot,\cdot\rangle_{l^2(Z,H)}$.

We now state the definition of our quantum walk as follows. The walk takes $l^2(Z,H)$ as its state space and its states satisfy the following evolution equation

$$\Phi_{n+1}(x) = \partial_n\Phi_n(x-1) + \partial_n^*\Phi_n(x+1), \qquad (3.3)$$

$x\in Z$, $n\geq 0$, where Φ_n denotes the state of the walk at time $n\geq 0$ and in particular Φ_0 denotes the initial state of the walk.

It is well known that $l^2(Z,H)\cong l^2(Z)\otimes H$. This just means that $l^2(Z)$ describes the position of the walk, while H describes its internal degrees of freedom. By convention, H is called the coin space of the walk. Clearly our walk has infinitely many internal degrees of freedom since its coin space H is infinitely dimensional.

Theorem 3.1. Let $n\geq 0$ and $\Phi\in l^2(Z,H)$. If a function $\Psi : Z\to H$ satisfies that

$$\Psi(x) = \partial_n\Phi(x-1) + \partial_n^*\Phi(x+1), \quad x\in Z. \qquad (3.4)$$

Then $\Psi\in l^2(Z,H)$ and $\|\Psi\|_{l^2(Z,H)} = \|\Phi\|_{l^2(Z,H)}$.

Proof. According to formula $\partial_n\partial_n = \partial_n^*\partial_n^* = 0$ and $\partial_n\partial_n^* + \partial_n^*\partial_n = I$, we have

$$\begin{aligned}
\sum_{x=-\infty}^{\infty} \|\Psi\|^2 &= \sum_{x=-\infty}^{\infty} \langle \partial_n\Phi(x-1) + \partial_n^*\Phi(x+1), \partial_n\Phi(x-1) + \partial_n^*\Phi(x+1)\rangle \\
&= \sum_{x=-\infty}^{\infty} [\langle\Phi(x-1), \partial_n^*\partial_n\Phi(x-1)\rangle + \langle\Phi(x+1), \partial_n\partial_n^*\Phi(x+1)\rangle] \\
&= \sum_{x=-\infty}^{\infty} \langle\Phi(x), \partial_n^*\partial_n\Phi(x)\rangle + \sum_{x=-\infty}^{\infty} \langle\Phi(x), \partial_n\partial_n^*\Phi(x)\rangle \\
&= \sum_{x=-\infty}^{\infty} \langle\Phi(x), (\partial_n^*\partial_n + \partial_n\partial_n^*)\Phi(x)\rangle \\
&= \sum_{x=-\infty}^{\infty} \|\Phi\|^2,
\end{aligned}$$

which together with $\Phi\in l^2(Z,H)$ implies that

$$\Psi\in l^2(Z,H) \text{ and } \|\Psi\|_{l^2(Z,H)} = \|\Phi\|_{l^2(Z,H)}.$$

Theorem 3.2. Let $n\geq 0$. Then there exists a unitary self-adjoint operator $w_n : l^2(Z,H)\to l^2(Z,H)$ such that

$$[w_n\Phi](x) = \partial_n\Phi(x-1) + \partial_n^*\Phi(x+1), \;\; x\in Z. \qquad (3.5)$$

Proof. For each $\Phi\in l^2(Z,H)$, we define the function $\Psi_\Phi : Z\to H$ as

$$\Psi_\Phi(x) = \partial_n\Phi(x-1) + \partial_n^*\Phi(x+1), \qquad x\in Z,$$

By Theorem 3.1, we have $\Psi_\Phi\in l^2(Z,H)$ and

$$\|\Psi_\Phi\|_{l^2(Z,H)} = \|\Phi\|_{l^2(Z,H)}$$

Thus we can define an isometric operator

$$w_n : l^2(Z,H)\to l^2(Z,H)$$

such that $w_n\Phi = \Psi_\Phi, \quad \Phi\in l^2(Z,H)$,

which means the w_n satisfies (3.5).

Next, we consider the adjoint w_n^* of w_n. Let $\Phi\in l^2(Z,H)$. Then, for any $x\in Z$ and $\sigma\in\Gamma$, we define a function $\Phi^\sigma\in l^2(Z,H)$ such that

$$\Phi^\sigma(y) = \begin{cases} Z_\sigma, & y = x \\ 0, & y\neq x, y\in Z. \end{cases}$$

Which gives

$$\begin{aligned}\langle [w_n{}^*\Phi](x), Z_\sigma \rangle &= \langle w_n{}^*\Phi, \Phi^\sigma \rangle_{l^2(Z,H)} \\ &= \langle \Phi, w_n \Phi^\sigma \rangle_{l^2(Z,H)} \\ &= \langle \Phi(x+1), \partial_n Z_\sigma \rangle + \langle \Phi(x-1), \partial_n^* Z_\sigma \rangle \\ &= \langle \partial_n^* \Phi(x+1) + \partial_n \Phi(x-1), Z_\sigma \rangle.\end{aligned}$$

Thus w_n is self-adjoint.

Finally we prove that w_n is a unitary operator. Since w_n is an isometric operator, thus we need only to prove that $w_n w_n{}^* = I$. Let $\Phi \in l^2(Z,H)$ and $\Psi = w_n{}^*\Phi$. Then,

$$\begin{aligned}[w_n\Psi](x) &= \partial_n \Psi(x-1) + \partial_n^* \Psi(x+1) \\ &= \partial_n \partial_n^* \Phi(x) + \partial_n^* \partial_n \Phi(x) \\ &= (\partial_n \partial_n^* + \partial_n^* \partial_n)\Phi(x) \\ &= \Phi(x),\end{aligned}$$

where $x \in Z$. Thus $w_n \Psi = \Phi$, which together with $\Psi = w_n{}^*\Phi$ and the arbitrariness of $\Phi \in l^2(Z,H)$ implies that $w_n w_n{}^* = I$. This completes the proof.

By using Theorem 3.2, one can easily come to the next result.

Theorem 3.3. Our quantum walk has a unitary representation of the following form

$$\Phi_n = (\prod_{k=0}^{n-1} w_k)\Phi_0, \qquad n \geq 1, \qquad (3.6)$$

where $w_k : l^2(Z,H) \to l^2(Z,H)$ is a unitary operator as indicated in (3.5).

Proof. Combining (3.3) with (3.5), we simply get

$$\Phi_n(x) = [w_{n-1}\Phi_{n-1}](x), \qquad x \in Z, \qquad n \geq 1,$$

which implies (3.6).

IV. REPRESENTATION IN TENSOR SPACE.

It is well known that $l^2(Z,H) \cong l^2(Z) \otimes H$. In this part, we show a representation of our walk in $l^2(Z) \otimes H$.

Definition 4.1. For $f \in l^2(Z)$ *and* $\xi \in H$, we define a function $\Phi_{f,\xi} : Z \to H$ as

$$\Phi_{f,\xi} = f(x)\xi, \qquad x \in Z. \qquad (4.1)$$

It is well know that $\{\Phi_{f,\xi} \mid f \in l^2(Z), \xi \in H\}$ is total in $l^2(Z,H)$.

Lemma 4.1. There exist a unitary linear isomorphism mapping $J : l^2(Z,H) \to l^2(Z) \otimes H$ satisfying that

$$J\Phi_{f,\xi} = f \otimes \xi, \qquad f \in l^2(Z), \quad \xi \in H. \qquad (4.2)$$

This lemma builds a bridge from $l^2(Z,H)$ to $l^2(Z) \otimes H$. Through this lemma we come to another conclusion.

Lemma 4.2. Define the operator $\tau : l^2(Z) \to l^2(Z)$ as :

$$[\tau f](x) = f(x-1), \qquad x \in Z, \ f \in l^2(Z). \qquad (4.3)$$

Then τ is a unitary operator and

$$\tau = \sum_{x \in Z} |\delta_x >< \delta_{x-1}|,$$

where $\{\delta_x \mid x \in Z\}$ is a OBN in $l^2(Z)$.

The following theorem describes the unitary operator on $l^2(Z) \otimes H$ that corresponds to the operator w_n on $l^2(Z,H)$.

Theorem 4.1. For each $n \geq 0$, there exist a unitary operator T_n on $l^2(Z) \otimes H$, such that

$$\begin{aligned}T_n &= J w_n J^{-1} = \tau \otimes \partial_n + \tau^{-1} \otimes \partial_n^* \\ &= \sum_{x \in Z} [|\delta_x >< \delta_{x-1}| \otimes \partial_n + |\delta_x >< \delta_{x+1}| \otimes \partial_n^*]. \qquad (4.4)\end{aligned}$$

Proof. For each $f \otimes \xi \in l^2(Z) \otimes H$, we have $J^{-1}(f \otimes \xi) = \Phi_{f,\xi}$, thus for all $x \in Z$,

$$\begin{aligned}w_n \Phi_{f,\xi}(x) &= \partial_n \Phi_{f,\xi}(x-1) + \partial_n^* \Phi_{f,\xi}(x+1) \\ &= \partial_n f(x-1)\xi + \partial_n^* f(x+1)\xi \\ &= f(x-1)\partial_n \xi + f(x+1)\partial_n^* \xi \\ &= [(\tau f)(x)]\partial_n \xi + [(\tau^{-1} f)(x)]\partial_n^* \xi \\ &= \Phi_{\tau f, \partial_n \xi}(x) + \Phi_{\tau^{-1} f, \partial_n^* \xi}(x) \\ &= [\Phi_{\tau f, \partial_n \xi} + \Phi_{\tau^{-1} f, \partial_n^* \xi}](x).\end{aligned}$$

Then $w_n \Phi_{f,\xi} = \Phi_{\tau f, \partial_n \xi} + \Phi_{\tau^{-1} f, \partial_n^* \xi}$, which implies that

$$\begin{aligned}T_n(f \otimes \xi) &= J w_n J^{-1}(f \otimes \xi) = J w_n \Phi_{f,\xi} \\ &= J(\Phi_{\tau f, \partial_n \xi} + \Phi_{\tau^{-1} f, \partial_n^* \xi}) \\ &= \tau f \otimes \partial_n \xi + \tau^{-1} f \otimes \partial_n^* \xi \\ &= (\tau \otimes \partial_n)(f \otimes \xi) + (\tau^{-1} \otimes \partial_n^*)(f \otimes \xi) \\ &= (\tau \otimes \partial_n + \tau^{-1} \otimes \partial_n^*)(f \otimes \xi).\end{aligned}$$

Therefore $T_n = J w_n J^{-1} = \tau \otimes \partial_n + \tau^{-1} \otimes \partial_n^*$.

We now describe our walk in the tensor space $l^2(Z) \otimes H$. Let $F_n \in l^2(Z) \otimes H$ be such that

$F_n = J\Phi_n$, where Φ_n is the state of the walk at time $n \geq 0$.

Then one has the following relations

$$F_{n+1} = T_n F_n, \qquad n \geq 0. \qquad (4.5)$$

Theorem 4.2. The quantum walk has the following representation in $l^2(Z) \otimes H$:

$$F_n = (\prod_{k=0}^{n-1} \sum_{x \in Z} [|\delta_x >< \delta_{x-1}| \otimes \partial_k + |\delta_x >< \delta_{x+1}| \otimes \partial_k^*]) F_0, \; n \geq 1. \quad (4.6)$$

Proof. From (4.4) and (4.5), we can immediately get (4.6).

ACKNOWLEDGEMENT

This work is supported by National Natural Science Foundation of China (Grant No. 11461061 and Grant No. 11861057).

REFERENCES

[1] G. Grimmett, S. Janson and P.F. Scudo, Weak limits for quantum random walks, Phys. Rev. E 69 (2004),026119.

[2] J. Kempe, Quantum random walks: an introductory overview, Contemp. Phys. 44 (2003), No. 4, 307C327.

[3] N. Konno, A new type of limit theorems for one-dimensional quantum random walk, J. Math. Soc. Jpn 57 (2005), 1179-1195.

[4] N. Konno, T. Luczak and E. Segawa, Limit measure of in homogeneous discrete-time quantum walks in one dimension, Quantum Inf. Process 12 (2013), 33-53.

[5] N. Konno, Quantum walks, In: U. Franz, M. Schürmann (eds.) Quantum Potential Theory, Lecture Notes in Mathematics, vol. 1954, pp. 309-452. Springer, Heidelberg (2008)

[6] N. Privault, Stochastic analysis of Bernoulli process, Probab. Surv. 5 (2008), 435.

[7] S.E. Venegas-Andraca, Quantum walks: a comprehensive review. Quantum Inf. Process. 11 (2012), 1015-1106.

[8] C. Wang, H. Chai and Y. Lu, Discrete-time quantum Bernoulli noises, J. Math. Phys. 51 (2010), no. 5, 053528.

[9] C. Wang and J. Chen, Characterization theorems for generalized functional of Discrete-time normal martingale, Journal of Function Spaces, Volume 2015, Article ID 714745.

[10] C. Wang. Y. Lu and H. Chai, An alternative approach to Privault's discrete- time chaotic calculus, J. Math. Anal. Appl. 373 (2011), 643–654.

[11] W. Feller, in An Introduction to Probability Theory and Its Applications Vol.1 (Wiley, New York 1968).

[12] Y. Aharonov, L. Davidovich and N. Zagury, Quantum random walks, Phys. Rev. A 48 (1993), 1687-1690.

[13] J. Kempe, Contemp. Phys. 44, 307 (2003).

[14] C. Wang and X. Ye, Quantum walk in terms of quantum Bernoulli noises, Quantum Inf. Process. 15 (2016), no. 5, 1897-1908.

[15] I. Sinayskiy and F. Petruccione, Phys. J. Special Topics 227, 1869-1883 (2019).

[16] N. Konno, A new time-series model based on quantum walk, Quantum Stud.: Math. Found. 6 (2019), 61-72.

Yin Zhang, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China, Mobile 86- 18368918324.

Caishi Wang, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China.

Jia Shi, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China.

Lixia Zhang, School of Mathematics and Statistics, Northwest Normal University, Lanzhou, China.