Two Regularization Methods Simultaneous Determination of Initial Value and the Source Term of the Heat Conduction Problem

Xue-Juan Ren, Cheng-Bo Jin

***Abstract*— In this paper, we investigate the inverse problem of determining a heat source and initial temperature simultaneously in a parabolic equation. Since, it is an ill-posed problem, we use the mollification method to solve source and use the modified method to solve initial value. And we obtain an optimal error estimate from an a priori parameter choice rule. Finally, two examples show that the proposed method is effective and efficient.**

***Index Terms*— Ill-posedness, Backward heat conduction problem, Inverse source, Simultaneous determination, Mollification method, modified method, Error estimate.**

I. INTRODUCTION

In this paper, we consider the following inverse problem: to find a pair of functions $(u(x,0), f(x))$ which satisfy the heat equation in a strip domain, where data is given at the final time $t=T(T>0)$ and middle time $\mathrm{t}=\mathrm{t}_0(0\le t\le T)$ is sought:

$$\begin{cases} u_t(x,t)=u_{xx}(x,t)+f(x), x\in\Re, t\in[0,T], \\ u(x,T)=\psi(x), & x\in\Re, t\in[0,T], \\ u(x,t_0)=\varphi(x), & x\in\Re, t_0\in[0,T]. \end{cases} \tag{1.1}$$

We want to retrieve the temperature distribution $u(x,0)$ and identify $f(x)$ from the additional data $\psi(x)$ and $\varphi(x)$ for $[0,T]$. Since the data $\psi(x)$ and $\varphi(x)$ in $L^2(\Re)$ is based on (physical) observations, there will be measurement errors, so we assume the measured data function $\psi_\delta(x)$ and $\varphi_\delta(x)$ all in $L^2(\Re)$, satisfying

$$\|\psi_\delta-\psi\|\le\delta,\ \|\varphi_\delta-\varphi\|\le\delta, \tag{1.2}$$

where $\|\cdot\|$ denotes the $L^2(\Re)$, and the constant $\delta>0$ represents a noise level. Let $\hat{\psi}(\xi)$ denote the Fourier transform of $\psi_\delta(x)\in L^2(\Re)$ and define it by

$$\hat{\psi}(\xi):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\xi x}\psi(x)dx, \tag{1.3}$$

because $u(x,0)$ and $f(x)$ are unknown, in order to solve initial value and heat source, let's

$$u(x,\mathrm{t})=v(x,\mathrm{t})+w(x,\mathrm{t}), \tag{1.4}$$

we can divide the equations into the following two equations:

$$\begin{cases} v_t(x,t)=v_{xx}(x,t)+f(x), x\in\Re, t\in[0,T], \\ v(x,0)=0, & x\in\Re, t\in[0,T], \\ u(x,t_0)=\varphi(x), & x\in\Re, t_0\in[0,T]. \end{cases} \tag{1.5}$$

$$\begin{cases} w_t(x,t)=w_{xx}(x,t), & x\in\Re, t\in[0,T], \\ w(x,T)=u(x,T)-v(x,T), & x\in\Re, t\in[0,T]. \end{cases} \tag{1.6}$$

We use the Fourier transform to solve the exact solution of the source term and initial value of the equation (1.5) and (1.6), as follow

$$\hat{f}(\xi)=\frac{\xi^2}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi), \tag{1.7}$$

$$\hat{v}(\xi,\ t)=\frac{1-e^{-\xi^2 t}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi), \tag{1.8}$$

$$\hat{w}(\xi,\ t)=e^{\xi^2(T-t)}\left[\hat{\psi}(\xi)-\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi)\right], \tag{1.9}$$

we can be concluded from (1.4)

$$\hat{u}(\xi,\ t)=\frac{1-e^{-\xi^2 t}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi)+e^{\xi^2(T-t)}\left[\hat{\psi}(\xi)-\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi)\right], \tag{1.10}$$

so

$$\hat{u}(\xi,0)=e^{\xi^2 T}\left[\hat{\psi}(\xi)-\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi)\right], \tag{1.11}$$

let

$$\hat{u}(\xi,0)=\hat{u}^{\psi}(\xi,0)-\hat{u}^{\varphi}(\xi,0), \tag{1.12}$$

where

$$\hat{u}^{\psi}(\xi,0)=e^{\xi^2 T}\hat{\psi}(\xi), \tag{1.13}$$

and

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$$\hat{u}^{\varphi}(\xi,0)=e^{\xi^2 T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(\xi). \tag{1.14}$$

From the right hand side of (1.7), we know that the unbounded function ξ^2 goes to infinity. So ξ^2 is a magnifying factor of $\hat{\varphi}(\xi)$. Moreover, function $e^{\xi^2(T-t_0)}$ approaches to infinity. When $|\xi|\to\infty$ in (1.11). Therefor, the exact date function $\hat{\psi}(\xi)$ and $\hat{\varphi}(\xi)$ must decay rapidly as $|\xi|\to\infty$. Small errors in high-frequency components can blow up and completely destroy the solution for $0\le t\le T$. However, the measured date function $\hat{\psi}(\xi)$ and $\hat{\varphi}(\xi)$ which are merely in $L^2(\Re)$, does not possess such a decay property in general. So the problem (1.1) is mildly ill-posed. In the following section, we will propose another different regularization methods to deal with the ill-posed problem.

In this years, most of the published works in this field used regularization methods, for example the Fourier regularization method [1, 2, 3], the simplified Tikhonov regularization [4, 5, 6], the wavelet dual least squares method [7], the modified regularization method (MRD) in [8, 9, 10], the Quasi-reversibility regularization method [11, 12], the mollification regularization method [13, 14]. In (1.1), no solution, which satisfies the heat conduction equation and the final data, exists, even if a solution did exist, it would not be continuously dependent on the final data, see [15]. So, the BHCP is impossible to solve using classical numerical methods and needs special techniques, we can see [16, 17]. For simultaneous determination of source term and initial distribution in bounded domain, we can refer to [18, 19, 20, 21].

The outline of this paper is as follows. Section 2 ,a mollification method to deal with the source term (1.7) and a valid error estimate is given. In Section 3, we give the regularization solution by using the modified regularization and give the stable error estimate between the regularization solution and the exact solution. In Section 4, some numerical examples are proposed to show the effectiveness of this tow method.

II. THE MOLLIFICATION METHOD FOR $f(x)$

In this part, the error estimate of the mollification regularization method will be given for the ill-posed problem(1.5)under the a priori parameter choice rule.before doing it, we must impose an a priori bound on the input data, i.e.,

$$\|f(\cdot)\|_{H^p(\Re)}\le E\ ,p>0, \tag{2.1}$$

where $E_1>0$ is a constant, and $\|\cdot\|_{H^p(\Re)}$ denotes the norm in the Sobole space. Let the following Gauss function as the mollifier kernel[23],

$$g_\mu(x):=\frac{1}{\mu\sqrt{\pi}}e^{-\frac{x^2}{\mu^2}}, \tag{2.2}$$

where $\mu>0$ is a constant. Define operator g_μ as

$$\hat{g}_\mu(\xi):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\xi x}g_\mu(x)dx=\frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2\xi^2}{4}}, \tag{2.3}$$

let operator G_μ as

$$G_\mu f(x):=g_\mu * f(x)$$
$$=\int_{-\infty}^{\infty}g_\mu(s)f(x-s)ds=\int_{-\infty}^{\infty}g_\mu(x-s)f(s)ds, \tag{2.4}$$

for $f(x)\in L^2(\Re)$. We use the new problem of searching its approximation $f_\delta^\mu(x)$ to replace the exact solution (1.7) of problem (1.1), which is defined by

$$f_\mu^\delta(x):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}\hat{g}_\mu * \hat{f}^\delta(\xi)d\xi, \tag{2.5}$$

where $\mu>0$ is a regularization parameter, use the exact solution (1.7) and by elementary calculations, we get

$$f_\mu^\delta(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ix\xi}\frac{\xi^2}{(1-e^{-\xi^2 t_0})e^{\frac{\mu^2\xi^2}{4}}}\hat{\varphi}_\delta(\xi)d\xi, \tag{2.6}$$

so

$$\hat{f}_\mu^\delta(\xi)=\frac{\xi^2}{(1-e^{-\xi^2 t_0})e^{\frac{\mu^2\xi^2}{4}}}\hat{\varphi}_\delta(\xi). \tag{2.7}$$

he valid error estimate of this section is the following theorem.

Theorem 2.1. Let $f(x)$ be the exact solution of (1.1) given by(1.7), and $f_\delta^\mu(x)$ be its regularization approximation given by(2.7). Let assumptions (1.2) and the priori condition (2.1) hold. If we select

$$\mu=\left(\frac{\delta}{E}\right)^{\frac{1}{p+2}}, \tag{2.8}$$

We will get the following error estimates

$$\left\|f(\cdot)-f_\mu^\delta(\cdot)\right\|$$
$$\le 4\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}(1/4\max\{(\frac{\delta}{E})^{\frac{2-p}{p+2}},1\}+1). \tag{2.9}$$

The method of proof is similar to the literature [23], which readers can refer to it.

III. A MODIFIED METHOD FOR U(X,0)

In this section ,we will analyzes the ill-posedness of BHCP (1.6). As an alternative way, we discuss the possibility of modifying (1.6) to obtain a stable approximation, i.e., we' ll consider the following problem

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + \mu w_{xxt}(x,t), & x\in\Re, t\in[0,T], \\ w(x,T) = u_\delta(x,T) - v_\delta(x,T), & x\in\Re, t\in[0,T]. \end{cases} \tag{3.1}$$

by taking the Fourier transform with respect to the space variable x in (3.1), we have

$$\begin{cases} \hat{w}_\mu^\delta(\xi,t) = -\dfrac{\xi^2}{1+\mu\xi^2}\hat{w}(\xi,t), & \xi\in\Re, t\in[0,T], \\ \hat{w}_\mu^\delta(\xi,T) = \hat{u}_\delta(\xi,T) - \hat{v}_\delta(\xi,T), & \xi\in\Re, t\in[0,T]. \end{cases} \tag{3.2}$$

the formal solution of (3.2) is also easily seen to be

$$\hat{w}_\mu^\delta(\xi,t) = e^{\frac{\xi^2}{1+\mu\xi^2}(T-t)}\left[\hat{\psi}_\delta(\xi) - \frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}_\delta(\xi)\right], \tag{3.3}$$

we can obtain the following regularization solution of $u(x,t)$ by (1.4)

$$\hat{u}_\mu^\delta(\xi,t) = \frac{1-e^{-\xi^2 t}}{1-e^{-\xi^2 t_0}}\hat{\varphi}_\delta(\xi) + e^{\frac{\xi^2}{1+\mu\xi^2}(T-t)}\left[\hat{\psi}_\delta(\xi) - \frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}_\delta(\xi)\right], \tag{3.4}$$

so, we can get the initial value regularization solution

$$\hat{u}_\mu^\delta(\xi,0) = e^{\frac{\xi^2}{1+\mu\xi^2}T}\hat{\psi}_\delta(\xi) - e^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}_\delta(\xi), \tag{3.5}$$

let

$$\hat{u}_\mu^\delta(\xi,0) = \hat{u}_\mu^{\psi_\delta}(\xi,0) - \hat{u}_\mu^{\varphi_\delta}(\xi,0), \tag{3.6}$$

where

$$\hat{u}_\mu^{\psi_\delta}(\xi,0) = e^{\frac{\xi^2}{1+\mu\xi^2}T}\hat{\psi}_\delta(\xi), \tag{3.7}$$

and

$$\hat{u}_\mu^{\varphi_\delta}(\xi,0) = e^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}_\delta(\xi), \tag{3.8}$$

In the next, we're going to give an error estimate about $u(x,0)$ and $u_\mu^\delta(x,0)$. However, before doing such a property we need to impose an a priori bound on the solution of problem (1.1) at t = 0, i.e.,

$$\|u^\psi(\cdot,0)\|_{H^p(\Re)} \le E,\quad \|u^\varphi(\cdot,0)\|_{H^p(\Re)} \le E, p>0. \tag{3.9}$$

where E is a finite positive constant. And $\|\cdot\|_{H^p(\Re)}$ denotes the norm in the Sobolev space.

Next an important theorem for this part

Theorem 3.1. Let $u(x,0)$ be the exact solution of (1.1) given by(1.11), and $u_\delta^\mu(x,0)$ be its regularization approximation given by(3.5). Let assumptions (1.2) and the priori condition (3.9) hold. If we select

$$\mu = \frac{T}{\ln\left(\frac{E}{\delta}\ln\left(\frac{E}{\delta}\right)^{-2p}\right)}, \tag{3.10}$$

we will get the following error estimates

$$\|u_\mu^\delta(\cdot,0) - u(\cdot,0)\| \le \left(\frac{3}{(\ln\frac{E}{\delta})^{2p}} + 2\varepsilon\right)E. \tag{3.11}$$

where $\varepsilon = \max\left\{\mu^{\frac{p}{4}}, T\mu^{\frac{p}{4}}, T\mu\right\}$.

Proof. Due to (1.2), (1.13), (1.14), (3.7), (3.8) and (3.10) using the a priori assumption (3.9), we get

$$\begin{aligned} \|u_\mu^\delta(x,0) - u(x,0)\| &= \|\hat{u}_\mu^\delta(\xi,0) - \hat{u}(\xi,0)\| \\ &= \|(\hat{u}_\mu^{\psi_\delta}(\xi,0) - \hat{u}_\mu^{\varphi_\delta}(\xi,0)) - (\hat{u}^\psi(\xi,0) - \hat{u}^\varphi(\xi,0))\| \\ &\le \|\hat{u}_\mu^{\psi_\delta}(\xi,0) - \hat{u}^\psi(\xi,0)\| + \|\hat{u}_\mu^{\varphi_\delta}(\xi,0) - \hat{u}^\varphi(\xi,0)\| \end{aligned}$$

Next, we can divide the calculation into two parts

$$\begin{aligned} I_1 &= \|\hat{u}_\mu^{\psi_\delta}(\xi,0) - \hat{u}^\psi(\xi,0)\| \\ &\le \|\hat{u}_\mu^{\psi_\delta}(\xi,0) - \hat{u}_\mu^\psi(\xi,0)\| + \|\hat{u}_\mu^\psi(\xi,0) - \hat{u}^\psi(\xi,0)\| \\ &\le \left\|e^{\frac{\xi^2}{1+\mu\xi^2}T}\hat{\psi}_\delta - e^{\frac{\xi^2}{1+\mu\xi^2}T}\hat{\psi}\right\| + \left\|e^{\frac{\xi^2}{1+\mu\xi^2}T}\hat{\psi} - e^{\xi^2 T}\hat{\psi}\right\| \\ &= \left\|e^{\frac{\xi^2}{1+\mu\xi^2}T}(\hat{\psi}_\delta - \hat{\psi})\right\| + \left\|\left(\frac{e^{\frac{\xi^2}{1+\mu\xi^2}T}}{e^{\xi^2 T}} - 1\right)e^{\xi^2 T}\hat{\psi}(1+\xi^2)^{\frac{p}{2}}(1+\xi^2)^{-\frac{p}{2}}\right\| \\ &\le \sup_{\xi\in\Re}\left|e^{\frac{\xi^2}{1+\mu\xi^2}T}\right|\delta + \sup_{\xi\in\Re}\left|(1-e^{-\frac{\mu\xi^4}{1+\mu\xi^2}T})(1+\xi^2)^{-\frac{p}{2}}\right|E \end{aligned}$$

We know the following inequalities hold by[24]

$$\sup_{\xi\in\Re}\left|e^{\frac{\xi^2}{1+\mu\xi^2}T}\right| \le e^{\frac{T}{\mu}}, \tag{3.12}$$

$$\sup_{\xi\in\Re}\left|(1-e^{-\frac{\mu\xi^4}{1+\mu\xi^2}T})(1+\xi^2)^{-\frac{p}{2}}\right| \le \max\left\{\mu^{\frac{p}{4}}, T\mu^{\frac{p}{4}}, T\mu\right\}, \tag{3.13}$$

so, we can get

$$I_1 \le e^{\frac{T}{\mu}}\delta + \max\left\{\mu^{\frac{p}{4}}, T\mu^{\frac{p}{4}}, T\mu\right\}E.$$

In the same way, let

$$I_{2=}\left\|\hat{u}_{\mu}^{\varphi_\delta}(\xi,0) - \hat{u}^{\varphi}(\xi,0)\right\|$$

$$\le \left\|\hat{u}_{\mu}^{\varphi_\delta}(\xi,0) - \hat{u}_{\mu}^{\varphi}(\xi,0)\right\| + \left\|\hat{u}_{\mu}^{\varphi}(\xi,0) - \hat{u}^{\varphi}(\xi,0)\right\|$$

$$\le \left\| \mathrm{e}^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}_\delta - \mathrm{e}^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}\right\|$$

$$+ \left\| \mathrm{e}^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi} - \mathrm{e}^{\xi^2 T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}\right\|$$

$$= \left\| \mathrm{e}^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}(\hat{\varphi}_\delta - \hat{\varphi})\right\|$$

$$+ \left\| \left(\frac{\mathrm{e}^{\frac{\xi^2}{1+\mu\xi^2}T}}{\mathrm{e}^{\xi^2 T}} - 1\right)\mathrm{e}^{\xi^2 T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\hat{\varphi}(1+\xi^2)^{\frac{p}{2}}(1+\xi^2)^{-\frac{p}{2}}\right\|$$

$$\le \sup_{\xi\in\Re}\left|\mathrm{e}^{\frac{\xi^2}{1+\mu\xi^2}T}\frac{1-e^{-\xi^2 T}}{1-e^{-\xi^2 t_0}}\right|\delta$$

$$+ \sup_{\xi\in\Re}\left|(1-\mathrm{e}^{-\frac{\mu\xi 4}{1+\mu\xi^2}T})(1+\xi^2)^{-\frac{p}{2}}\right|E$$

$$\le 2e^{\frac{T}{\mu}}\delta + \max\left\{\mu^{\frac{p}{4}}, T\mu^{\frac{p}{4}}, T\mu\right\}E,$$

so,

$$\left\|u_{\mu}^{\delta}(x,0) - u(x,0)\right\| = I_1 + I_2$$

$$\le 3e^{\frac{T}{\mu}}\delta + 2\max\left\{\mu^{\frac{p}{4}}, T\mu^{\frac{p}{4}}, T\mu\right\}E$$

$$= \left(\frac{3}{(\ln\frac{E}{\delta})^{2P}} + 2\varepsilon\right)E$$

where $\varepsilon = \max\left\{\mu^{\frac{p}{4}}, T\mu^{\frac{p}{4}}, T\mu\right\}$. The proof of Theorem 3.1 is completed. □

IV. SOME NUMERICAL EXAMPLES

In this section, we present two examples numerical intended to illustrate computational performances and limitations. Follow the steps outlined below: first we selected $u(x,T) = \psi(x)$ and $u(x,t_0) = \varphi(x)$ when $t = T$ and $t = t_0$, $x \in [-10,10]$. Next we added a random distributed perturbation to the data function obtaining vector ψ_δ and φ_δ, i.e.,

$$\psi_\delta = \psi + \varepsilon randn(size(\psi)), \quad (4.1)$$

$$\varphi_\delta = \varphi + \varepsilon randn(size(\varphi)), \quad (4.2)$$

where

$$\begin{aligned} &\psi = (\psi(x_{-n}),\dots,\psi(x_0),\dots,\psi(x_n))^T, \\ &x_i = i\Delta x, \Delta x = 2\pi/n, i = -n,\dots,0,\dots,n, \end{aligned} \quad (4.3)$$

$$\begin{aligned} &\varphi = (\varphi(x_{-n}),\dots,\varphi(x_0),\dots,\varphi(x_n))^T, \\ &x_i = i\Delta x, \Delta x = 2\pi/n, i = -n,\dots,0,\dots,n, \end{aligned} \quad (4.4)$$

$$\delta = \left\|\varphi_\delta - \varphi\right\|_{L^2} = \left(\frac{1}{2n+1}\sum_{i=-n}^{n}\left|\varphi_\delta(\mathrm{x_i}) - \varphi(x_i)\right|^2\right)^{1/2}. \quad (4.5)$$

The function"randn(.)" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. Finally , we need to make the vector φ_δ and ψ_δ periodical and then we take the discrete Fourier transform for the vector φ_δ and ψ_δ . The approximation of the regularization solution are computed by using the FFT algorithm [25].

Example 1.

$$u(x,t) = \begin{cases} \dfrac{x}{(t+1)^{\frac{3}{2}}}e^{\frac{-x^2}{4(t+1)}} - xe^{-\frac{x^2}{4}} + e^{-t}\sin x, & t > 0, \\ 0, & t < 0. \end{cases} \quad (4.6)$$

and the function

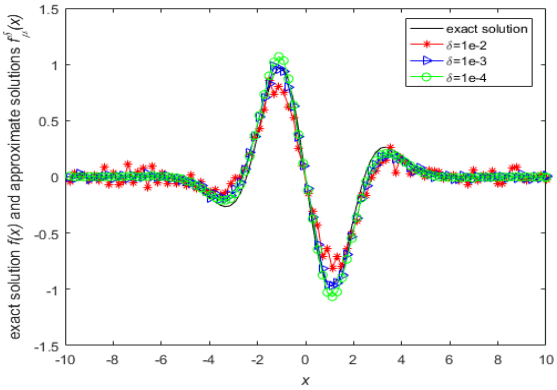
$$f(x) = \left(\frac{x^3}{4} - \frac{3}{2}x\right)e^{\frac{-x^2}{4}}, \quad (4.7)$$

$$u(x,0) = \sin x, \quad (4.8)$$

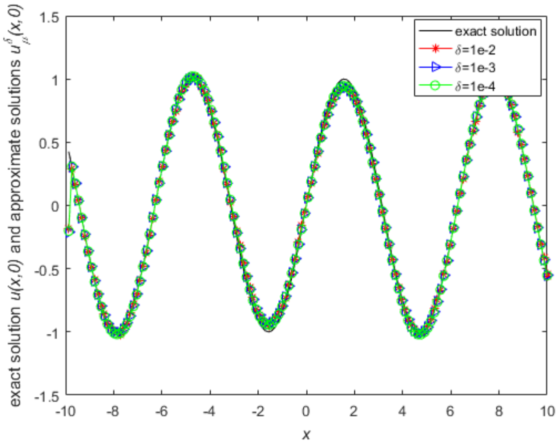
are satisfied with the problem (1.1) with exact data. The measurement function is as follows

$$\varphi(x) = \frac{x}{(t_0+1)^{\frac{3}{2}}}e^{\frac{-x^2}{4(t_0+1)}} - xe^{-\frac{x^2}{4}} + e^{-t_0}\sin x, \quad (4.9)$$

$$\psi(x) = \frac{x}{(T+1)^{\frac{3}{2}}}e^{\frac{-x^2}{4(T+1)}} - xe^{-\frac{x^2}{4}} + e^{-T}\sin x, \quad (4.10)$$



$(a)\, p = 20, t_0 = 0.5, T = 1$



$(b)\, p = 20, t_0 = 0.5, T = 1$

Example 2.

Consider the following function

$$u(x,t) = 2e^{-\pi^2 t}\cos \pi x + \cos \pi x, \qquad (4.11)$$

by simple calculation, we get

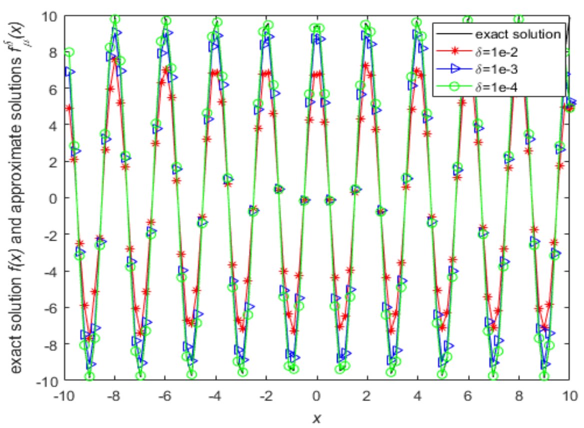
$$f(x) = \pi^2 \cos \pi x, \qquad (4.12)$$

$$u(x,0) = 3\cos \pi x, \qquad (4.13)$$

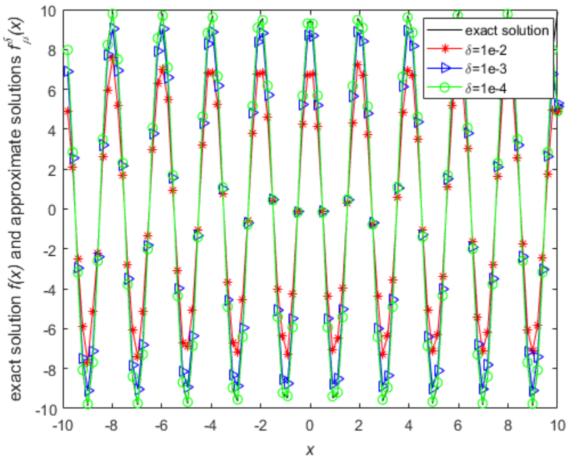
the measurement function is as follows

$$\varphi(x) = 2e^{-\pi^2 t_0}\cos \pi x + \cos \pi x, \qquad (4.14)$$

$$\psi(x) = 2e^{-\pi^2 T}\cos \pi x + \cos \pi x. \qquad (4.15)$$



$(c)\, p = 2, t_0 = 0.01, T = 0.5$



$(d)\, p = 2, t_0 = 0.01, T = 0.5$

For examples 1-2, we know the original problem, and the initial value and source term of the problem are solved by simple calculation.

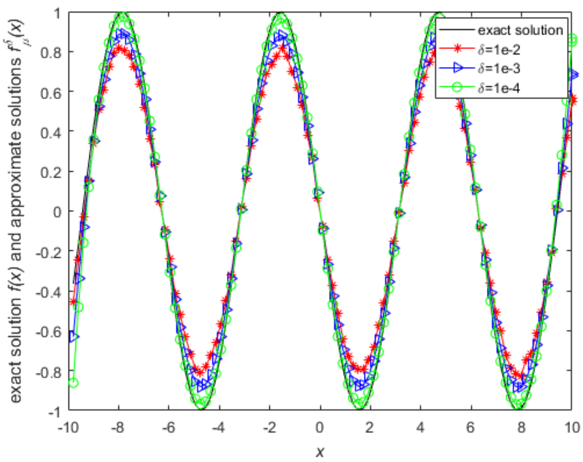
Example 3.

Consider the follow smooth heat source:

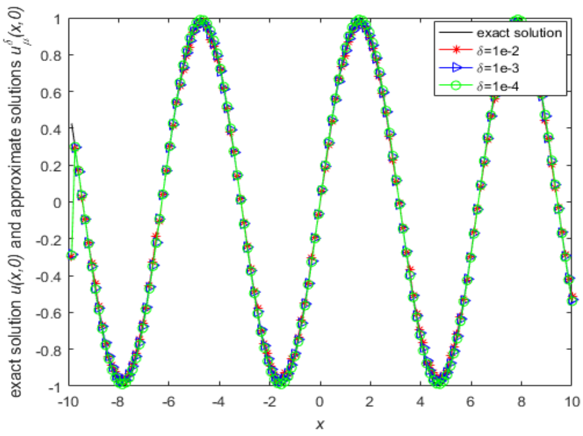
$$f(x) = -\sin x, \qquad (4.16)$$

$$u(x,0) = \sin x, \qquad (4.17)$$

We obtain the dates φ_δ and ψ_δ by solving the direct problem.



$(e)\, p = 15, t_0 = 0.5, T = 2$



$(f)\, p = 15, t_0 = 0.5, T = 2$

For examples 1-3, We can obtain that the smaller δ is, the better the computed approximation is. So, the numerical experiments show that our proposed regularization method is effective and stable.

V. Conclusion

In this paper, we solve a ill-posed problem for identifying the unknown source term and initial in a strip domain. An analytical solution is deduced based on the function expansion. We propose use two different regularization methods for overcoming its ill-posed and obtaining a regularized solution. The numerical examples for three examples show that our proposed method is effective and stable for reconstructing smooth and non smooth source term and initial data.

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