Global Behavior of the Fourth – Order Difference Equation

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***Abstract*— In this article, the global asymptotic stability of the fourth order differential equation**

$$x_{n+1} = \frac{\alpha x_n x_{n-2}}{x_{n-1}(\beta + \gamma x_{n-3})}$$

where α, β and γ are positive constants and non-negative initial conditions is examined.

***Index Terms*— Difference Equation, Asymptotic Stability, Global Behavior,**
2010 Mathematics Subject Classification: 35B40, 39A10;

I. Introduction

The disciplines that examine subjects such as population, finance, probability, genetics need difference equations in mathematical models that have been created to explain real-life situations. Therefore, the necessity of some techniques that can be used in the investigation of the resulting equations reveals the importance of difference equations. Therefore, interest towards difference equations has increased.

Even if difference equations appear in a simple form, it is actually quite difficult to fully understand the behavior of their solutions, see [1–10] and the references cited therein. Until now, many studies have been done on the stability of nonlinear difference equations. For example:

Yang et al. [1] investigated the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2} + x_{n-3} + a}{x_{n-1} + x_{n-2}x_{n-3} + a}.$$

Kulenovic, Ladas and Sizer et al. [2] studied the behavior of rational recursive sequence

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}.$$

Elabbasy and colleagues et al. [3] investigated and study some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-l} x_{n-k}}{b x_{n-p} - c x_{n-q}}.$$

Abdul Khaliq and Elsayed et al. [4] studied behavior and obtained some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_n x_{n-l}}{\beta x_{n-m} + \gamma x_{n-l}}.$$

See also [5 – 14]. Our aim is examine the global behavior of the following fourth-order rational difference equation

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$$x_{n+1} = \frac{\alpha x_n x_{n-2}}{x_{n-1}(\beta + \gamma x_{n-3})} \qquad (1.1)$$

where α, β and γ are positive constants and non-negative initial conditions that will serve as the basis for such modeling.

Definition 1.1. [15] Let I be some interval of real numbers and let

$$f : I^4 \to I$$

be a continuously differentiable function.

Then for every set of initial conditions $x_0, x_{-1}, x_{-2}, x_{-3} \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}), \; n = 0,1,2,\cdots \qquad (1.2)$$

has a **unique solution** $\{x_n\}_{n=-1}^{\infty}$.

A point $\bar{x} \in I$ is called an **equilibrium point** of Eq.(1.2) if

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \bar{x});$$

that is,

$$x_n = \bar{x} \text{ for } n \geq 0$$

is a solution of Eq.(1.2), or equivalently, $\bar{x}$ is **fixed point** of f.

Definition 1.2. [15] Let $\bar{x}$ be an equilibrium point of Eq.(1.2)

(i) The equilibrium $\bar{x}$ of Eq.(1.2) is called **locally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0, x_{-1}, x_{-2}, x_{-3} \in I$ with $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| + |x_{-2} - \bar{x}| + |x_{-3} - \bar{x}| < \delta$, we have

$$|x_n - \bar{x}| < \epsilon \text{ for all } n \geq -3.$$

(ii) The equilibrium $\bar{x}$ of Eq.(1.2) is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_0, x_{-1}, x_{-2}, x_{-3} \in I$ with $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| + |x_{-2} - \bar{x}| + |x_{-3} - \bar{x}| < \gamma$, we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) The equilibrium $\bar{x}$ of Eq.(1.2) is called **global attractor** if for every $x_0, x_{-1}, x_{-2}, x_{-3} \in I$ we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) The equilibrium $\bar{x}$ of Eq.(1.2) is called **global asymptotically stable** if it is locally stable and a global attractor.
(v) The equilibrium $\bar{x}$ of Eq.(1.2) is called **unstable** if it is not stable.
(vi) The equilibrium $\bar{x}$ of Eq.(1.2) is called **source** or a **repeller**, if there exists $r > 0$ such that for all $x_0, x_{-1}, x_{-2}, x_{-3} \in I$ with $0 < |x_0 - \bar{x}| + |x_{-1} - \bar{x}| + |x_{-2} - \bar{x}| + |x_{-3} - \bar{x}| < r$, there exists $N \geq 1$ such that

$$|x_N - \bar{x}| \geq r.$$

The linearized equation of (1.2) about the equilibrium point $\bar{x}$ is

$$y_{n+1} = p_1 y_n + p_2 y_{n-1} + p_3 y_{n-2} + p_4 y_{n-3},\ n = 0,1,2,\cdots \quad (1.3)$$

where

$$p_1 = \frac{\partial f}{\partial x_n}(\bar{x},\bar{x},\bar{x},\bar{x})$$

$$p_2 = \frac{\partial f}{\partial x_{n-1}}(\bar{x},\bar{x},\bar{x},\bar{x})$$

$$p_3 = \frac{\partial f}{\partial x_{n-2}}(\bar{x},\bar{x},\bar{x},\bar{x})$$

$$p_4 = \frac{\partial f}{\partial x_{n-3}}(\bar{x},\bar{x},\bar{x},\bar{x})$$

The characteristic equation of (1.3) is

$$\lambda^4 - p_1\lambda^3 - p_2\lambda^2 - p_3\lambda - p_4 = 0 \qquad (1.4)$$

Theorem 1.1. (The Linearized Stability Theorem)
(i) If all roots of (1.4) have absolute values less than one, then the equilibrium point $\bar{x}$ of (1.2) is locally asymptotically stable.
(ii) If at least one of the roots of (1.4) has absolute value greater than one, then the equilibrium point $\bar{x}$ of (1.2) is unstable.
(iii) The equilibrium point $\bar{x}$ of (1.2) is called saddle point if (1.4) has roots both inside and outside the unit disk.

Theorem 1.2. [15] Assume that $p_i \in \mathbb{R}, i = 1,2,\cdots$. Then

$$\sum_{i=1}^{4} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of (1.4).

Theorem 1.3. [15] Let $[p,q]$ be an interval of real numbers and assume that $f : [p,q]^4 \to [p,q]$ is a continuous function satisfying the following properties:

(a) $f(x,y,z,t)$ is non-decreasing in $x,z \in [p,q]$ for each $y,t \in [p,q]$, and non-increasing in $y,t \in [p,q]$ for each $x,z \in [p,q]$;
(b) If $(m,M) \in [p,q] \times [p,q]$ is a solution of the system

$$M = f(M,m,M,m) \text{ and } m = f(m,M,m,M)$$

so $m = M$.

Then Eq.(1.2) has a unique equilibrium $\bar{x} \in [p,q]$ and every solution of Eq.(1.2) converges to $\bar{x}$.

II. Dynamics of Eq.**Error! Reference source not found.**

In this section, we investigate the dynamics of **Error! Reference source not found.** under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The Eq.**Error! Reference source not found.** has the equilibrium points obtained from the equation

$$\bar{x} = \frac{\alpha \bar{x}^2}{\bar{x}(\beta + \gamma\bar{x})},$$

so,

$$\bar{x}^2(\gamma\bar{x} + \beta - \alpha) = 0,$$

then the equilibrium points are $\bar{x}_1 = 0$ and $\bar{x}_2 = \dfrac{\alpha - \beta}{\gamma}$.

Theorem 2.1. The statements given below are true.

(i) If $3\alpha < \beta$, then the equilibrium point $\bar{x}_1 = 0$ of Eq.**Error! Reference source not found.** is locally asymptotically stable.

(ii) If $3\alpha > \beta$, then the equilibrium point $\bar{x}_1 = 0$ of Eq.**Error! Reference source not found.** is a saddle point.

(iii) The equilibrium point $\bar{x}_2 = \dfrac{\alpha - \beta}{\gamma}$ of Eq.**Error! Reference source not found.** is unstable.

Proof. Let $f : (0,\infty)^4 \to (0,\infty)$ be a function defined by

$$f(u,v,w,t) = \frac{\alpha u w}{v(\beta + \gamma t)}.$$

So,

$$\frac{\partial f}{\partial u}(\bar{x},\bar{x},\bar{x},\bar{x}) = \frac{\alpha}{\beta + \gamma\bar{x}},$$

$$\frac{\partial f}{\partial v}(\bar{x},\bar{x},\bar{x},\bar{x}) = -\frac{\alpha}{\beta + \gamma\bar{x}},$$

$$\frac{\partial f}{\partial w}(\bar{x},\bar{x},\bar{x},\bar{x}) = \frac{\alpha}{\beta + \gamma\bar{x}},$$

$$\frac{\partial f}{\partial t}(\bar{x},\bar{x},\bar{x},\bar{x}) = -\frac{\alpha\gamma\bar{x}}{(\beta + \gamma\bar{x})^2}.$$

The linearized equation of Eq.**Error! Reference source not found.** about the equilibrium point $\bar{x}_1 = 0$ is

$$x_{n+1} - \frac{\alpha}{\beta}x_n + \frac{\alpha}{\beta}x_{n-1} - \frac{\alpha}{\beta}x_{n-2} = 0$$

so, the characteristic equation of Eq.**Error! Reference source not found.** about the equilibrium point $\overline{x}_1 = 0$ is

$$\lambda^4 - \frac{\alpha}{\beta}\lambda^3 + \frac{\alpha}{\beta}\lambda^2 - \frac{\alpha}{\beta}\lambda = 0.$$

From Theorem 1.2, Eq.**Error! Reference source not found.** about the equilibrium point $\overline{x}_1 = 0$ is locally asymptotically stable if

$$\left|\frac{\alpha}{\beta}\right| + \left|\frac{\alpha}{\beta}\right| + \left|\frac{\alpha}{\beta}\right| < 1.$$

Thus $3\alpha < \beta$. The proof of (ii) follows Theorem 1.1.

For (iii), the linearized equation of Eq.**Error! Reference source not found.** about the equilibrium point $\overline{x}_2 = \frac{\alpha-\beta}{\gamma}$ has the form

$$x_{n+1} - x_n + x_{n-1} - x_{n-2} + \frac{\alpha-\beta}{\alpha}x_{n-3} = 0$$

so, the characteristic equation of Eq.**Error! Reference source not found.** about the equilibrium point $\overline{x}_2 = \frac{\alpha-\beta}{\gamma}$ is

$$\lambda^4 - \lambda^3 + \lambda^2 - \lambda + \frac{\alpha-\beta}{\alpha} = 0.$$

From Definition 1.2 and Theorem 1.2, it is clear that Eq.**Error! Reference source not found.** about the equilibrium point $\overline{x}_2 = \frac{\alpha-\beta}{\gamma}$ is unstable.

Then the proof is complete.

Theorem 2.2. The equilibrium point $\overline{x}_1 = 0$ of Eq.**Error! Reference source not found.** is globally asymptotically stable.

Proof. Let $g:[p,q]^4 \to [p,q]$ is a function defined as $g(x,y,z,t) = \frac{\alpha xz}{y(\beta+\gamma t)}$. It can easily be seen that the $g(x,y,z,t)$ function is increased with respect to x,z and decreasing with respect to y,t. Let us assume that (m,M) is the solution of the system

$$M = g(M,m,M,m) \quad \text{and} \quad m = g(m,M,m,M).$$

So

$$M = \frac{\alpha M^2}{m(\beta+\gamma m)}, \quad m = \frac{\alpha m^2}{M(\beta+\gamma M)}$$

and

$$Mm\beta + \gamma Mm^2 = \alpha M^2$$
$$mM\beta + \gamma mM^2 = \alpha m^2$$

If we solve linear equation system, we obtain

$$(M-m)\left(\alpha(M+m)+\gamma mM\right) = 0$$

Since $\left(\alpha(M+m)+\gamma mM\right) > 0$, $M = m$.

From Theorem 1.3, the equilibrium point $\overline{x}_1 = 0$ is a global attractor of Eq.**Error! Reference source not found.** and then the equilibrium point $\overline{x}_1 = 0$ of Eq.**Error! Reference source not found.** is globally asymptotically stable by Definition 1.2.

Then the proof is complete.

Theorem 2.3. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.**Error! Reference source not found.** is bounded if $\alpha \le 1$.

Proof. From Eq.**Error! Reference source not found.**

$$x_1 = \frac{\alpha x_0 x_{-2}}{x_{-1}(\beta+\gamma x_{-3})} < \alpha x_0 x_{-2}$$

$$x_2 = \frac{\alpha x_1 x_{-1}}{x_0(\beta+\gamma x_{-2})} = \frac{\alpha \frac{\alpha x_0 x_{-2}}{x_{-1}(\beta+\gamma x_{-3})} x_{-1}}{x_0(\beta+\gamma x_{-2})} = \frac{\alpha^2 x_{-2}}{(\beta+\gamma x_{-2})(\beta+\gamma x_{-3})}$$
$$< \alpha^2 x_{-2}$$

$$x_3 = \frac{\alpha x_2 x_0}{x_1(\beta+\gamma x_{-1})} = \frac{\alpha \frac{\alpha^2 x_{-2}}{(\beta+\gamma x_{-2})(\beta+\gamma x_{-3})} x_0}{\frac{\alpha x_0 x_{-2}}{x_{-1}(\beta+\gamma x_{-3})}(\beta+\gamma x_{-1})}$$
$$= \frac{\alpha^2 x_{-1}}{(\beta+\gamma x_{-1})(\beta+\gamma x_{-2})} < \alpha^2 x_{-1}$$

$$x_4 = \frac{\alpha x_3 x_1}{x_2(\beta+\gamma x_0)} = \frac{\alpha \frac{\alpha^2 x_{-1}}{(\beta+\gamma x_{-1})(\beta+\gamma x_{-2})} \frac{\alpha x_0 x_{-2}}{x_{-1}(\beta+\gamma x_{-3})}}{\frac{\alpha^2 x_{-2}}{(\beta+\gamma x_{-2})(\beta+\gamma x_{-3})}(\beta+\gamma x_0)}$$
$$= \frac{\alpha^2 x_0}{(\beta+\gamma x_0)(\beta+\gamma x_{-1})} < \alpha^2 x_0$$

$$x_5 = \frac{\alpha x_4 x_2}{x_3(\beta+\gamma x_1)} = \frac{\alpha \frac{\alpha^2 x_0}{(\beta+\gamma x_0)(\beta+\gamma x_{-1})} \frac{\alpha^2 x_{-2}}{(\beta+\gamma x_{-2})(\beta+\gamma x_{-3})}}{\frac{\alpha^2 x_{-1}}{(\beta+\gamma x_{-1})(\beta+\gamma x_{-2})}(\beta+\gamma x_1)}$$
$$= \frac{\alpha^2 \frac{\alpha x_0 x_{-2}}{x_{-1}(\beta+\gamma x_{-3})}}{(\beta+\gamma x_1)(\beta+\gamma x_0)} = \frac{\alpha^2 x_1}{(\beta+\gamma x_1)(\beta+\gamma x_0)} < \alpha^2 x_1$$

By iteration we obtain

$$\left.\begin{array}{r} x_{4k+1} > x_{4k+5} \\ x_{4k} > x_{4k+4} \\ x_{4k-1} > x_{4k+3} \\ x_{4k-2} > x_{4k+2} \end{array}\right\} \text{for } k = 0,1,2,\cdots \text{ if } \alpha \le 1.$$

Then the subsequences $\{x_n\}_{n=-3}^{\infty}$ are decreasing and they are bounded to $M=\{x_{-2},x_{-1},x_0,x_1\}$ from above. This completes the proof.

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