Strongly Gorenstein injective complexes with respect to a cotorsion pair

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***Abstract*-This paper introduces strongly Gorenstein injective complexes with respect to a cotorsion pair $(\mathcal{A},\mathcal{B})$, that is, strongly Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ -injective complexes, and give some properties and equivalent characterizations. Firstly, we define Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$-injective complexes, that is the complexes of Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ –injective modules, and discuss some properties. Secondly, we define strongly Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ -injective complexes, and prove Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ -injective complex is a summand of some strongly Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ -injective complex.**

***Index Terms*—Cotorsion pair, Gorenstein injective complexes with respect to a cotorsion pair, Strongly Gorenstein injective complexes with respect to a cotorsion pair, stability.**

I. INTRODUCTION

As a generalization of projective, injective and flat complexes, Enochs et al introduced Gorenstein projective, injective and flat complexes, and developed Gorenstein homological algebra in the category of complexes in [2]. Renyu Zhao and Pengju Ma introduced Gorenstein projective complexes with respect to a cotorsion pair, that is Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ -projective complexes in [5], and investigate stability of Gorenstein $(\mathcal{A},\mathcal{A}\cap\mathcal{B})$ -projective complexes.

Definition 1.1 ([1. Definition 3.3]) Let R is a ring, X is a complexe.

(1) A complex X is called $\mathcal{A}$ complex, if X is exact and $Z_n(X)\in\mathcal{A}, n\in\mathbb{Z}$.

(2) A complex X is called $\mathcal{B}$ complex, if X is exact and $Z_n(X)\in\mathcal{B}, n\in\mathbb{Z}$.

(3) A complex X is called dg-$\mathcal{A}$ complex, if $X_n\in\mathcal{A}$ and $\mathrm{Hom}_R(X,B)$is exact, $B\in\mathcal{B}$.

(4) A complex X is called dg-$\mathcal{B}$ complex, if $X_n\in\mathcal{B}$ and $\mathrm{Hom}_R(X,A)$ is exact, $A\in\mathcal{A}$.

Lemma 1.2. Let $(\mathcal{A},\mathcal{B})$ is a complete and hereditary cotorsion pair in R-Mod, then derived cotorsion pair $(\tilde{\mathcal{A}},dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}},\tilde{\mathcal{B}})$ is complete and hereditary in $\mathrm{C(R)}$. Furthermore, $dg\tilde{\mathcal{A}}\cap\varepsilon=\tilde{\mathcal{A}}$, $dg\tilde{\mathcal{B}}\cap\varepsilon=\tilde{\mathcal{B}}$, where ε is the class of exact complexes.

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II. GORENSTEIN INJECTIVE COMPLEXES WITH RESPECT TO A COTORSION PAIR

Definition 2.1. A complex C is called Gorenstein injective complex with respect to a cotorsion pair, that is Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$ -injective complex, if there exists a $\mathrm{Hom}_{C(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}},-)$-exact exact sequence

$\cdots\to Y^1\to Y^0\to Y^{-1}\to\cdots$

such that$C\cong\mathrm{Im}(Y^0\to Y^{-1})$, whenever $Y^i\in\tilde{\mathcal{B}}$.

We use $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ express the class of Gorenstein $(\mathcal{A}\cap\mathcal{B},\mathcal{B})$-injective complexes.

Remark 2.2. (1) It is clear that $\tilde{\mathcal{B}}\subseteq GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$. If

$$\mathbf{Y}=\cdots\to Y^1\to Y^0\to Y^{-1}\to\cdots$$

is a $\mathrm{Hom}_{C(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}},-)$ -exact exact sequence of complexes, then all Kernels, Imagrs, Cokernels of $\mathbf{Y}$ belong to $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$.

(2) If $(\mathcal{A},\mathcal{B})=(\mathrm{R-Mod},\mathcal{I})$, then we have the fact that Gorenstein injective complexes are Gorenstein injective complexes with respect to the cotorsion pair $(\mathrm{R-Mod},\mathcal{I})$ by [2].

(3) If $(\mathcal{A},\mathcal{B})=(\mathcal{P},\mathrm{R-Mod})$, then every complex is Gorenstein injective complex with respect to the cotorsion pair $(\mathcal{P},\mathrm{R-Mod})$.

Lemma 2.3. Let $\cdots\to Y^1\to Y^0\to Y^{-1}\to\cdots$ is a $\mathrm{Hom}_{C(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}},-)$ -exact exact sequence, then the sequence $\cdots\to Y_n^1\to Y_n^0\to Y_n^{-1}\to\cdots$ is $\mathrm{Hom}_{\mathrm{R}}(\mathcal{A}\cap\mathcal{B},-)$-exact, where $n\in\mathbb{Z}$.

Proof. Let $K\in\mathcal{A}\cap\mathcal{B}$ and $n\in\mathbb{Z}$, then we have $K[n]\in\widetilde{\mathcal{A}\cap\mathcal{B}}$ by [3. Lemma 3.4], then we have the following exact sequence

$\cdots\to\mathrm{Hom}_{C(R)}(\overline{K}[n],B^1)\to\mathrm{Hom}_{C(R)}(\overline{K}[n],B^0)\to\mathrm{Hom}_{C(R)}(\overline{K}[n],B^{-1})\to\cdots$

we have the following exact sequence by standard adjnction in [3. Lemma 3.1(2)].

$\cdots\to\mathrm{Hom}_R(K,B_n^1)\to\mathrm{Hom}_{C(R)}(K,B_n^0)\to\mathrm{Hom}_{C(R)}(K,B_n^{-1})\to\cdots$

Theorem 2.4. Let C is a complex, then $C\in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ if and only if $C_n\in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$.

Proof. $\Rightarrow$) Assume that $C\in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$. Then there exists an $\mathrm{Hom}_{C(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}},-)$ -exact exact sequence

$\cdots\to Y^1\to Y^0\to Y^{-1}\to\cdots$

with each $Y^i\in\tilde{\mathcal{B}}$ such that $C\cong\mathrm{Im}\,(Y^0\to Y^{-1})$. Now for any but fixed $n\in\mathbb{Z}$, by Lemma 2.3, we have the following $\mathrm{Hom}_R(\mathcal{A}\cap\mathcal{B},-)$ -exact exact sequence of modules in Y

$\cdots \to Y_n^1 \to Y_n^0 \to Y_n^{-1} \to \cdots$

such that $C_n = \operatorname{Im}(Y_n^0 \to Y_n^{-1})$, then $C_n \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$.

$\Leftarrow$) Suppose that $C_n \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$ for all n $\in \mathbb{Z}$. Then for any n $\in \mathbb{Z}$, there exists an exact sequence

$$0 \to L_n \to B_n \to C_n \to 0$$

where $B_n \in \mathcal{B}$, $L_n \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$, $n \in \mathbb{Z}$. These exact sequences induce a short exact sequences of complexes

$$0 \to \oplus_{n\in\mathbb{Z}} \overline{L_n}[n] \to \oplus_{n\in\mathbb{Z}} \overline{B_n}[n] \to \oplus_{n\in\mathbb{Z}} \overline{C_n}[n] \to 0$$

Put $B^0 = \oplus_{n\in\mathbb{Z}} \overline{B_n}[n]$. It is easy to see that $B^0 \in \tilde{\mathcal{B}}$. On the other hand, there is an obvious short exact sequence

$$0 \to C[-1] \xrightarrow{\binom{1}{-\delta}} \oplus_{n\in\mathbb{Z}} \overline{C_n}[n-1] \xrightarrow{(\delta \ \ 1)} C \to 0$$

where δ is the differential of C, Now Let $\alpha: B^0 \to C$ be the composite

$$\oplus_{n\in\mathbb{Z}} \overline{B_n}[n] \to \oplus_{n\in\mathbb{Z}} \overline{C_n}[n] \to C$$

Then α is monomorphism since it is the composite of two injection. Denote Cokerα by C^0. Then by Snake Lemma, we have a short exact sequence

$$0 \to \oplus_{n\in\mathbb{Z}} \overline{L_n}[n] \to C^0 \to C[-1] \to 0$$

Since each degree of $\oplus_{n\in\mathbb{Z}} \overline{L_n}[n]$ and $C[-1]$ are in $GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$. Let K $\in \widetilde{\mathcal{A}\cap\mathcal{B}}$. Thus K $\cong \oplus_{n\in\mathbb{Z}} \overline{Z_n(K)}[n]$ by [4. Lemma 4.1.]. Hence

$Ext^1_{\mathcal{C}(R)}(K, C^0) \cong \Pi_{n\in\mathbb{Z}} Ext^1_{\mathcal{C}(R)}(\overline{Z_n(K)}[n], C^0) \cong \Pi_{n\in\mathbb{Z}} Ext^1_{\mathcal{C}(R)}(Z_n(K), C_n^0) = 0$

Where the second isomorphism follows from [5. Lemma 3.1.(2)] and the last equality follows from Proposition 2.3. This implies that $0 \to C^0 \to B^0 \to C \to 0$ is $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact. Notice that C^0 has the same property as C, so we can use the same procedure to construct a $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact exact sequence

$$\cdots \to B^{-1} \to B^0 \to C \to 0 \quad (*)$$

where each B^i is an $\mathcal{B}$-complexes.

Since $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is complete cotorsion pair, we have a short exact sequence

$$0 \to C \to B^{-1} \to C^{-1} \to 0$$

where $B^{-1} \in \tilde{\mathcal{B}}$ and $C^{-1} \in dg\tilde{\mathcal{A}}$. Note that $C_n \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$ for any n $\in \mathbb{Z}$, this sequence is $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact by a similarly discussion as above. Also, it follows from exact sequence and [6. Proposition 3.3.(1)] that each $C_n^{-1} \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$ for any $n \in \mathbb{Z}$. Thus we can continuously use the same method to contruct a $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact exact sequence

$$0 \to C \to B^{-1} \to B^{-2} \to \cdots \quad (**)$$

where each B^i is a $\mathcal{B}$-complexes.

Finally, gluing the sequence $(*)$ and $(**)$ together, one has a $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact exact sequence of complexes

$$\cdots \to B^1 \to B^0 \to B^{-1} \to \cdots$$

with all $B^1 \in \tilde{\mathcal{B}}$ such that $C \cong \operatorname{Im}(B^0 \to B^{-1})$. Hence $C \in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$.

Corollary 2.5. $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ is injectively coresolving.

Proof. Clearly, $\tilde{I} \subseteq \tilde{\mathcal{B}} \subseteq GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ Let

$0 \to C' \to C \to C'' \to 0$

be a short exact sequence of complexes with $C' \in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$. Then for any $n \in \mathbb{Z}$, in the exact sequence $0 \to C_n' \to C_n \to C_n'' \to 0$, $C_n'' \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$. so $C_n'' \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$ if and only if $C_n \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$ by [6. Proposition 3.4.(1)]. Hence $C'' \in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ if and only if $C \in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ by Theorem 2.4. Now the result follows.

Corollary 2.6. Let $0 \to C' \to C \to C'' \to 0$

be a short exact sequence of complexes. If C', C are belong to $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$, then $C'' \in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ if and only if $Ext^1_{\mathcal{C}(R)}(C'', K)$ for any K $\in \widetilde{\mathcal{A}\cap\mathcal{B}}$.

Proof. $\Rightarrow$) it is obvious.

$\Leftarrow$) Let $n \in \mathbb{Z}$. Consider the exact sequence of R-modules

$$0 \to C_n' \to C_n \to C_n'' \to 0$$

By theorem 2.4, C_n, C_n'' are belong to $GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$. Let K $\in \mathcal{A}\cap\mathcal{B}$. Then $\overline{K}[n] \in \widetilde{\mathcal{A}\cap\mathcal{B}}$. Thus

$Ext^1_R(K, C_n') \cong Ext^1_{\mathcal{C}(R)}(\overline{K}[n], C')=0$

by [1. Lemma 3.1.(1)] and the hypothesis. Hence $C_n' \in GI_{(\mathcal{A}\cap\mathcal{B},\mathcal{B})}$ by [6. Proposition 3.4.(2)]. Therefore $C_n' \in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ by Theorem 2.4.

Proposition 2.7. Let C be a complex of R-modules.Then the following statements are equivalent:

(1) C $\in GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$.

(2) $Ext^i_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, C)=0$ for any $i \geq 1$ and there exists a $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact exact sequence

$$\cdots \to B^1 \to B^0 \to C \to 0$$

where $B^i \in \tilde{\mathcal{B}}$.

Proof. It is similar to the proof of proposition 2.3.

Lemma 2.8. $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ is closed under direct products and direct summands for any a ring R.

Proof. $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ is closed under direct products by Proposition 2.7. since $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ is injectively coresolving and closed under direct products, then $GI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ is closed under direct summands by [8. Proposition 1.4.].

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III. Strongly Gorenstein Injective Complexes with Respect to a Cotorsion Pair

Definition 3.1. A complex C is called Strongly Gorenstein injective complexes with respect to a cotorsion pair, that is Strongly Gorenstein $(\mathcal{A}\cap\mathcal{B}, \mathcal{B})$-injective complexes, if there exists an $\operatorname{Hom}_{\mathcal{C}(R)}(\widetilde{\mathcal{A}\cap\mathcal{B}}, -)$ -exact exact sequence

$\cdots \to Y \to Y \to Y \to \cdots$

such that $C = \operatorname{Im}(Y \to Y)$, whenever $Y \in \tilde{\mathcal{B}}$.

We use $SGI_{(\widetilde{\mathcal{A}\cap\mathcal{B}},\tilde{\mathcal{B}})}$ express the class of Strongly Gorenstein $(\mathcal{A}\cap\mathcal{B}, \mathcal{B})$ -injective complexes

Proposition 3.2. Let C be a complex of R-modules.Then the following statements are equivalent:

(1) C ∈ $SGI_{(\widetilde{A\cap B},\tilde{B})}$.

(2) there exists a short exact sequence of complexes

$$0 \to C \to B \to C \to 0$$

where $B \in \tilde{B}$ such that $Ext^{i}_{C(R)}(Q, C) = 0$ for any Q∈ $\widetilde{A \cap B}$, i≥ 1.

(3) there exists a short exact sequence of complexes

$$0 \to C \to B \to C \to 0$$

where $B \in \tilde{B}$ such that $Ext^{1}_{C(R)}(Q, C) = 0$ for any Q∈ $\widetilde{A \cap B}$.

(4) there exists a short exact sequence of complexes

$$0 \to C \to B \to C \to 0$$

where $B \in \tilde{B}$ such that $Ext^{i}_{C(R)}(Q, C) = 0$ for any Q∈ $\widetilde{A \cap B}$, some i≥ 1.

Proof. It is similar to the proof of [7. Proposition 2.10.] and [4. Remark 2.3.].

Theorem 3.3. A complex C is Gorenstein $(A \cap B, B)$ -injective complex if and only if C is a summand of some Strongly Gorenstein$(A \cap B, B)$ -injective complex.

Proof. $\Leftarrow$) since $SGI_{(\widetilde{A\cap B},\tilde{B})} \subseteq GI_{(\widetilde{A\cap B},\tilde{B})}$, we have the fact $GI_{(\widetilde{A\cap B},\tilde{B})}$ is closed under direct summands by Lemma 2.8.so C is Gorenstein $(A \cap B, B)$ -injective complexes.

$\Rightarrow$) Assume C is Gorenstein $(A \cap B, B)$ -injective complex, there exists an $\mathrm{Hom}_{C(R)}(\widetilde{A \cap B}, -)$ -exact exact sequence

$$\boldsymbol{B} = \cdots \xrightarrow{d_B^2} B^1 \xrightarrow{d_B^1} B^0 \xrightarrow{d_B^0} B^{-1} \xrightarrow{d_B^{-1}} \cdots$$

such that $C \cong$ Im (d_B^0), whenever $B^i \in \tilde{B}$. $(\Sigma^m \boldsymbol{B})^i = B_{i-m}$ and $d^i_{\Sigma^m \boldsymbol{B}} = d_B^{i-m}$, i ∈ $\mathbb{Z}$, consider exact sequence

$$\mathbf{N} = \oplus(\Sigma^m \boldsymbol{B}) = \cdots \xrightarrow{\oplus d_B^i} N = \oplus B_i \xrightarrow{\oplus d_B^i} N = \oplus B_i \xrightarrow{\oplus d_B^i} N = \oplus B_i \xrightarrow{\oplus d_B^i} \cdots$$

since Im$(\oplus d_B^i) \cong \oplus \mathrm{Im} d_B^i$, then C is a summand of Im$(\oplus d_B^i)$. **N** is $\mathrm{Hom}_{C(R)}(\widetilde{A \cap B}, -)$ -exact by [8 proposition 20.2.]. so C is a summand of Strongly Gorenstein $(A \cap B, B)$ -injective complex Im$(\oplus d_B^i)$.

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