Gorenstein $FP_n$ injective modules

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Abstract— In this paper, we introduce and study Gorenstein $FP_n$ injective modules and investigate the homological properties of them.

Index Terms— Gorenstein $FP_n$ injective modules, dimensions, (n-1)-coherent ring.

I. INTRODUCTION

The flat modules and FP-injective modules play an important role in characterizing coherent rings. Naturally, many literature articles generalized these notions in relative homological algebra. In [5], Costa introduced absolutely clean and level modules. In [6], Chen and Ding introduced n-flat and n-FP injective modules. In 2015, Wei and coauthors call them $FP_n$-injective and $FP_n$-flat, respectively. In 2017, Bravo and others investigate n-coherent and give some equivalent characterizations of (n-1)-coherent ring [7]. On the other hand, Enochs and Jenda introduced Gorenstein projective, injective, Gorenstein flat modules, and developed Gorenstein homological algebra in [2, 3.4]. Later, many scholars further studied these modules and introduced various generalizations of these modules. In [11], Mao and Ding gave a definition of Gorenstein FP-injective modules. However, under their definition these Gorenstein FP-injective modules are stronger than the Gorenstein injective modules. In 2014, Bravo et al. introduced in [1] the notion of Gorenstein AC-projective (resp., Gorenstein AC-injective) modules and established the “Gorenstein AC-homological algebra” over an arbitrary ring.

Inspired by aforementioned work, we introduce the concept of Gorenstein $FP_n$-injective modules as a generalization of above Gorenstein homological modules. Then we character when a left module is Gorenstein $FP_n$-injective over (n-1)-coherent rings. In the following, we recall some notions that will be used throughout the paper.

Definition 1.1[3] A left $R$-module $M$ is called Gorenstein FP-injective, if there exists an exact sequence

$$E = \cdots \rightarrow E_i \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of injective left $R$-module with $M = \text{Im}(E_0 \rightarrow E^0)$ such that the functor $\text{Hom}(Q, -)$ leaves the sequence exact whenever $Q$ is FP-injective.

Definition 1.2[5] A ring $R$ is called n-coherent ring, if every finite $n$-presented module coincides with finite $(n+1)$-presented module.

Definition 1.3[5] A $R$-modules $M$ is called finite $n$-presented, if there exists an exact sequence of left $R$-modules

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where $P_i$ is finitely generated projective for $0 \leq i \leq n$. Such exact sequence is called a finite $n$-presentation of $M$.

Definition 1.4[7] A right $R$-modules $N$ is called $FP_n$-flat if $\text{Tor}^R_n(N,F) = 0$ for any finite $n$-presented module $F$.

A left $R$-modules $M$ is called $FP_n$-injective if $\text{Ext}^1_R(F, M) = 0$ for any finite $n$-presented module $F$.

II. GORENSTEIN $FP_n$ INJECTIVE MODULES

Definition 2.1 A left $R$-module $M$ is called Gorenstein $FP_n$ injective, if there exists an exact sequence

$$E = \cdots \rightarrow E_i \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of injective left $R$-module with $M = \text{Im}(E_0 \rightarrow E^0)$ such that the functor $\text{Hom}(Q, -)$ leaves the sequence exact whenever $Q$ is $FP_n$-injective.

Remark 2.2 (1) It is clear that each injective module is Gorenstein $FP_n$ injective. (2) If $M$ is a Gorenstein $FP_n$ injective module, by symmetric all the kernels, the images, and the cokernels of $E$ are Gorenstein $FP_n$ injective module. (3) Gorenstein AC injective $\subseteq$ Gorenstein $FP_n$ injective $\subseteq$ Ding injective $\subseteq$ Gorenstein injective. (4) If $n = 0$, then Gorenstein injective modules are Gorenstein $FP_n$ injective. (5) If $R$ is n-coherent, then Gorenstein $FP_n$ injective modules are Gorenstein injective; If $R$ is coherent, then Ding injective modules are Gorenstein $FP_n$ injective. (6) The class of Gorenstein $FP_n$ injectives is closed under direct summands.

Theorem 2.3 The following assertions are equivalent for a left $R$-module $M$. (1) $M$ is Gorenstein $FP_n$ injective. (2) $M$ has an exact injective resolution which is $\text{Hom}(Q, -)$-exact all $FP_n$-injective left $R$-modules $Q$. $\text{Ext}^1_R(Q, M) = 0$ for all $i \geq 1$. (3) There exist a short exact sequence of left $R$-modules $0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0$, where $E$ is injective and $K$ is Gorenstein $FP_n$ injective. Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) is clear by the definition of Gorenstein $FP_n$ injective module.

Definition 3(2) Since $K$ is Gorenstein $FP_n$ injective, there exist an exact sequence

$$\cdots \rightarrow E_i \rightarrow E_0 \rightarrow K \rightarrow 0,$$

which is $\text{Hom}(Q, -)$-exact, where $Q$ is $FP_n$-injective and $E_i$ are injective for all $i \geq 0$.

Note that the exact sequence of left $R$-modules $0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0$ is $\text{Hom}(Q, -)$-exact, so we obtain an left injective resolution of $M$

$$\cdots \rightarrow E_i \rightarrow E_0 \rightarrow E \rightarrow M \rightarrow 0.$$

On the other hand, for all $FP_n$-injective $Q$, we have an exact sequences of left $R$-modules

$$\cdots \rightarrow \text{Ext}^1_R(Q, E) \rightarrow \text{Ext}^1_R(Q,M) \rightarrow \text{Ext}^1_R(Q,K) \rightarrow \cdots.$$

By dimension shifting, $\text{Ext}^1_R(Q,K) = \text{Ext}^1_R(Q,E) = 0$ for all $i \geq 1$, therefore $\text{Ext}^1_R(Q,M) = 0$. So $M$ is Gorenstein $FP_n$-injective by (1) $\Rightarrow$ (2).

Proposition 2.4 Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an short exact sequence of left $R$-modules. (1) If $A$ and $C$ are Gorenstein $FP_n$ injective, then so is $B$. (2) If $A$ and $B$ are Gorenstein $FP_n$ injective, then so is $C$. 

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(3) If B and C are Gorenstein FP}_n injective, then A is Gorenstein FP}_n injective if and only if Ext}^1_R(A, B) = 0 for all FP}_n-injective left R-modules Q.

Proof. This is similar to the proof of [4, Theorems 2.8, 2.11].

Lemma 2.5 Let M be a left R-module. Consider two exact sequences of left R-modules,

0 → M → G_n → · · · → G_3 → G_2 → 0,

and

0 → M → H_n → · · · → H_3 → H_2 → 0,

where G_0, · · · , G_n−1 and H_0, · · · , H_n−1 are Gorenstein FP}_n injective, then G_n is Gorenstein FP}_n injective if and only if H_n is Gorenstein FP}_n injective.

Proof. It is obtained by Proposition 2.4 and [10, Lemma 2.1].

Proposition 2.6 Let n > 1. Then the following are true for any (n−1)-coherent ring R.

(1) Ext}^i_R(F, M) = 0 for all finitely n-presented left R-modules F.

(2) If 0 → T → M → L → 0 is a short exact sequence of left R-modules with T and M FP}_n-injective, then L is FP}_n-injective.

Proof. Let F be a finitely n-presented left R-module. There exists an exact sequence 0 → T → P → F → 0, with P finitely generated and T finitely (n−1)-presented. Consider exact sequences

⋯ → Ext}^i_R(T, N) → Ext}^i_R(F, N) → Ext}^i_R(P, N) → · · · ,

since R is (n−1)-coherent, Ext}^i_R(T, N) = 0. Ext}^i_R(F, N) = 0. By dimension shifting Ext}^i_R(F, N) = 0.

(2) Let 0 → T → M → L → 0 be a short exact sequence. If N and M are FP}_n-injective, consider the exact sequence

⋯ → Ext}^i_R(F, M) → Ext}^i_R(F, L) → Ext}^i_R(F, N) → · · · ,

By (1) we can get Ext}^i_R(F, N) = 0, therefore Ext}^i_R(L, F) = 0. so L is FP}_n-injective.

Definition 2.7 Let M be a left R-modules and n > 1. Put FP}_n-id(M) = inf {m | 0 → M → E_0 → · · · → E_{m−1} → E_m → 0 is an FP}_n-injective of M).

If no such m exists, set FP}_n-id(M) = ∞.

Then we call FP}_n-id(M) the FP}_n-injective dimension of M.

Definition 2.8 Let n > 1 and N a left R-modules. Put FP}_n-fd(N) = inf {m | 0 → F_m → F_{m−1} → · · · → F_0 → N → 0 is an FP}_n-flat of N).

If no such m exists, set FP}_n-fd(N) = ∞.

Then we call FP}_n-fd(N) the FP}_n-flat dimension of N.

Proposition 2.9. Let R be an (n−1)-coherent ring and n > 1.

Then the following conditions are equivalent for any left R-module M.

(1) FP}_n-id(M) ≤ m.

(2) Ext}^i_R(Q, M) = 0 for all FP}_n-injective R-modules Q.

(3) Ext}^i_R(Q, M) = 0 for all k ≥ 1, and all finite n-presented R-modules F.

(4) For every exact sequence 0 → M → E_0 → · · · → E_{m−1} → K → 0 where E_0, · · · , E_{m−1} are FP}_n-injective, then also K is FP}_n-injective.

Proof. It is easy to prove by dimension shifting and Proposition 2.6.

Proposition 2.10. Let R be an (n−1)-coherent ring and n > 1.

Then following conditions are equivalent for N is a left R-modules.

(1) FP}_n-fd(N) ≤ m;

(2) Tor}^k_R(N, F) = 0 for all finite n-presented R-modules F.

(3) Tor}^k_R(N, F) = 0 for all k ≥ 1, and all finite n-presented R-modules F.

(4) For every exact sequence 0 → K → F_{m−1} → · · · → F_0 → N → 0 where F_0, · · · , F_{m−1} are FP}_n-flat, then also K is FP}_n-flat.

Proof. It is similar to the proof of Proposition 2.9.

Proposition 2.11. Let R be an (n−1)-coherent ring and n > 1.

Then following conditions are equivalent for C is a left R-modules.

(1) FP}_n-fd(C) = FP}_n-id(C);

(2) FP}_n-id(C) = FP}_n-fd(C).

Proof. This follows from the definition and [7, Proposition 3.5].

Theorem 2.12. Let R be an (n−1)-coherent ring and n > 1.

Then following conditions are equivalent for a left R-modules M.

(1) M is Gorenstein FP}_n injective.

(2) M has an exact left FP}_n resolution and Ext}^i_R(Q, M) = 0 for all left R-modules Q with FP}_n-id(Q) < ∞ and all i ≥ 1.

(3) M has an exact left FP}_n resolution and Ext}^i_R(Q, M) = 0 for all all FP}_n-injective left R-modules Q and all i ≥ 1.

Moreover, if FP}_n-id(R) < ∞, then the above conditions are equivalent to

(4) Ext}^i_R(Q, M) = 0 for all FP}_n-injective left R-modules Q, and all i ≥ 1.

Proof. (1) ⇒ (2) is clear. (2) ⇒ (3) hold by dimension shifting.

(2) ⇒ (4) Obvious.

(3) ⇒ (1) Let f : E_0 → M be an FP}_n-injective cover of M. Consider the short exact sequence

0 → E_0 → E → C_0 → 0,

where E is injective and C_0 is FP}_n-injective. Denote i : E_0 → E.

Consider the exact sequence

0 → Hom_R(C_0, M) → Hom_R(E, M) → Hom_R(E_0, M) → Ext}^1_R(C_0, M) = 0.

For every f : E_0 → M , there exists g : E_0 → M such that g = f. Since f is cover, there exists a homomorphism h : E → E_0 such that fh = g. Therefore f = h, and h is an isomorphism. It follows that E_0 is injective. Thus, for any FP}_n-injective Q, there is the exact sequence

Hom_R(Q, E_0) → Hom_R(Q, Imf) → Ext}^1_R(Q, Kerf) → 0.

In addition, the exactness of 0 → Kerf → E_0 → Imf → 0 yields the exact sequence

Hom_R(Q, E_0) → Hom_R(Q, Imf) → 0. Hence Ext}^1_R(Q, Kerf) = 0.

Hence Ext}^1_R(Q, Kerf) = 0. So Kerf has FP}_n-injective cover E_1 → Kerf with E_1 is injective. Continuing this process, we can get a Hom_R(Q, −) exact complex

⋯ → E_1 → E_0 → M → 0

with E_i is injective. Note that Ext}^i_R(RM) = 0 for all i ≥ 1 and Ext}^i_R(RM) = M since M has an exact left FP}_n resolution. So the complex

⋯ → E_1 → E_0 → M → 0

is exact. On the other hand Ext}^i_R(QM) = 0 for all FP}_n-injective Q and all i ≥ 1. So M is Gorenstein FP}_n injective.

(4) ⇒ (1) By the proof of (3) ⇒ (1), we obtain an exact complex

ε = · · · → E_1 → E_0 → E → E_1 → · · · such that M = Im{E_0 → E_0}, and for all FP}_n-injective Q.Hom(Q, ε) is exact. Next we will show that Hom(Q, ε) is exact for any left R-module Q with FP}_n-id(Q) < ∞. We proceed by induction on m. The case m = 0 is clear. Let m ≥ 1.

There is an exact sequence

0 → Q → H → L → 0

with H injective, which induces an exact sequence

0 → Hom(L, ε) → Hom(H, ε) → Hom(Q, ε) → 0
of complexes. Note that \( \text{FP}_n \text{-id}(L) = m - 1 \), so \( \text{Hom}(L, \epsilon) \) is exact. Thus \( \text{Hom}(Q, \epsilon) \) is exact. In particular, since \( \text{FP}_n \text{-id}(\mathbb{R}) < \infty \), \( \text{Hom}(\mathbb{R}, \epsilon) \) is an exact. Therefore \( \epsilon \) is an exact sequence. So \( M \) is Gorenstein \( \text{FP}_n \) injective.

III. CONCLUSION

We give some equivalent characterizations of Gorenstein \( \text{FP}_n \) injective modules in \( (n-1) \)-coherent ring.

REFERENCES