Gorenstein FP$_n$ injective modules

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Abstract— **In this paper, we introduce and study Gorenstein FPn injective modules and investigate the homological properties of them.**

***Index Terms*— Gorenstein FPn injective modules, dimensions, (n-1)-coherent ring.**

I. INTRODUCTION

The flat modules and FP-injective modules play an important role in characterizing coherent rings. Naturally, many literature articles generalized these notations in relative homological algebra. In [5], Costa introduced absolutely clean and level modules. In [6], Chen and Ding introduced *n*-flat and *n*-FP injective modules. In 2015, Wei and coauthors call them FP$_n$-injective and FP$_n$-flat , respectively. In 2017, Bravo and others investigate *n*-coherent and give some equivalent characterizations of (*n*-1)-coherent ring [7]. On the other hand, Enochs and Jenda introduced Gorenstein projective, injective, Gorenstein flat modules, and developed Gorenstein homological algebra in [2, 3.4]. Later, many scholars further studied these modules and introduced various generalizations of these modules. In [11], Mao and Ding gave a definition of Gorenstein FP-injective modules. However, under their definition these Gorenstein FP-injective modules are stronger than the Gorenstein injective modules. In 2014, Bravo et al. introduced in [1] the notion of Gorenstein AC-projective (resp., Gorenstein AC-injective) modules and established the "Gorenstein AC-homological algebra" over an arbitrary ring.

Inspired by aforementioned work, we introduce the concept of Gorenstein FP$_n$-injective modules as a generalization of above Gorenstein homological modules. Then we character when a left module is Gorenstein FP$_n$-injective over (*n*-1)-coherent rings. In the following, we recall some notions that will be used throughout the paper.

Definition 1.1[3] A left *R*-module *M* is called Gorenstein FP-injective, if there exists an exact sequence

E=$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$

of injective left *R*-module with $M = \mathrm{Im}\,(E_0 \to E^0)$ such that the functor $\mathrm{Hom}_R(Q, -)$ leaves the sequence exact whenever *Q* is FP-injective.

Definition 1.2[5] A ring *R* is called *n*-coherent ring, if every finite *n*-presented module coincident with finite (*n*+1)-presented module.

Definition 1.3[5] A *R*-modules *M* is called finite *n*-presented, if there exists an exact sequence of left *R*-modules

$P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0,$

where P_i is finitely generated projective for $0 \le i \le n$. Such exact sequence is called a finite *n*-presentation of *M*.

Definition 1.4[7] A right *R*-modules *N* is called FP$_n$-flat if $\mathrm{Tor}^R_1(N,F) = 0$ for any finite *n*-presented module F.

A left *R*-modules *M* is called FP$_n$-injective if $\mathrm{Ext}^1_R(F, M) = 0$ for any finite *n*-presented module F.

II. GORENSTEIN FP$_N$ INJECTIVE MODULES

Definition 2.1 A left *R*-module *M* is called Gorenstein FP$_n$ injective, if there exists an exact sequence

E=$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$

of injective left *R*-module with $M = \mathrm{Im}\,(E_0 \to E^0)$ such that the functor $\mathrm{Hom}_R(Q, -)$ leaves the sequence exact whenever *Q* is FP$_n$-injective.

Remark 2.2 (1) It is clear that each injective module is Gorenstein FP$_n$ injective.

(2) If *M* is a Gorenstein FP$_n$ injective module, by symmetic all the kernels, the images, and the cokernels of **E** are Gorenstein FP$_n$ injective module.

(3) Gorenstein AC injective ⊆ Gorenstein FP$_n$ injective ⊆ Ding injective ⊆ Gorenstein injective.

(4) If $n = 0$, then Gorenstein injective modules are Gorenstein FP$_n$ injective.

(5) If *R* is *n*-coherent, then Gorenstein FP$_n$ injective modules are Gorenstein injective; If *R* is coherent, then Ding injective modules are Gorenstein FP$_n$ injective.

(6) The class of Gorenstein FP$_n$ injectives is closed under direct summands.

Theorem 2.3 The following assertions are equivalent for a left *R*-module *M*.

(1) *M* is Gorenstein FP$_n$ injective.

(2) *M* has an exact left injective resolution which is $\mathrm{Hom}_R(Q, -)$-exact all FP$_n$-injective left *R*-modules *Q*, $\mathrm{Ext}^i_R(Q, M) = 0$ for all $i \geq 1$.

(3) There exist a short exact sequence of left *R*-modules $0 \to K \to E \to M \to 0$, where *E* is injective and *K* is Gorenstein FP$_n$ injective.

Proof. (1) ⇔ (2), (1) ⇒ (3) is clear by the definition of Gorenstein FP$_n$ injective module.

(3) ⇒ (2) Since *K* is Gorenstein FP$_n$ injective, there exist an exact sequence

$\cdots \to E_1 \to E_0 \to K \to 0,$

which is $\mathrm{Hom}_R(Q, -)$ –exact, where Q is FP$_n$-injective and E_i are injective for all $i \geq 0$.

Note that the exact sequence of left *R*-modules $0 \to K \to E \to M \to 0$ is $\mathrm{Hom}_R(Q, -)$-exact, so we obtain an left injective resolution of *M*

$\cdots \to E_1 \to E_0 \to E \to M \to 0.$

On the other hand, for all FP$_n$-injective *Q*, we have an exact sequences of left *R*-modules

$\cdots \to \mathrm{Ext}^i_R(Q,E) \to \mathrm{Ext}^i_R(Q,M) \to \mathrm{Ext}^{i+1}_R(Q,K) \to \cdots.$

By dimension shifting, $\mathrm{Ext}^{i+1}_R(Q, K) = \mathrm{Ext}^i_R(Q, E) = 0$ for all $i \geq 1$, therefore $\mathrm{Ext}^i_R(Q,M) = 0$. So *M* is Goren stein FP$_n$-injective by (1) ⇔(2).

Proposition 2.4 Let $0 \to A \to B \to C \to 0$ be an short exact sequence of left *R*-modules.

(1) If *A* and *C* are Gorenstein FP$_n$ injective, then so is *B* .

(2) If *A* and *B* are Gorenstein FP$_n$ injective, then so is *C*.

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(3) If B and C are Gorenstein FP_n injective, then A is Gorenstein FP_n injective if and only if $Ext^1_R(Q, A) = 0$ for all FP_n-injective left R-modules Q.

Proof. This is similar to the proof of [4, Theorems 2.8, 2.11].

Lemma 2.5 Let M be a left R-module. Consider two exact sequences of left R-modules,

$0 \to M \to G_0 \to \cdots \to G_{n-1} \to G_n \to 0,$

and

$0 \to M \to H_0 \to \cdots \to H_{n-1} \to H_n \to 0,$

where $G_0, \cdots, G_{n-1}$ and $H_0, \cdots, H_{n-1}$ are Gorenstein FP_n injective , then G_n is Gorenstein FP_n injective if and only if H_n is Gorenstein FP_n injective .

Proof. It is obtained by Proposition 2.4 and [10, Lemma 2.1].

Proposition 2.6 Let $n > 1$. Then the following are ture for any (n-1)-coherent ring R.

(1) $Ext^i_R(F, M) = 0$ for all finitely n-presented left R-modules F.

(2) If $0 \to T \to M \to L \to 0$ is a short exact sequence of left R-modules with T and M FP_n-injective, then L is FP_n-injective.

Proof. Let F be a finitely n-presented left R-module. There exists an exact sequence $0 \to T \to P \to F \to 0$, with P finitely generated projective and T finitely (n-1)-presented. Consider exact sequences

$\cdots \to Ext^1_R(T, N) \to Ext^2_R(F, N) \to Ext^2_R(P, N) \to \cdots.$

since R is (n-1)-coherent, $Ext^1_R(T, N) = 0$. $Ext^2_R(F, N) = 0$. By dimension shifting $Ext^i_R(F,N) = 0$.

(2) Let $0 \to T \to M \to L \to 0$ be a short exact sequence. If N and M areFP_n-injective, consider the exact sequence

$\cdots \to Ext^1_R(F, M) \to Ext^1_R(F, L) \to Ext^2_R(F, N) \to \cdots.$

By (1) we can get $Ext^2_R(F, N) = 0$, therefore $Ext^1_R(F, L) = 0$. so L is FP_n-injective.

Definition 2.7 Let M be a left R-modules and $n > 1$. Put

$FP_n\text{-id}(M) = \inf\{m \mid 0 \to M \to E_0 \to \cdots \to E_{m+1} \to E_m \to 0 \text{ is an } FP_n \text{ injective of } M\}$.

If no such m exists, set $FP_n\text{-id}(M) = \infty$.

Then we call FP_n-id(M) the FP_n-injective dimension of M.

Definition 2.8 Let $n > 1$ and N a left R-modules. Put

$FP_n\text{-fd}(N) = \inf\{m \mid 0 \to F_m \to F_{m-1} \to \cdots \to F_0 \to N \to 0 \text{ is an } FP_n \text{ flat of } N\}$

If no such m exists, set $FP_n\text{-fd}(N) = \infty$.

Then we call FP_n-id(N) the FP_n-flat dimension of N.

Proposition 2.9. Let R be an (n-1)-coherent ring and $n > 1$. Then the following conditions are equivalent for any left R-module M.

(1) $FP_n\text{-id}(M) \le m$.

(2) $Ext^m_R(Q, M) = 0$ for all FP_n-injective R-modules Q .

(3) $Ext^{m+k}_R(Q, M) = 0$ for all $k \ge 1$, and all finite n-presented R-modules F.

(4) For every exact sequence $0 \to M \to E_0 \to \cdots \to E_{m-1} \to K \to 0$ where $E_0, \cdots, E_{m-1}$ are FP_n-injective, then also K is FP_n-injective .

Proof. It is easy to prove by dimension shifting and Proposition 2.6.

Proposition 2.10. Let R be an (n-1)-coherent ring and $n > 1$. Then following conditions are equivalent for N is a left R-modules.

(1) $FP_n\text{-fd}(N) \le m$;

(2) $Tor^R_{m+1}(N, F) = 0$ for all finite n-presented R-modules F.

(3) $Tor^R_{m+k}(N, F) = 0$ for all $k \ge 1$, and all finite n-presented R-modules F.

(4) For every exact sequence $0 \to K \to F_{m-1} \to \cdots \to F_0 \to N \to 0$ where $F_0, \cdots, F_{m-1}$ are FP_n-flat , then also K is FP_n-flat.

Proof. It is similar to the proof of Proposition 2.9.

Proposition 2.11 Let R be an (n-1)-coherent ring and $n > 1$. Then following conditions are equivalent for C is a left R-modules.

(1) $FP_n\text{-fd}(C) = FP_n\text{-id}(C^+)$;

(2) $FP_n\text{-id}_R(C) = FP_n\text{-fd}(C^+)$.

Proof. This follows from the definition and [7, Proposition 3.5].

Theorem 2.12 Let R be an (n-1)-coherent ring and $n > 1$. Then following conditions are equivalent for a left R-modules M .

(1) M is Gorenstein FP_n injective.

(2) M has an exact left FP_n resolution and $Ext^i_R(Q, M) = 0$ for all left R-modules Q with $FP_n\text{-id}(Q) < \infty$ and all $i \ge 1$.

(3) M has an exact left FP_n resolution and $Ext^i_R(Q, M) = 0$ for all FP_n-injective left R-modules Q and all $i \ge 1$.

Moreover, if $FP_n\text{-id}(R) < \infty$, then the above conditions are equivalent to

(4) $Ext^i_R(Q, M) = 0$ for all FP_n-injective left R-modules Q, and all $i \ge 1$.

Proof. (1) ⇒ (2) is clear. (2) ⇒ (3) hold by dimension shifting. (2) ⇒ (4) Obvious.

(3) ⇒ (1) Let $f : E_0 \to M$ be an FP_n-injective cover of M. Consider the short exact sequence

$0 \to E_0 \to E \to C_0 \to 0,$

where E is injective and C_0 is FP_n-injective. Denote $i : E_0 \to E$. Consider the exact sequence

$0 \to Hom_R(C_0, M) \to Hom_R(E, M) \to Hom_R(E_0, M) \to Ext^1_R(C_0, M) = 0.$

For every $f : E_0 \to M$, there exists $g: E_0 \to M$ such that gi = f. Since f is cover, there exists a homomorphism $h : E \to E_0$ such that fh = g. Therefore fhi = f, and hi is an isomorphism. It follows that E_0 is injective. Thus, for any FP_n-injective Q, there is the exact sequence

$Hom_R(Q, E_0) \to Hom_R(Q, \mathrm{Imf}) \to Ext^1_R(Q, \mathrm{Kerf}) \to 0$.

In addition, the exactness of $0 \to \mathrm{Kerf} \to E_0 \to \mathrm{Imf} \to 0$ yields the exact sequence

$Hom_R(Q, E_0) \to Hom_R(Q, \mathrm{Imf}) \to 0$, Hence $Ext^1_R(Q, \mathrm{Kerf}) = 0$.

Hence $Ext^1_R(Q, \mathrm{Kerf}) = 0$. So Kerf has FP_n-injective cover $E_1 \to \mathrm{Kerf}$ with E_1 is injective. Continuing this process, we can get a $Hom_R(Q, -)$ exact complex

$\cdots \to E_1 \to E_0 \to M \to 0$

with E_i is injective. Note that $Ext^i_R(R, M) = 0$ for all $i \ge 1$ and $Ext^0_R(R, M) = M$ since M has an exact left FP_n resolution. So the complex

$\cdots \to E_1 \to E_0 \to M \to 0$

is exact. On the other hand $Ext^i_R(Q, M) = 0$ for all FP_n-injective Q and all $i \ge 1$. So M is Gorenstein FP_n injective.

(4) ⇒ (1) By the proof of (3) ⇒ (1), we obtain an exact complex

$\varepsilon = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$

such that $M = \mathrm{Im}(E_0 \to E^0)$, and for all FP_n-injective Q, $Hom(Q, \varepsilon)$ is exact. Next we will show that $Hom(Q, \varepsilon)$ is exact for any left R-module Q with $FP_n\text{-id}(Q) < \infty$ .We proceed by induction on m. The case $m = 0$ is clear. Let $m \ge 1$. There is an exact sequence

$0 \to Q \to H \to L \to 0$

with H injective, which induces an exact sequence

$0 \to Hom(L, \varepsilon) \to Hom(H, \varepsilon) \to Hom(Q, \varepsilon) \to 0$

of complexes. Note that FP_n-id(L) = $m-1$, so Hom(L, ε) is exact. Thus Hom (Q, ε) is exact. In particular, since FP_n-id($_RR$) < ∞, Hom($_RR$, ε) is an exact. Therefore ε is an exact sequence. So M is Gorenstein FP_n injective.

III. Conclusion

We give some equivalent characterizations of Gorenstein FP_n injective modules in (n-1)-coherent ring.

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