Control design using backstepping technique for a cart-inverted pendulum system

Tran Thien Dung, Nguyen Nam Trung, Nguyen Van Lanh

Abstract—The cart-inverted pendulum system is one of the classical experimental systems that fully converges the complex properties of nonlinear control problems. It represents a class of real world systems such as two-wheeled mobile robots, pendulums, missile launchers and many more. The problems associated with it are always challenging topics in the field of control systems. This paper presents a novel technique to control this system stabilizing at a vertical upright position, its unstable equilibrium point. Simulation and experimental results will show a better performance of the proposed controller in comparison with Quadratic Optimal Regulator method under disturbance and change in mass.

Index Terms—Cart-inverted pendulum, Backstepping, DC motor, Quadratic Optimal Regulator.

I. INTRODUCTION

The cart-inverted pendulum system has two equilibrium points [1], [8] the stable point is at which the pendulum is pointing downwards and the unstable one is at which the pendulum is pointing upwards. The aim of designing a controller is to move and balance the pendulum from the stable equilibrium point to the unstable one. This is a challenging control problem because the system is highly unstable, nonlinear and underactuated. Different control algorithms are studied by many researchers, from classical PID controllers [2], [13] to advanced controllers such as fuzzy control [3], [14] neural networks [4], [15] and genetic algorithms [5], [16]. Recently, optimal control approach is one of the potential solutions for a given set of performance objectives [17], [21], with detail review in [6]. In [7] and [20], state space control using Linear Quadratic Regulator (LQR) is presented and successfully conducted.

The goal of this article is to design controllers to swing up and balance the pendulum from a pending position to the vertical upward point. Swinging up the pendulum can be achieved by using an energy control [8], [18], [22]. At the vertical position, another controller is used to stabilize the pendulum. In this paper, a stabilizing controller based on backstepping technique [9], [10], [19], is designed and compared to the Quadratic optimal controller [11], [12], [23]. A switch is used to change controllers. This means, when the pendulum approaches a certain area, the stabilizing controller will replaces the swinging up controller to balance the pendulum at the vertical upward position.

The paper is organized as follows. System model is provided in section II, including nonlinear dynamic model of the system, linearized model in state-space form and permanent magnet DC motor dynamics. Section III presents controller design. Then, section IV shows simulation and experimental results. Finally, Section 5 concludes this paper.

II. SYSTEM MODELS

A. Nonlinear Dynamic Model

In our research, the model of inverted pendulum system is pre-designed and simulated on 3D Solidworks software. Then, an experimental setup is built as shown in Fig. 1. The setup consists of a movable cart driven by a DC motor according to the control voltage. The cart can move along a horizontal track. A pendulum is mounted on the cart and can freely rotate around its axis.

![Fig. 1: Snapshot of Real plant](image)

![Fig. 2: Reference frames and parameters of pendulum](image)

The inverted pendulum is an open-loop, unstable and highly nonlinear system. The objective of the controller is to balance the pendulum at its upward position. Parameters of the system are showed in Table 1.
Figure 2 shows the reference frames and parameters of the system. The movement of the cart is constrained in the x-horizontal direction, and the pendulum can rotate in the x-y plane. The system has two DOF and can be fully represented using two coordinates: horizontal displacement of the cart, s; and rotational displacement of the pendulum, \( \varphi \). Coordinates of the Centre of Gravity (CoG) of the pendulum is given by:

\[
\begin{align*}
\mathbf{c}_1 &= \left[ s - \alpha_1 \sin \varphi \quad \alpha_1 \cos \varphi \quad 0 \right]^T \\
\mathbf{c}_2 &= \left[ s - \alpha_1 \phi \cos \varphi - \alpha_1 \phi \sin \varphi \quad 0 \right]^T
\end{align*}
\]

(1)

Table 1: Parameters of the inverted pendulum

<table>
<thead>
<tr>
<th>Variable</th>
<th>Unit</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi )</td>
<td>rad</td>
<td>Angular displacement of the pendulum from the vertical upright position.</td>
</tr>
<tr>
<td>( s )</td>
<td>m</td>
<td>Cart displacement</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>kg.m^2</td>
<td>Moment of inertia of the pendulum.</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>kg</td>
<td>Mass of the pendulum</td>
</tr>
<tr>
<td>( m )</td>
<td>kg</td>
<td>Mass of the cart</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>m</td>
<td>Distance from the CoG of the pendulum to the pivot.</td>
</tr>
<tr>
<td>( g )</td>
<td>m/s^2</td>
<td>Acceleration of gravity</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>Nm.s</td>
<td>Friction coefficient with the rail</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>Nm.s</td>
<td>Friction coefficient of pendulum</td>
</tr>
<tr>
<td>( R_m )</td>
<td>( \Omega )</td>
<td>Armature resistance of motor</td>
</tr>
<tr>
<td>( L_m )</td>
<td>H</td>
<td>Armature inductance of motor</td>
</tr>
<tr>
<td>( K_m )</td>
<td>Wb</td>
<td>Pully constant</td>
</tr>
<tr>
<td>( r )</td>
<td>m</td>
<td>Pully radius</td>
</tr>
</tbody>
</table>

Applying Euler-Lagrangian equation to the system yields:

\[
d \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = F - \frac{\partial R}{\partial \dot{\varphi}}
\]

where \( L \) is the Lagrangian function defined as the difference between kinetic (T) and potential (V) energies: \( L = T - V \). \( T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} m_1 \left[ \dot{s} - \alpha_1 \sin \varphi \right]^2 + \alpha_1^2 \phi^2 \sin^2 \varphi + \frac{1}{2} I \phi^2, V = m_1 g a_1 \cos \varphi; R = \frac{1}{2} (d_1 + d_0) \dot{\varphi} \)

\( q = [\varphi \quad \dot{s}]^T; \quad F = [0 \quad \tau]^T, \quad \tau = M / R \)

\[
\begin{align*}
\frac{dL}{d\varphi} &= (J + m_2 \dot{\varphi}^2) \dot{\varphi} + (-m_1 \alpha_1 \sin \varphi) \ddot{\varphi} \\
\frac{dL}{d\dot{\varphi}} &= (J + m_2 \dot{\varphi}^2) \dot{\varphi} + (-m_1 \alpha_1 \cos \varphi) \ddot{\varphi} + (m_2 \alpha_1 \phi \sin \varphi) \dddot{\varphi} \\
\frac{dL}{ds} &= (-m_1 \alpha_1 \sin \varphi) \dot{\varphi} + (m_1 + m_2) \dot{\varphi} \\
\frac{dL}{d\dot{s}} &= (m_1 \alpha_1 \cos \varphi) \dot{\varphi} + (m_1 + m_2) \dot{\varphi} + m_1 \alpha_1 \phi \ddot{\varphi} + (m + m_2) \dot{s} \ddot{\varphi} + m_1 \alpha_1 \phi \dddot{\varphi} + m + m_2 = 0
\end{align*}
\]

(3)

\[
\mathbf{D}(q) \ddot{q} + \mathbf{C}(q, \dot{q}) \dot{q} + \mathbf{G} \dot{q} + \mathbf{g}(q) = F
\]

Linearizing the model, the following approximations are applied: \( \varphi \approx 0 \Rightarrow \sin \varphi \approx \varphi, \cos \varphi = 1 \)

Defining the state variables as below:

\( x_1 = q, x_2 = \dot{q}, x_3 = s; x_4 = \dot{s}; x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T \)

The linearized model, thus, becomes:

\[
\begin{align*}
\dot{x}_1 &= \frac{(m + m_1)(-d_1 x_2 + m_1 a_1 g x_3) + m_1 a_1 (\frac{M}{R} - d_0 x_4)}{(J_1 + m_1 a_1^2)(m + m_2) - m_1 a_1^2} \\
\dot{x}_2 &= \frac{(m + m_1)(-d_1 x_2 + m_1 a_1 g x_3) + m_1 a_1 (\frac{M}{R} - d_0 x_4)}{(J_1 + m_1 a_1^2)(m + m_2) - m_1 a_1^2} \\
\end{align*}
\]

B. Linearized Model in State-Space Form

Linearizing the inverted pendulum system results in:

\[
A = \begin{bmatrix}
\frac{0}{0} & 1 & 0 & 0 \\
\frac{-m_1}{m_1} & \frac{0}{0} & 1 & 0 \\
\frac{0}{0} & \frac{-m_1}{m_1} & \frac{0}{0} & 1 \\
\frac{0}{0} & \frac{1}{m_1} & \frac{0}{0} & \frac{-m_1}{m_1} \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\frac{0}{0} & \frac{m_1}{m_1} & \frac{0}{0} & \frac{-m_1}{m_1} \\
\frac{0}{0} & \frac{J_1}{m_1} & \frac{0}{0} & \frac{-m_1}{m_1} \\
\end{bmatrix}
\]

Table 2: List of Parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_i )</td>
<td>0.0052</td>
<td>kg.m^2</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>0.43</td>
<td>kg</td>
</tr>
<tr>
<td>( m )</td>
<td>1.3</td>
<td>kg</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>0.157</td>
<td>m</td>
</tr>
<tr>
<td>( g )</td>
<td>9.81</td>
<td>m/s^2</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>0.147</td>
<td>Nm.s</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>0.00243</td>
<td>Nm.s</td>
</tr>
</tbody>
</table>

Substituting the parameters given in Table 1 into (4), we obtain:
In the classical sense, a PI controller has the following transfer function:

\[
W_s = K_p \left( 1 + \frac{1}{s} \right) = 0.32 \left( 1 + \frac{1}{660.16 \frac{1}{s}} \right)
\]  

The inner loop needs a fast response. Using PI controller with the above parameters, the system has a Settling Time of 0.008s. Therefore, the designed PI controller meets the requirement.

### III. DESIGN OF CONTROLLERS

#### A. Design and Simulation of inverted pendulum Quadratic optimal regulator problem

The system equation in the state space is represented as:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

We determine the matrix \( K \) of the optimal control vector \( u = -Kx \) to minimize the performance index:

\[
J = \frac{1}{2} \int_0^\infty \left( x^T Q x + u^T R u \right) dt \rightarrow \text{min}
\]

Where \( Q \) and \( R \) are weighting matrices. In this problem, we assume that the control vector \( u(t) \) is unconstrained. The linear control law given by Eq. (8) is the optimal control law. The matrix \( K \) is determined by minimizing the performance index \( J \), then \( u(t) = -Kx(t) \) is the optimal control signal for any initial state \( x(0) \). The block diagram is shown in Fig 4.

#### B. Backstepping linear design

The new control variables are defined as:

\[
\begin{align*}
z_i &= x_i - k_i x_3 \\
z_i &= x_2 - k_i x_4
\end{align*}
\]

Define \( x_2 \)
as the virtual control variable, for which the stabilizing function is chosen: \( \alpha_i = -c_i z_i + k_i x_i \), where \( c_i \) is positive. In addition, the corresponding error state variable is defined as \( z_2 = x_1 - \alpha_i \). So, we have: \( \dot{z}_2 = x_1 - k_i \dot{x}_1 - z_2 - c_i z_i \). The derivative of \( z_2 \) is computed as follows:
\[
\dot{z}_2 = \ddot{x}_2 - \dot{\alpha}_i = \ddot{x}_2 - k_i \dot{x}_1 - c_i \dot{z}_1 + c_i x_2 - c_i k_i \dot{x}_1.
\]
However, the desired dynamics of \( z_2 \) can be defined:
\[
\ddot{z}_2 = -c_i \dot{z}_2 + c_i x_2 - c_i k_i \dot{x}_1.
\]
From these above equations, we design a controller as below:
\[
u_1 = \frac{1}{2.9642 - k_1} (h_k x_1 + h_k x_2 + h_k x_3 + h_k x_4) (10)
\]
\[
h_1 = -51.307 + k_1 \cdot 1.9637 - c_i c_2
\]
\[
h_2 = 0.1846 - k_2 \cdot 0.007203 - (c_1 + c_3)
\]
\[
h_3 = k_3 (1 + c_i c_4)
\]
\[
h_4 = 0.4357 - k_1 (0.102 - (c_1 + c_3))
\]
Analyzing stability of the system, we have:
\[
\dot{V}_2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 = z_1 \dot{z}_1 + z_2 \dot{z}_2 = z_1 (z_2 - c_i z_2) + + z_2 (x_2 - k_i x_1 + c_i x_2 - c_i k_i \dot{x}_1) = -c_i \ddot{z}_2 - c_i \ddot{z}_2 \leq 0
\]
This implies that \( z_1, z_2 \) are stable, the state trajectory approaches to the origin, so \( x_1, x_2 \) are also stable. Note that it is important to choose \( k_i \) appropriately to stabilize the closed-loop system. This means \( k_i \) is chosen so that \( x_1, x_2 \) are also approaches to zero. As a result, the backstepping controller not only keeps the pendulum at the vertical upright position, but also moves the cart to its original position.

<table>
<thead>
<tr>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( k_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100</td>
<td>0.03</td>
</tr>
</tbody>
</table>

C. Swing-Up Control
Neglecting frictions and assuming pendulum as a rigid body, we obtain the equation of motion of the pendulum:
\[
\sin \phi = \frac{1}{m \mu g} \left( (J_1 + m \mu_1^2) \ddot{\phi} - m \mu_1 \cos \phi \dot{u} \right)
\]
We choose the energy of the system as zero in the lower position, and normalize it by \(-m \mu_1 g\), which is the energy required to raise the pendulum from the hanging down position to the horizontal position. The normalized energy can be then written as below:
\[
E = \frac{1}{2} \left( J_1 + m \mu_1^2 \right) \dot{\phi}^2 + m \mu_1 g \cos \phi + 1
\]
Computing the derivative of \( E \) with respect to time we find:
\[
\dot{E} = \dot{\phi} \left( J_1 + m \mu_1^2 \right) \ddot{\phi} - m \mu_1 g \phi \sin \phi
\]
\[
\dot{E} = -m \mu_1 g u \phi \cos \phi \quad u = 0.117 \%
\]
Define the desired energy as \( E_0 = 2 m \mu_1 g \). The following control is a strategy for achieving the desired energy
\[
u = -m \mu_1 g \left( E - E_0 \right) \phi \cos \phi
\]
To change the energy fast, the magnitude of the control signal should be as large as possible. This is achieved with the control law:
\[
u = -k_2 \cdot \text{sign} \left( (E - E_0) \phi \cos \phi \right) \quad (11)
\]
where \( k_2 \) is a design parameter.

IV. SIMULATION AND EXPERIMENTAL RESULTS
A. Simulation results
Block diagram and simulation result of the controller using swing up in combination with Quadratic optimal control are shown in Figure 6 and Figure 7.

Block diagram and simulation result of the controller using swing up combined with backstepping control are shown in Figure 8 and Figure 9.
Fig. 7 and Fig. 9 show that the transition time of the system using Quadratic optimal regulator is nearly 4 seconds, while using backstepping control is only 2.2 seconds. This means that the Backstepping control is much better than the Quadratic optimal regulator.

B. Experimental results

![Block diagram of experimental setup](image)

**Fig. 10:** Block diagram of experimental setup

**Figure 10 shows the block diagram of the experimental setup. Experimental results of the controller using swing up combined with Quadratic optimal control in Figure 11 and with the backstepping control in Figure 12. It can be seen that control input $u$ from a combination of a swing up controller and a stabilizing controller is able to move and balance the pendulum from its stable equilibrium point, $x=[\pi,0,0,0]^T$, to its unstable equilibrium point, $x=[0,0,0,0]^T$. We also see that the backstepping controller can guarantee a faster and smoother stabilizing process with less oscillation and more robustness than the Quadratic optimal regulator design.**

**Figure 11: Experimental Swing-Up & Stabilization using Quadratic Optimal Regulator**

**Figure 12: Experimental Swing-Up & Stabilization using Backstepping control**

V. CONCLUSION

The proposed controller has achieved that the closed-loop system is able not only to swing up and balance the pendulum from downward position to the upward equilibrium point, but also to return the cart to its original position on the rail. The pendulum is stable at its upward position. This proves that the control algorithm is effective. In additions, the performance of controller using backstepping technique is significantly better than that using Quadratic optimal regulator.

Simulation and experimental results are almost similar. In experimental results, however, the pendulum still oscillates slightly around the equilibrium position. This could be due to the dynamic uncertainty, pinion backlash, motor dead-zone, magnetic hysteresis, and other mechanical imperfections. More details about the experiment and its results can be found at: https://m.youtube.com/watch?v=RfKzVgG2Z0.

Our future research is control design for the triple link inverted pendulum system, as shown in Fig. 13.

**Figure 13: 3D Solidworks Triple inverted pendulum system**

REFERENCES


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