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Spectral Analysis of Quantum Bernoulli Noise

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***Abstract*— Quantum Bernoulli noise is the family of annihilation and creation operators acting on Bernoulli functionals,which satisfy the anti-commutation relations (ACR) in equal-time . In this paper , we study spectral structure of operators related to quantum Bernoulli noise . Among others, we obtain spectral theorems for these operators.**

***Index Terms*—Quantum Bernoulli noise, Spectral theorem. MSC(2010):—60H40, 47B38.**

I. INTRODUCTION

Quantum Bernoulli noise [5] is the family of annihilation and creation operators acting on Bernoulli functionals (Bernoulli annihilators and creators for short) , which satisfy the anti-commutation relations (ACR) in equal-time, and can be viewed as a discrete-time analog of quantum white noise [2]. In the past two decades, quantum Bernoulli noise has found wide application in many problems in mathematical physics . Privault [4] used Bernoulli annihilators to define his gradient operation on Bernoulli functionals. Nourdin , Peccati and Reinert [3] investigated normal approximation of Rademacher functionals (a special case of Bernoulli functionals) with the help of Bernoulli annihilators. In 2016, Wang and Ye [8] constructed a quantum walk model in terms of quantum Bernoulli noise . The same year , Wang and Chen [6] applied quantum Bernoulli noise to the study of quantum Markov semigroups . There are other interesting application results of quantum Bernoulli noise (see, e.g[1]).

Let $\{\partial_k, \partial_k^* \mid k \geq 0\}$ be quantum Bernoulli noise. In a recent paper [7] , it was shown that there exists a close link between operators of form $\sum_{k=0}^{n}(\partial_k + \partial_k^*)$ and probability distributions of the quantum walk introduced in [8]. In this note , we study operators of form $\sum_{k=0}^{n}(\partial_k + \partial_k^*)$ and other operators related to quantum Bernoulli noise from a viewpoint of spectral theory . Among others , we obtain spectral theorems for these operators.

Notation and conventions.Throughout, Z always denotes the set of all integers , while N means the set of all nonnegative integers.We denote by Γ the finite power set of N, namely

$$\Gamma = \{\tau \mid \tau \subset \mathrm{N}, \#(\tau) < \infty\}, \tag{1.1}$$

where $\#(\tau)$ means the cardinality of τ as a finite set .Unless otherwise stated , letters like j , k and n stand for nonnegative integers , namely elements of N.

II. FUNDAMENTALS ABOUT QBN

In the present section , we recall some fundamental notions and facts about quantum Bernoulli noises.

Let $\sum = \{-1,1\}^{\mathrm{N}}$ be the set of all mappings $\omega : \mathrm{N} \mapsto \{-1,1\}$, and $(\zeta_n)_{n\geq 0}$ the sequence of canonical projections on $\sum$ given by

$$\zeta_n(\omega) = \omega(n), \omega \in \Sigma.$$

Denote byA the σ -field on $\sum$ generated by the sequence $(\zeta_n)_{n\geq 0}$. Let $(p_n)_{n\geq 0}$ be a given sequence of positive numbers with the property that $0 < p_n < 1$ for all $n \geq 0$ .It is known [4] that there exists a unique probability measure μ on A such that $\mu \circ (\zeta_{n_1}, \zeta_{n_2} \cdots, \zeta_{n_k})^{-1}\{(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)\} = \prod_{j=1}^{k} p_j^{\frac{1+\varepsilon_j}{2}} (1-p_j)^{\frac{1-\varepsilon_j}{2}}$. for $n_j \in \mathrm{N}$, $\varepsilon_j \in \{-1,1\}(1 \leq j \leq k)$ with $n_i \neq n_j$ when $i \neq j$ and $k \in \mathrm{N}$ with $k \geq 1$. Thus we come to a probability measure space $(\Sigma, \mathrm{A}, \mu)$, which is referred to as the Bernoulli space and random variables on it are known as Bernoulli functionals.

Let $\mathrm{Z} = (\mathrm{Z}_n)_{n\geq 0}$ be the sequence of random variables on $(\Sigma, \mathrm{A}, \mu)$ defined by

$$\mathrm{Z}_n = \frac{\zeta_n + q_n - p_n}{2\sqrt{p_n q_n}}, n \geq 0.$$

where $q_n = 1 - p_n$. Clearly $\mathrm{Z} = (\mathrm{Z}_n)_{n\geq 0}$ is an independent sequence of random variables , and , for each $n \geq 0, \mathrm{Z}_n$ has a probability distribution

$$\mu\{\mathrm{Z}_n = \theta_n\} = p_n,\ \mu\{\mathrm{Z}_n = -1/\theta_n\} = q_n, n \geq 0$$

with $\theta_n = \sqrt{q_n / p_n}$ .To be convenient , we set $\mathrm{A}_n = \sigma(\mathrm{Z}_k; 0 \leq k \leq n)$, the σ -field over $\sum$ generated by $(\mathrm{Z}_k)_{0\leq k\leq n}$, for $n \geq 0$ and $\mathrm{A}\varphi_1 = \{\varnothing, \Sigma\}$.

Let $L^2(\mathrm{Z})$ be the space of square integrable complex -valued Bernoulli functionals, namely

$$L^2(\mathrm{Z}) = L^2(\Sigma, \mathrm{A}, \mu).$$

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We denote by $\langle\cdot,\cdot\rangle$ the usual inner product of the space $L^2(Z)$ and by $\|\cdot\|$ the corresponding norm . It is known [4] that Z has the chaotic representation property . Thus $L^2(Z)$ has $\{Z_\tau \mid \tau\in\Gamma\}$ as its orthonormal basis , where $Z_\varnothing=1$ and $Z_\tau=\prod_{i\in\tau}Z_i$, $\tau\in\Gamma$, $\tau\neq\varnothing$, which shows that $L^2(Z)$ is an infinite dimensional, separable complex Hilbert space.

For integer $k\geq 0$, there exists a bounded linear operator ∂_k on $L^2(Z)$ such that

$$\partial_k Z_\tau=\mathbf{1}_\tau(k)Z_{\tau\setminus k},\tau\in\Gamma,$$

where $\tau\setminus k=\tau\setminus\{k\}$ and $\mathbf{1}_\tau(k)$ the indicator of τ as a subset of N. Denoting by ∂_k^* the adjoint operator of ∂_k , one has

$$\partial_k^* Z_\tau=(1-\mathbf{1}_\tau(k))Z_{\tau\cup k},\tau\in\mathrm{T},$$

where $\tau\bigcup k=\tau\bigcup\{k\}$. In the language of physics , the operator ∂_k and its adjoint ∂_k^* are referred to as the annihilation operator and creation operator at site k , respectively.

The family $\{\partial_k,\partial_k^* \mid k\geq 0\}$ is known as quantum Bernoulli noise (QBN) . The next lemma shows that QBN satisfies the canonical anti-commutation relations (CAR) in equal-time.

Lemma 2.1 [5] Let $k,l\in\mathrm{N}$. Then it holds true that

$$\partial_k\partial_l=\partial_l\partial_k,\partial_k^*\partial_l^*=\partial_l^*\partial_k^*,\partial_k^*\partial_l=\partial_l\partial_k^*(k\neq l) \qquad (2.1)$$

and

$$\partial_k\partial_k=\partial_k^*\partial_k^*=0,\ \partial_k\partial_k^*+\partial_k^*\partial_k=I, \qquad (2.2)$$

where I is the identity operator on $L^2(Z)$.

III. Main Results

In the present section , we state and prove our main results about spectral structure of operators related to quantum Bernoulli noise.

In the following , for convenience , we write $\Xi_k=\partial_k^*+\partial_k$ for each $k\geq 0$. We note that

$$\Xi_k^*=\Xi_k,\Xi_k^2=I,\Xi_j\Xi_k=\Xi_k\Xi_j,\ j,k\geq 0. \qquad (3.1)$$

In other words , the family $\{\Xi_k \mid k\geq 0\}$ is a commutative family consisting of self-adjoint unitary operators on $L^2(Z)$.

Proposition 3.1 Let $k\geq 0$ and define $P_k=\frac{1}{2}(I+\Xi_k)$, $Q_k=\frac{1}{2}(I-\Xi_k)$. Then both P_k and Q_k are projection operators on $L^2(Z)$, and moreover they satisfy relations $P_k+Q_k=I$ and $P_kQ_k=Q_kP_k=0$.

Proof. Clearly , by their definitions , we immediately have $P_k+Q_k=I$, $P_k^*=P_k$ and $Q_k^*=Q_k$. On the other hand , a straightforward computation gives

$$P_k^2=\frac{1}{4}(I+2\Xi_k+\Xi_k^2)=\frac{1}{4}(2I+2\Xi_k)=P_k,$$

which together with $P_k^*=P_k$ implies that P_k is a projection operator . Similarly we can show that Q_k is also a projection operator . Finally , we have

$$P_kQ_k=\frac{1}{4}(I+\Xi_k)(I-\Xi_k)=\frac{1}{4}(I-\Xi_k+\Xi_k-\Xi_k^2)=0,$$

which implies that $Q_kP_k=0$.

By convention , we denote by $\sigma(A)$ the spectrum of an operator A, and by $\rho(A)$ its resolvent set . The next proposition gives the spectrum and spectral decomposition of operator Ξ_k.

Proposition 3.2 Let $k\geq 0$ .Then, for each $\lambda\in C\setminus\{-1,1\}$, the operator $\lambda I-\Xi_k$ has a bounded inverse operator of the following form

$$(\lambda I-\Xi_k)^{-1}=\frac{\lambda}{\lambda^2-1}I+\frac{1}{\lambda^2-1}\Xi_k. \qquad (3.2)$$

And moreover Ξ_k has spectrum $\sigma(\Xi_k)=\{-1,1\}$ and spectral decomposition $\Xi_k=P_k-Q_k$.

Proof. For each $\lambda\in C\setminus\{-1,1\}$, by computation we find

$$\begin{aligned}(\lambda I-\Xi_k)\left(\frac{\lambda}{\lambda^2-1}I+\frac{1}{\lambda^2-1}\Xi_k\right)\\ =\left(\frac{\lambda}{\lambda^2-1}I+\frac{1}{\lambda^2-1}\Xi_k\right)(\lambda I-\Xi_k)\\ =I,\end{aligned}$$

which means that $\lambda I-\Xi_k$ has a bounded inverse operator and its inverse operator takes the form

$$(\lambda I-\Xi_k)^{-1}=\frac{\lambda}{\lambda^2-1}I+\frac{1}{\lambda^2-1}\Xi_k.$$

Thus $C\setminus\{-1,1\}\subset\rho(\Xi_k)$, which implies that $\sigma(\Xi_k)\subset\{-1,1\}$. On the other hand , for $\tau\in\Gamma$ with k $\in\tau$, we have

$$\Xi_k(Z_\tau-Z_{\tau\setminus k})=-(Z_\tau-Z_{\tau\setminus k}),$$
$$\Xi_k(Z_\tau+Z_{\tau\setminus k})=Z_\tau+Z_{\tau\setminus k},$$

which implies that $-1,1\in\sigma(\Xi_k)$. Thus $\sigma(\Xi_k)=\{-1,1\}$. It follows immediately from the definitions of P_k and Q_k that $\Xi_k=P_k-Q_k$.

It follows easily from (3.2) that the family $\{P_k,Q_k \mid k\geq 0\}$ is a commutative one in the sense that

$$P_jP_k=P_kP_j,\ P_jQ_k=Q_kP_j,\ Q_jQ_k=Q_kQ_j,\ \ j,k\geq 0. \qquad (3.3)$$

In view of this , we introduce the following notation: $P_\varnothing=Q_\varnothing=I$, where I denotes the empty set , and

$$P_\tau=\prod_{k\in\tau}P_k,\ Q_\tau=\prod_{k\in\tau}Q_k \qquad (3.4)$$

for $\tau\in\Gamma$ with $\tau\neq\varnothing$. Clearly, the family $\{P_\tau,Q_\tau \mid \tau\in\Gamma\}$

is also commutative , namely

$$P_\tau P_\gamma = P_\gamma P_\tau,\ P_\tau Q_\gamma = Q_\gamma P_\tau,\ Q_\tau Q_\gamma = Q_\gamma Q_\tau \qquad (3.5)$$

hold for all $\tau,\gamma \in \Gamma$.

Proposition 3.3 Let $n \ge 0$ and $\mathrm{N}_n = \{0,1,\cdots,n\}$. Then $P_\tau Q_{\mathrm{N}_n\setminus\tau}$ is a projection operator on $L^2(\mathrm{Z})$ for each $\tau \subset \mathrm{N}_n$. Moreover, $P_\tau Q_{\mathrm{N}_n\setminus\tau} P_\gamma Q_{\mathrm{N}_n\setminus\gamma} = 0$ whenever $\tau,\gamma \subset \mathrm{N}_n$ with $\tau \ne \gamma$, and $\sum_{\tau\subset \mathrm{N}_n} P_\tau Q_{\mathrm{N}_n\setminus\tau} = I$. (3.6)

Proof. Let $\tau \subset \mathrm{N}_n$. Then , by Proposition 3.1 and relations (3.3), we know that both P_τ and $Q_{\mathrm{N}_n\setminus\tau}$ are projection operators , which together with relations (3.5) implies that $P_\tau Q_{\mathrm{N}_n\setminus\tau}$ is also a projection operator . If $\tau,\gamma \subset \mathrm{N}_n$ with $\tau\ne\gamma$, then $\tau \cap (\mathrm{N}_n \setminus \gamma) \ne \emptyset$ or $\gamma \cap (\mathrm{N}_n\setminus\tau) \ne \emptyset$, which together with Proposition 3.1 implies that $P_\tau Q_{\mathrm{N}_n\setminus\gamma} = 0$ or $P_\gamma Q_{\mathrm{N}_n\setminus\tau} = 0$, which then implies

$$P_\tau Q_{\mathrm{N}_n\setminus\tau} P_\gamma Q_{\mathrm{N}_n\setminus\gamma} = 0.$$

Finally , by using the relations described in Proposition 3.1 as well as commutative Relations (3.5), we come to

$$I = I^{n+1} = \prod_{k=0}^{n}(P_k + Q_k) = \sum_{\tau\subset \mathrm{N}_n} P_\tau Q_{\mathrm{N}_n\setminus\tau}.$$

This completes the proof.

Proposition 3.4 Let $n \ge 0$ and $\mathrm{N}_n = \{0,1,\cdots,n\}$. Define an operator-valued mapping $\pi_n(\cdot)$ on $\{2j-n-1 \mid 0 \le j \le n+1\}$ as

$$\pi_n(2j-n-1) = \sum_{\#(\tau)=j,\tau\subset \mathrm{N}_n} P_\tau Q_{\mathrm{N}_n\setminus\tau}, 0 \le j \le n+1, \qquad (3.7)$$

where $\#(\tau)$ denotes the cardinality of τ as a finite set.Then $\pi_n(\cdot)$ is a projection operator-valued measure on $\{2j-n-1 \mid 0\le j\le n+1\}$, namely it satisfies that

(i) $\pi_n(2j-n-1)$ is a projection operator for each j with $0 \le j \le n+1$;

(ii) $\pi_n(2j-n-1)\pi_n(2k-n-1) = 0$ whenever $j \ne k$, $0 \le j,k \le n+1$;

(iii) and $\sum_{j=0}^{n=1} \pi_n(2j-n-1) = I$.

Proof. This is an immediate consequence of Proposition 3.3.

Theorem3.5 $n \ge 0$ and $\mathrm{N}_n = \{0,1,\cdots,n\}$ .Define $S_n = \sum_{k=0}^{n} \Xi_k$. Then S_n has a spectral decomposition of the following form

$$S_n = \sum_{j=0}^{n+1}(2j-n-1)\pi_n(2j-n-1). \qquad (3.8)$$

In particular , the spectrum $\sigma(S_n)$ of S_n coincides with $\{2j-n-1 \mid 0 \le j \le n+1\}$.

Proof. For each $k \in \mathrm{N}_n$, define a function $\varepsilon_k : \mathrm{N}_n \to \{-1,1\}$ as $\varepsilon_k(k) = -1$ and $\varepsilon_k(j) = 1$ for $j \in \mathrm{N}_n$ with $j \ne k$. Then by Proposition 3.1 we have

$$\begin{aligned}
&\prod_{j=0,j\ne k}^{n}\left(P_j + \varepsilon_k(j)Q_j\right)\\
&= \prod_{j=0,j\ne k}^{n}\left(P_j + Q_j\right)\\
&= I^n\\
&= I,\ 0 \le k \le n. \qquad (3.9)
\end{aligned}$$

On the other hand , for $k \in \mathrm{N}_n$, by using relations (3.9) we get

$$\begin{aligned}
\Xi_k &= P_k + \varepsilon_k(k)Q_k\\
&= \prod_{j=0}^{n}\left(P_j + \varepsilon_k(j)Q_j\right)\\
&= \sum_{j=0}^{n+1}\sum_{\#(\tau)=j,\tau\subset \mathrm{N}_n} f_k(\mathrm{N}_n\setminus\tau) P_\tau Q_{\mathrm{N}_n\setminus\tau},
\end{aligned}$$

where $f_k(\mathrm{N}_n\setminus\tau) = \prod_{i\in \mathrm{N}_n\setminus\tau}\varepsilon_k(i)$.

Thus

$$\begin{aligned}
S_n &= \sum_{k=0}^{n}\Xi_k\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n+1}\sum_{\#(\tau)=j,\tau\subset \mathrm{N}_n} f_k(\mathrm{N}_n\setminus\tau) P_\tau Q_{\mathrm{N}_n\setminus\tau} \qquad (3.10)\\
&= \sum_{j=0}^{n+1}\sum_{\#(\tau)=j,\tau\subset \mathrm{N}_n}\left(\sum_{k=0}^{n} f_k(\mathrm{N}_n\setminus\tau)\right) P_\tau Q_{\mathrm{N}_n\setminus\tau}.
\end{aligned}$$

Note that , for $\tau \subset \mathrm{N}_n$, a careful computation gives

$$\begin{aligned}
&\sum_{k=0}^{n} f_k(\mathrm{N}_n\setminus\tau)\\
&= \sum_{k=0}^{n}\prod_{i\in \mathrm{N}_n\setminus\tau}\varepsilon_k(i)\\
&= \#(\tau) + (-1)\#(\mathrm{N}_n\setminus\tau)\\
&= 2\#(\tau) - n - 1,
\end{aligned}$$

which together with (3.10) and (3.7) implies that

$$\begin{aligned}
S_n &= \sum_{j=0}^{n+1}\sum_{\#(\tau)=j,\tau\subset \mathrm{N}_n}(2j-n-1) P_\tau Q_{\mathrm{N}_n\setminus\tau}\\
&= \sum_{j=0}^{n+1}(2j-n-1)\pi_n(2j-n-1).
\end{aligned}$$

This completes the proof.

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