

Spectral Analysis of Quantum Bernoulli Noise

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Abstract— Quantum Bernoulli noise is the family of annihilation and creation operators acting on Bernoulli functionals, which satisfy the anti-commutation relations (ACR) in equal-time. In this paper, we study spectral structure of operators related to quantum Bernoulli noise. Among others, we obtain spectral theorems for these operators.

Index Terms—Quantum Bernoulli noise, Spectral theorem. MSC(2010):—60H40, 47B38.

I. INTRODUCTION

Quantum Bernoulli noise [5] is the family of annihilation and creation operators acting on Bernoulli functionals (Bernoulli annihilators and creators for short), which satisfy the anti-commutation relations (ACR) in equal-time, and can be viewed as a discrete-time analog of quantum white noise^[21]. In the past two decades, quantum Bernoulli noise has found wide application in many problems in mathematical physics. Privault [4] used Bernoulli annihilators to define his gradient operation on Bernoulli functionals. Nourdin, Peccati and Reinert [3] investigated normal approximation of Rademacher functionals (a special case of Bernoulli functionals) with the help of Bernoulli annihilators. In 2016, Wang and Ye [8] constructed a quantum walk model in terms of quantum Bernoulli noise. The same year, Wang and Chen [6] applied quantum Bernoulli noise to the study of quantum Markov semigroups. There are other interesting application results of quantum Bernoulli noise (see, e.g.[1]).

Let $\{\partial_k, \partial_k^* \mid k \geq 0\}$ be quantum Bernoulli noise. In a recent paper [7], it was shown that there exists a close link between operators of form $\sum_{k=0}^n (\partial_k + \partial_k^*)$ and probability distributions of the quantum walk introduced in^[8]. In this note, we study operators of form $\sum_{k=0}^n (\partial_k + \partial_k^*)$ and other operators related to quantum Bernoulli noise from a viewpoint of spectral theory. Among others, we obtain spectral theorems for these operators.

Notation and conventions. Throughout, \mathbb{Z} always

denotes the set of all integers, while \mathbb{N} means the set of all nonnegative integers. We denote by Γ the finite power set of \mathbb{N} , namely

$$\Gamma = \{\tau \mid \tau \subset \mathbb{N}, \#(\tau) < \infty\}, \quad (1.1)$$

where $\#(\tau)$ means the cardinality of τ as a finite set. Unless otherwise stated, letters like j, k and n stand for nonnegative integers, namely elements of \mathbb{N} .

II. FUNDAMENTALS ABOUT QBN

In the present section, we recall some fundamental notions and facts about quantum Bernoulli noises.

Let $\Sigma = \{-1, 1\}^{\mathbb{N}}$ be the set of all mappings $\omega: \mathbb{N} \mapsto \{-1, 1\}$, and $(\zeta_n)_{n \geq 0}$ the sequence of canonical projections on Σ given by

$$\zeta_n(\omega) = \omega(n), \omega \in \Sigma.$$

Denote by \mathcal{A} the σ -field on Σ generated by the sequence $(\zeta_n)_{n \geq 0}$. Let $(p_n)_{n \geq 0}$ be a given sequence of positive numbers with the property that $0 < p_n < 1$ for all $n \geq 0$. It is known [4] that there exists a unique probability measure μ on \mathcal{A} such that

$$\mu \circ (\zeta_{n_1}, \zeta_{n_2}, \dots, \zeta_{n_k})^{-1} \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)\} = \prod_{j=1}^k p_j^{\frac{1+\varepsilon_j}{2}} (1-p_j)^{\frac{1-\varepsilon_j}{2}}.$$

for $n_j \in \mathbb{N}$, $\varepsilon_j \in \{-1, 1\}$ ($1 \leq j \leq k$) with $n_i \neq n_j$ when $i \neq j$ and $k \in \mathbb{N}$ with $k \geq 1$. Thus we come to a probability measure space $(\Sigma, \mathcal{A}, \mu)$, which is referred to as the Bernoulli space and random variables on it are known as Bernoulli functionals.

Let $Z = (Z_n)_{n \geq 0}$ be the sequence of random variables on $(\Sigma, \mathcal{A}, \mu)$ defined by

$$Z_n = \frac{\zeta_n + q_n - p_n}{2\sqrt{p_n q_n}}, n \geq 0.$$

where $q_n = 1 - p_n$. Clearly $Z = (Z_n)_{n \geq 0}$ is an independent sequence of random variables, and, for each $n \geq 0$, Z_n has a probability distribution

$$\mu\{Z_n = \theta_n\} = p_n, \mu\{Z_n = -1/\theta_n\} = q_n, n \geq 0$$

with $\theta_n = \sqrt{q_n/p_n}$. To be convenient, we set $A_n =$

$\sigma(Z_k; 0 \leq k \leq n)$, the σ -field over Σ generated by $(Z_k)_{0 \leq k \leq n}$, for $n \geq 0$ and $\mathcal{A}_\emptyset = \{\emptyset, \Sigma\}$.

Let $L^2(\mathbb{Z})$ be the space of square integrable complex-valued Bernoulli functionals, namely

$$L^2(\mathbb{Z}) = L^2(\Sigma, \mathcal{A}, \mu).$$

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We denote by $\langle \cdot, \cdot \rangle$ the usual inner product of the space $L^2(\mathbf{Z})$ and by $\|\cdot\|$ the corresponding norm . It is known [4] that \mathbf{Z} has the chaotic representation property . Thus $L^2(\mathbf{Z})$ has $\{Z_\tau \mid \tau \in \Gamma\}$ as its orthonormal basis , where $Z_\emptyset = 1$ and $Z_\tau = \prod_{i \in \tau} Z_i$, $\tau \in \Gamma$, $\tau \neq \emptyset$, which shows that $L^2(\mathbf{Z})$ is an infinite dimensional, separable complex Hilbert space.

For integer $k \geq 0$, there exists a bounded linear operator ∂_k on $L^2(\mathbf{Z})$ such that

$$\partial_k Z_\tau = \mathbf{1}_\tau(k) Z_{\tau \setminus k}, \tau \in \Gamma,$$

where $\tau \setminus k = \tau \setminus \{k\}$ and $\mathbf{1}_\tau(k)$ the indicator of τ as a subset of \mathbf{N} . Denoting by ∂_k^* the adjoint operator of ∂_k , one has

$$\partial_k^* Z_\tau = (1 - \mathbf{1}_\tau(k)) Z_{\tau \cup k}, \tau \in \Gamma,$$

where $\tau \cup k = \tau \cup \{k\}$. In the language of physics , the operator ∂_k and its adjoint ∂_k^* are referred to as the annihilation operator and creation operator at site k , respectively.

The family $\{\partial_k, \partial_k^* \mid k \geq 0\}$ is known as quantum Bernoulli noise (QBN) . The next lemma shows that QBN satisfies the canonical anti-commutation relations (CAR) in equal-time.

Lemma 2.1 ^[5] Let $k, l \in \mathbf{N}$. Then it holds true that

$$\partial_k \partial_l = \partial_l \partial_k, \partial_k^* \partial_l^* = \partial_l^* \partial_k^*, \partial_k^* \partial_l = \partial_l \partial_k^* (k \neq l) \quad (2.1)$$

and

$$\partial_k \partial_k = \partial_k^* \partial_k^* = 0, \partial_k \partial_k^* + \partial_k^* \partial_k = I, \quad (2.2)$$

where I is the identity operator on $L^2(\mathbf{Z})$.

III. MAIN RESULTS

In the present section , we state and prove our main results about spectral structure of operators related to quantum Bernoulli noise.

In the following , for convenience , we write $\Xi_k =$

$\partial_k^* + \partial_k$ for each $k \geq 0$. We note that

$$\Xi_k^* = \Xi_k, \Xi_k^2 = I, \Xi_j \Xi_k = \Xi_k \Xi_j, j, k \geq 0. \quad (3.1)$$

In other words , the family $\{\Xi_k \mid k \geq 0\}$ is a commutative family consisting of self-adjoint unitary operators on $L^2(\mathbf{Z})$.

Proposition 3.1 Let $k \geq 0$ and define $P_k = \frac{1}{2}(I + \Xi_k)$, $Q_k = \frac{1}{2}(I - \Xi_k)$. Then both P_k and Q_k are projection operators on $L^2(\mathbf{Z})$, and moreover they satisfy relations $P_k + Q_k = I$ and $P_k Q_k = Q_k P_k = 0$.

Proof. Clearly , by their definitions , we immediately have $P_k + Q_k = I$, $P_k^* = P_k$ and $Q_k^* = Q_k$. On the other hand , a straightforward computation gives

$$P_k^2 = \frac{1}{4}(I + 2\Xi_k + \Xi_k^2) = \frac{1}{4}(2I + 2\Xi_k) = P_k,$$

which together with $P_k^* = P_k$ implies that P_k is a projection operator . Similarly we can show that Q_k is also a projection operator . Finally , we have

$$P_k Q_k = \frac{1}{4}(I + \Xi_k)(I - \Xi_k) = \frac{1}{4}(I - \Xi_k + \Xi_k - \Xi_k^2) = 0,$$

which implies that $Q_k P_k = 0$.

By convention , we denote by $\sigma(A)$ the spectrum of an operator A , and by $\rho(A)$ its resolvent set . The next proposition gives the spectrum and spectral decomposition of operator Ξ_k .

Proposition 3.2 Let $k \geq 0$. Then, for each $\lambda \in C \setminus \{-1, 1\}$, the operator $\lambda I - \Xi_k$ has a bounded inverse operator of the following form

$$(\lambda I - \Xi_k)^{-1} = \frac{\lambda}{\lambda^2 - 1} I + \frac{1}{\lambda^2 - 1} \Xi_k. \quad (3.2)$$

And moreover Ξ_k has spectrum $\sigma(\Xi_k) = \{-1, 1\}$ and spectral decomposition $\Xi_k = P_k - Q_k$.

Proof. For each $\lambda \in C \setminus \{-1, 1\}$, by computation we find

$$\begin{aligned} (\lambda I - \Xi_k) \left(\frac{\lambda}{\lambda^2 - 1} I + \frac{1}{\lambda^2 - 1} \Xi_k \right) &= \left(\frac{\lambda}{\lambda^2 - 1} I + \frac{1}{\lambda^2 - 1} \Xi_k \right) (\lambda I - \Xi_k) \\ &= I, \end{aligned}$$

which means that $\lambda I - \Xi_k$ has a bounded inverse operator and its inverse operator takes the form

$$(\lambda I - \Xi_k)^{-1} = \frac{\lambda}{\lambda^2 - 1} I + \frac{1}{\lambda^2 - 1} \Xi_k.$$

Thus $C \setminus \{-1, 1\} \subset \rho(\Xi_k)$, which implies that $\sigma(\Xi_k) \subset \{-1, 1\}$. On the other hand , for $\tau \in \Gamma$ with $k \in \tau$, we have

$$\begin{aligned} \Xi_k (Z_\tau - Z_{\tau \setminus k}) &= -(Z_\tau - Z_{\tau \setminus k}) \\ \Xi_k (Z_\tau + Z_{\tau \setminus k}) &= Z_\tau + Z_{\tau \setminus k}, \end{aligned}$$

which implies that $-1, 1 \in \sigma(\Xi_k)$. Thus $\sigma(\Xi_k) = \{-1, 1\}$. It follows immediately from the definitions of P_k and

Q_k that $\Xi_k = P_k - Q_k$.

It follows easily from (3.2) that the family $\{P_k, Q_k \mid k \geq 0\}$ is a commutative one in the sense that $P_j P_k = P_k P_j$, $P_j Q_k = Q_k P_j$, $Q_j Q_k = Q_k Q_j$, $j, k \geq 0$.

$$(3.3)$$

In view of this , we introduce the following notation: $P_\emptyset = Q_\emptyset = I$, where I denotes the empty set , and

$$P_\tau = \prod_{k \in \tau} P_k, Q_\tau = \prod_{k \in \tau} Q_k \quad (3.4)$$

for $\tau \in \Gamma$ with $\tau \neq \emptyset$. Clearly, the family $\{P_\tau, Q_\tau \mid \tau \in \Gamma\}$

is also commutative, namely

$$P_\tau P_\gamma = P_\gamma P_\tau, P_\tau Q_\gamma = Q_\gamma P_\tau, Q_\tau Q_\gamma = Q_\gamma Q_\tau \quad (3.5)$$

hold for all $\tau, \gamma \in \Gamma$.

Proposition 3.3 Let $n \geq 0$ and $N_n = \{0, 1, \dots, n\}$. Then

$$P_\tau Q_{N_n \setminus \tau} \text{ is a projection operator on } L^2(Z) \text{ for each } \tau \subset N_n. \text{ Moreover, } P_\tau Q_{N_n \setminus \tau} P_\gamma Q_{N_n \setminus \gamma} = 0 \text{ whenever } \tau, \gamma \subset N_n \text{ with } \tau \neq \gamma, \text{ and } \sum_{\tau \subset N_n} P_\tau Q_{N_n \setminus \tau} = I. \quad (3.6)$$

Proof. Let $\tau \subset N_n$. Then, by Proposition 3.1 and relations (3.3), we know that both P_τ and $Q_{N_n \setminus \tau}$ are projection operators, which together with relations (3.5) implies that $P_\tau Q_{N_n \setminus \tau}$ is also a projection operator. If $\tau, \gamma \subset N_n$ with $\tau \neq \gamma$, then $\tau \cap (N_n \setminus \gamma) \neq \emptyset$ or $\gamma \cap (N_n \setminus \tau) \neq \emptyset$, which together with Proposition 3.1 implies that $P_\tau Q_{N_n \setminus \gamma} = 0$ or $P_\gamma Q_{N_n \setminus \tau} = 0$, which then implies

$$P_\tau Q_{N_n \setminus \tau} P_\gamma Q_{N_n \setminus \gamma} = 0.$$

Finally, by using the relations described in Proposition 3.1 as well as commutative Relations (3.5), we come to

$$I = I^{n+1} = \prod_{k=0}^n (P_k + Q_k) = \sum_{\tau \subset N_n} P_\tau Q_{N_n \setminus \tau}.$$

This completes the proof.

Proposition 3.4 Let $n \geq 0$ and $N_n = \{0, 1, \dots, n\}$. Define an operator-valued mapping $\pi_n(\cdot)$ on

$$\{2j - n - 1 \mid 0 \leq j \leq n + 1\} \text{ as} \\ \pi_n(2j - n - 1) = \sum_{\#(\tau)=j, \tau \subset N_n} P_\tau Q_{N_n \setminus \tau}, 0 \leq j \leq n + 1, \quad (3.7)$$

where $\#(\tau)$ denotes the cardinality of τ as a finite set. Then $\pi_n(\cdot)$ is a projection operator-valued measure on $\{2j - n - 1 \mid 0 \leq j \leq n + 1\}$, namely it satisfies that

- (i) $\pi_n(2j - n - 1)$ is a projection operator for each j with $0 \leq j \leq n + 1$;
- (ii) $\pi_n(2j - n - 1)\pi_n(2k - n - 1) = 0$ whenever $j \neq k, 0 \leq j, k \leq n + 1$;
- (iii) and $\sum_{j=0}^{n+1} \pi_n(2j - n - 1) = I$.

Proof. This is an immediate consequence of Proposition 3.3.

Theorem 3.5 $n \geq 0$ and $N_n = \{0, 1, \dots, n\}$. Define $S_n = \sum_{k=0}^n \Xi_k$. Then S_n has a spectral decomposition of the following form

$$S_n = \sum_{j=0}^{n+1} (2j - n - 1) \pi_n(2j - n - 1). \quad (3.8)$$

In particular, the spectrum $\sigma(S_n)$ of S_n coincides with $\{2j - n - 1 \mid 0 \leq j \leq n + 1\}$.

Proof. For each $k \in N_n$, define a function

$\varepsilon_k : N_n \rightarrow \{-1, 1\}$ as $\varepsilon_k(k) = -1$ and $\varepsilon_k(j) = 1$ for $j \in N_n$ with $j \neq k$. Then by Proposition 3.1 we have

$$\begin{aligned} & \prod_{j=0, j \neq k}^n (P_j + \varepsilon_k(j) Q_j) \\ &= \prod_{j=0, j \neq k}^n (P_j + Q_j) \\ &= I^n \\ &= I, 0 \leq k \leq n. \end{aligned} \quad (3.9)$$

On the other hand, for $k \in N_n$, by using relations (3.9) we get

$$\begin{aligned} \Xi_k &= P_k + \varepsilon_k(k) Q_k \\ &= \prod_{j=0}^n (P_j + \varepsilon_k(j) Q_j) \\ &= \sum_{j=0}^{n+1} \sum_{\#(\tau)=j, \tau \subset N_n} f_k(N_n \setminus \tau) P_\tau Q_{N_n \setminus \tau}, \end{aligned}$$

where $f_k(N_n \setminus \tau) = \prod_{i \in N_n \setminus \tau} \varepsilon_k(i)$.

$$\begin{aligned} \text{Thus } S_n &= \sum_{k=0}^n \Xi_k \\ &= \sum_{k=0}^n \sum_{j=0}^{n+1} \sum_{\#(\tau)=j, \tau \subset N_n} f_k(N_n \setminus \tau) P_\tau Q_{N_n \setminus \tau} \quad (3.10) \\ &= \sum_{j=0}^{n+1} \sum_{\#(\tau)=j, \tau \subset N_n} \left(\sum_{k=0}^n f_k(N_n \setminus \tau) \right) P_\tau Q_{N_n \setminus \tau}. \end{aligned}$$

Note that, for $\tau \subset N_n$, a careful computation gives

$$\begin{aligned} & \sum_{k=0}^n f_k(N_n \setminus \tau) \\ &= \sum_{k=0}^n \prod_{i \in N_n \setminus \tau} \varepsilon_k(i) \\ &= \#(\tau) + (-1) \#(N_n \setminus \tau) \\ &= 2\#(\tau) - n - 1, \end{aligned}$$

which together with (3.10) and (3.7) implies that

$$\begin{aligned} S_n &= \sum_{j=0}^{n+1} \sum_{\#(\tau)=j, \tau \subset N_n} (2j - n - 1) P_\tau Q_{N_n \setminus \tau} \\ &= \sum_{j=0}^{n+1} (2j - n - 1) \pi_n(2j - n - 1). \end{aligned}$$

This completes the proof.

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