Number Operator on Functionals of Discrete-Time Normal Noises

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Abstract—Let $Z$ be a discrete-time normal noise that has the chaotic representation property. In this paper, we show that the number operator in the space of square integrable functionals of $Z$ can be extended to a continuous operator on the generalized functional space of $Z$.

Index Terms—Discrete-time normal noise, Generalized functional, Fock transform, Number operator.

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I. INTRODUCTION

Let $Z = \{Z_n\}$ be a discrete-time normal noise, and $L^2(Z)$ the space of square integrable functionals of $Z$.

Then the number operator $N$ in $L^2(Z)$ is defined as

$$N \xi = \sum_{\sigma \in \Gamma} \| \sigma \| \langle Z_{\sigma}, \xi \rangle Z_{\sigma}, \quad \xi \in \text{Dom} N,$$

where $\{Z_{\sigma} \mid \sigma \in \Gamma\}$ is the canonical orthonormal basis of $L^2(Z)$, $\# \sigma$ denotes the cardinality of $\sigma$ as a finite set, and $\text{Dom} N$ means the domain of $N$ given by

$$\text{Dom} N = \left\{ \xi \in L^2(Z) \mid \sum_{\sigma \in \Gamma} \| \sigma \| \langle Z_{\sigma}, \xi \rangle^2 < \infty \right\}.$$

The operator $N$ plays an important role in functional analysis of discrete-time normal noises. For example, $-N$ generates the Ornstein-Uhlenbeck semigroup of operators on $L^2(Z)^{[6]}$. In a recent paper [2], $N$ is used to study the regularity of solutions to the stochastic Schrödinger equation. However, $N$ is not defined on whole $L^2(Z)$, namely

$\text{Dom} N \neq L^2(Z)$, which may cause inconvenience in its application.

On the other hand, as is shown in [7], one can use the canonical orthonormal basis of $L^2(Z)$ to construct a nuclear space $S(Z)$ such that $S(Z)$ is densely contained in $L^2(Z)$. Thus, by identifying $L^2(Z)$ with its dual, one can get a Gel’fand triple

$$S(Z) \subset L^2(Z) \subset S^*(Z),$$

where $S^*(Z)$ is the dual of $S(Z)$, which is endowed with the strong topology, which can not be induced by any norm [3]. As usual, $S(Z)$ is called the testing functional space of $Z$, while $S^*(Z)$ is called the generalized functional space of $Z$. It turns out [4] that the generalized functional space $S^*(Z)$ can accommodate many quantities of theoretical interest that can not be covered by $L^2(Z)$.

In this paper, we would like to extend the number operator $N$ to generalized functionals of $Z$. More precisely, we will define the number operator on the generalized functional space $S^*(Z)$.

Throughout this paper, $N$ designates the set of all nonnegative integers and $\Gamma$ the finite power set of $N$, namely

$$\Gamma = \{ \sigma \mid \sigma \subset N, \#(\sigma) < \infty \},$$

where $\# \sigma$ means the cardinality of $\sigma$ as a finite set. If $k \in N$ and $\sigma \in \Gamma$, then we simply write $\sigma \cup \{k\}$ for $\sigma \cup \{k\}$. Similarly, we use $\sigma \setminus k$.

II. PRELIMINARY

In what follows, we always assume that $(\Omega, F, P)$ is a given probability space. We use $E$ to mean the expectation with respect to $P$. As usual, $L^2(\Omega, F, P)$ denotes the Hilbert space of square integrable complex-valued measurable functions on $(\Omega, F, P)$. We use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to mean the inner product and norm of $L^2(\Omega, F, P)$, respectively. By convention, $\langle \cdot, \cdot \rangle$ is conjugate-linear in its first argument and linear in its second argument.

Define 2.1 A sequence $Z = \{Z_n\}_{n \in \mathbb{N}}$ of integrable random variables on $(\Omega, F, P)$ is called a discrete-time normal noise if it satisfies:

(i) $E[Z_n \mid F_{n-1}] = 0$ for $n \geq 0$;

(ii) $E[Z_n^2 \mid F_{n-1}] = 1$ for $n \geq 0$.

Here $F_n = \sigma(\{\Omega\} \cup F_n)$ for $n \in N$ and $E[ \cdot \mid F_n]$ means the conditional expectation given $F_n$.

For a discrete-time normal noise $Z = \{Z_n\}_{n \in \mathbb{N}}$ on $(\Omega, F, P)$, one can construct a corresponding family $\{Z_{\sigma} \mid \sigma \in \Gamma\}$ of random variables on $(\Omega, F, P)$ in the following manner.

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such that the inclusion
\[ \{Z_\sigma \mid \sigma \in \Gamma \} \]
forms a countable orthonormal system in $L^2(\Omega, F, P)$. Let $F_\infty = \sigma(Z_n ; n \in N)$ be the $\sigma$-field over $\Omega$ generated by a discrete-time normal noise $Z = (Z_n)_{n \in N}$ on $(\Omega, F, P)$. Then the canonical functional system $\{Z_\sigma \mid \sigma \in \Gamma \}$ is also a countable orthonormal system in the space $L^2(\Omega, F_\infty, P)$ of square integrable complex-valued measurable functions on $(\Omega, F_\infty, P)$.

In the literature, $F_\infty$-measurable functions on $\Omega$ are also known as functionals of $Z$. Thus elements of $L^2(\Omega, F_\infty, P)$ are naturally called square integrable functionals of $Z$.

**Lemma 2.2** Let $Z = (Z_n)_{n \in N}$ be a discrete-time normal noise on $(\Omega, F, P)$. Then its canonical functional system $\{Z_\sigma \mid \sigma \in \Gamma \}$ is total in $L^2(\Omega, F_\infty, P)$, where $F_\infty = \sigma(Z_n ; n \in N)$.

Thus, if a discrete-time normal noise $Z$ has the chaotic representation property, then its canonical functional system $\{Z_\sigma \mid \sigma \in \Gamma \}$ is actually an orthonormal basis of $L^2(\Omega, F_\infty, P)$.

From now on, we always assume that $Z = (Z_n)_{n \in N}$ is a given discrete-time normal noise on $(\Omega, F, P)$ that has the chaotic representation property.

For brevity, we use $L^2(Z)$ to denote the space of square integrable functionals of $Z$, namely
\[ L^2(Z) = L^2(\Omega, F_\infty, P), \]
where $F_\infty = \sigma(Z_n ; n \in N)$. For $k \geq 0$, we denote by $F_k$ the $\sigma$-field generated by $\{Z_j ; 0 \leq j \leq k\}$, namely
\[ F_k = \sigma(Z_j ; 0 \leq j \leq k) \]
We note that $L^2(Z)$ shares the same inner product $\langle \cdot, \cdot \rangle$ and norm with $L^2(\Omega, F, P)$, and moreover the canonical functional system $\{Z_\sigma \mid \sigma \in \Gamma \}$ of $Z$ forms a countable orthonormal basis for $L^2(Z)$, which we call the canonical orthonormal basis of $L^2(Z)$.

**Lemma 2.2** Let $\sigma \mapsto \lambda_\sigma$ be the $N$-valued function on $\Gamma$ given by
\[
\lambda_\sigma = \begin{cases} 
\prod_{k=1}^{n}(k \notin \Omega) & \sigma \neq \phi, \sigma \in \Gamma; \\
1 & \sigma = \phi, \sigma \in \Gamma.
\end{cases}
\]

Then, for $\rho > 0$, the positive series $\sum_{\sigma \in \Gamma} \lambda_\rho^\rho$ converges and moreover
\[
\sum_{\sigma \in \Gamma} \lambda_\rho^\rho \leq \exp\left(\sum_{k=1}^{\infty} -\rho^k \right) < \infty
\]

Using the $N$-valued function defined by (2.1), we can construct a chain of Hilbert spaces consisting of functionals of $Z$ as follows. For $p \geq 0$, we put
\[
S_p(Z) = \left\{ \xi \in L^2(Z) \mid \sum_{\sigma \in \Gamma} \lambda_\rho^\rho \left( \|Z_\sigma, \xi \|^2 \right) < \infty \right\}
\]
and define
\[
\langle \xi, \eta \rangle_p = \sum_{\sigma \in \Gamma} \lambda_\rho^\rho \langle Z_\sigma, \xi \rangle \langle Z_\sigma, \eta \rangle, \xi, \eta \in S_p(Z).
\]

It is not hard to check that, with $\langle \cdot, \cdot \rangle_p$ as the inner product, $S_p(Z)$ becomes a Hilbert space. We write $\|\xi\|_p = \sqrt{\langle \xi, \xi \rangle_p}$ for $\xi \in S_p(Z)$. Clearly, it holds that
\[
\|\xi\|_p^2 = \sum_{\sigma \in \Gamma} \lambda_\rho^\rho \left( \|Z_\sigma, \xi \|^2 \right), \xi \in S_p(Z)
\]

**Lemma 2.3** For $p \geq 0$, $\{Z_\sigma \mid \sigma \in \Gamma \} \subset S_p(Z)$ and moreover the system $\left\{ \lambda_\rho^\rho Z_\sigma \mid \sigma \in \Gamma \right\}$ forms an orthonormal basis for $S_p(Z)$.

It is easy to see that $\lambda_\rho \geq 1$ for all $\sigma \in \Gamma$. This implies that $\|\xi\|_p \leq \|\xi\|_p$ and $S_q(Z) \subset S_p(Z)$ whenever $0 \leq p \leq q$. Thus we actually get a chain of Hilbert spaces of functionals of $Z$:
\[
\cdots \subset S_{p+1}(Z) \subset \cdots \subset S_p(Z) \subset S_q(Z) = L^2(Z)
\]
We now put
\[
S(Z) = \sum_{p=0}^{\infty} S_p(Z)
\]
and endow it with the topology generated by the norm sequence $\|\cdot\|_{p \geq 0}$. Note that, for each $p \geq 0$, $S_p(Z)$ is just the completion of $S(Z)$ with respect to $\|\cdot\|_p$.

Thus $S(Z)$ is a countably-Hilbert space. The next lemma, however, shows that $S(Z)$ even has a much better property.

**Lemma 2.4** The space $S(Z)$ is a nuclear space, namely for any $p \geq 0$, there exists $q \geq p$ such that the inclusion mapping $i_{pq} : S_q(Z) \to S_p(Z)$ defined by $i_{pq}(\xi) = \xi$ is a Hilbert-Schmidt operator.

For $p \geq 0$, we denote by $S'_p(Z)$ the dual of $S_p(Z)$ and $\|\cdot\|_{-p}$ the norm of $S'_p(Z)$. Then $S'_p(Z) \subset S'_q(Z)$ and...
The Riesz mapping $\rho^\sigma$ for $\rho \neq \rho^\sigma$, its Fock transform is the generalized functional of $Z$ and only if $\bar{\Phi} = \overline{\Phi}$. Thus a generalized functional of $Z$ is completely determined by its Fock transform. The following theorem characterizes generalized functionals of $Z$ through their Fock transforms.

**Lemma 2.7** [4] Let $F$ be a function on $\Gamma$. Then $F$ is the Fock transform of an element $\Phi$ of $S^*(Z)$ if and only if it satisfies

$$\|F^{\sigma}\| \leq C \|\rho^\sigma\|^{p/2}, \quad \sigma \in \Gamma$$

for $q > p + 1/2$, one has

$$\|\Phi\|_{q} \leq C\left(\sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2(q-\rho)}\right)^{1/2}$$

and in particular $\Phi \in \mathcal{S}^\sigma(Z)$.

**Remark 2.1** There exists a continuous linear mapping $R : L^2(Z) \to S^*(Z)$ such that

$$\langle R\eta, \xi \rangle = \langle \eta, \xi \rangle, \quad \eta \in L^2(Z), \xi \in S(Z)$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $S^*(Z) \times S(Z)$.

I. MAIN RESULT

In this section, we define our number operator on the generalized functional space $S^*(Z)$ and show its links with the number operator $N$ in $L^2(Z)$. Our main tool is the Fock transforms of generalized functionals of $Z$.

Recall that $\# \sigma$ means the cardinality of $\sigma$ as a finite set. The following lemma gives an inequality concerning $\# \sigma$, which will be used later.

**Lemma 3.1** For all $\sigma \in \Gamma$, it holds that $\# \sigma \leq \lambda_{\sigma}$.

**Proof.** Let $\sigma \in \Gamma$. Clearly, $\# \sigma \leq \lambda_{\sigma}$ holds for the case of $\sigma = \phi$. For the case of $\sigma \neq \phi$, by assuming $\sigma = \{k_1, k_2, \ldots, k_n\}$ with $k_1 < k_2 < \cdots < k_n$, we have

$$\lambda_{\sigma} = (k_1 + 1)(k_2 + 1) \cdots (k_n + 1) \geq 1 \times 2 \times \cdots \times n = \# \sigma$$

This completes the proof.

**Proposition 3.2** There exists a linear operator $n : S^*(Z) \to S^*(Z)$ such that

$$n\Phi(\sigma) = \# \sigma \bar{\Phi}(\sigma), \quad \Phi \in S^*(Z), \sigma \in \Gamma$$

where $\Phi$ and $n\Phi$ are Fock transforms of $\Phi$ and $n \Phi$, respectively.

**Proof.** Let $\Phi \in S^*(Z)$. Then, by Lemma 2.7, there exist constants $C \geq 0$ and $p \geq 0$ such that

\[ \|\Phi\|_q \leq C \left(\sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2(q-p)}\right)^{1/2} \]

and for $q > p + 1/2$, one has

\[ \|\Phi\|_q \leq C \left(\sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2(q-p)}\right)^{1/2} \]
which together with Lemma 3.1 gives
\[ |\Phi(\sigma)| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \]
which together Lemma 2.7 implies that there exists a unique generalized functional \( \Phi' \in S^*(Z) \) such that
\[ \Phi(\sigma) = \#\sigma \Phi(\sigma), \quad \sigma \in \Gamma. \]  (3.2)

Thus we have a mapping \( \Phi \to \Phi' \) from \( S^*(Z) \) into itself, which we denote by \( n \), namely
\[ n\Phi = \Phi', \quad \Phi \in S^*(Z) \]
Obviously \( n \) satisfies (3.1). Now let \( \Phi_1, \Phi_2 \in S^*(Z) \) and \( \alpha_1, \alpha_2 \) be any complex number. Then, by letting \( \Psi = \alpha_1 \Phi_1 + \alpha_2 \Phi_2 \), we have
\[ n\Psi(\sigma) = \#\sigma \Psi(\sigma) = \alpha_1 \#\sigma \Phi_1(\sigma) + \alpha_2 \#\sigma \Phi_2(\sigma) = \alpha_1 n\Phi_1(\sigma) + \alpha_2 n\Phi_2(\sigma), \quad \sigma \in \Gamma \]
which implies that \( n\Psi = \alpha_1 n\Phi_1 + \alpha_2 n\Phi_2 \). Thus \( n \) is a linear operator on \( S^*(Z) \).

**Proposition 3.3** The operator \( n \) is continuous on \( S^*(Z) \).

**Proof.** We need only to show that the composition \( n \circ j_p : S^*_p(Z) \to S^*(Z) \) is continuous for each \( p \geq 0 \), where \( j_p : S^*_p(Z) \to S^*(Z) \) denotes the natural embedding.

Let \( p \geq 0 \) and take \( q > p + \frac{3}{2} \). Then, for each \( \Phi \in S^*_p(Z) \), it follows from the inequality
\[ \#\sigma \leq \lambda_\sigma, \quad \sigma \in \Gamma \]
that
\[ n\Phi(\sigma) = \#\sigma \Phi(\sigma) = \#\sigma \left< \Phi, Z_\sigma \right> \leq \lambda_\sigma \|\Phi\|_{p} \|Z_\sigma\|_p \leq \|\Phi\|_{p} \lambda_\sigma^{p+1}, \quad \forall \sigma \in \Gamma. \]
which together with the characterization theorem (Lemma 2.7) implies that \( n\Phi \in S^*_q(Z) \) and
\[ \|nj_p\|_q \leq \left[ \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p-1)} \|\Phi\|_{p} \right]^{\frac{1}{2}}. \]  (3.3)

Thus, \( \forall \Phi \in S^*_p(Z) \), we have \( n \circ j_p(\Phi) = n\Phi \in S^*_q(Z) \), and moreover
\[ \|n \circ j_p(\Phi)\|_q \leq \left[ \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p-1)} \|\Phi\|_{p} \right]^{\frac{1}{2}} \|\Phi\|_{p}. \]  (3.4)
which means that \( n \circ j_p \) is a continuous linear operator from \( S^*_p(Z) \) to \( S^*_q(Z) \), hence a continuous linear operator from \( S^*_p(Z) \) to \( S^*(Z) \).

From Proposition 3.2 and Proposition 3.3, we see that \( n \) is actually a continuous linear operator from \( S^*(Z) \) to itself.

Recall that the number operator \( N \) in \( L^2(Z) \) is defined by
\[ N\xi = \sum_{\sigma \in \Gamma} \#(\sigma, Z_\sigma) \xi |Z_\sigma\|^2, \quad \xi \in \text{Dom}N \]

\[ \{Z_\sigma | \sigma \in \Gamma\} \]

is the canonical orthonormal basis of \( \text{Dom} \), and means the domain of given by
\[ \text{Dom}N = \left\{ \xi \in L^2(Z) | \sum_{\sigma \in \Gamma} \#(\sigma, \xi)^2 |Z_\sigma|^2 < \infty \right\} \]

\[ n \cdot N \]

The next proposition then shows the link between \( n \) and \( \Phi \).

**Proposition 3.4** Let \( R : L^2(Z) \to S^*(Z) \) be the Riesz mapping. Then it holds that
\[ nR\xi = RN\xi, \quad \forall \xi \in \text{Dom}N. \]  (3.5)

where \( N \) is the number operator in \( L^2(Z) \).

**Proof.** Let \( \xi \in \text{Dom}N \). Then, \( \forall \sigma \in \Gamma \), we have
\[ nR\xi(\sigma) = n(R\xi)(\sigma) = \#\sigma R\xi(\sigma) = \#\sigma \left< R\xi, Z_\sigma \right> = \#\sigma \xi, Z_\sigma \]
which together with
\[ RN\xi(\sigma) = R(N\xi)(\sigma) = \left< R(N\xi), Z_\sigma \right> = \left< N\xi, Z_\sigma \right> \]

\[ = \left< \xi, N\sigma \right> = \#\sigma \xi, Z_\sigma \]
gives
\[ nR\xi(\sigma) = RN\xi(\sigma) \]

follows from the arbitrariness of \( \sigma \in \Gamma \).

**Remark 3.1** In view of Proposition 3.3, we may think of \( n \) as the extension of the number operator \( N \) to the generalized functional space \( S^*(Z) \) and call it the number operator on

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