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Number Operator on Functionals of Discrete-Time Normal Noises

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***Abstract*— Let Z be a discrete-time normal noise that has the chaotic representation property. In this paper, we show that the number operator in the space of square integrable functionals of Z can be extended to a continuous operator on the generalized functional space of Z.**

***Index Terms*— Discrete-time normal noise, Generalized functional, Fock transform, Number operator.**

***MSC(2010):*— 60H40, 47B38**

I. INTRODUCTION

Let $Z=(Z_k)$ be a discrete-time normal noise, and $L^2(Z)$ the space of square integrable functionals of Z. Then the number operator N in $L^2(Z)$ is defined as

$$N\xi=\sum_{\sigma\in\Gamma}\#\sigma\langle Z_\sigma,\xi\rangle Z_\sigma,\qquad \xi\in DomN,$$

where $\{Z_\sigma \mid \sigma\in\Gamma\}$ is the canonical orthonormal basis of $L^2(Z)$, $\#\sigma$ denotes the cardinality of σ as a finite set, and $DomN$ means the domain of N given by

$$DomN=\left\{\xi\in L^2(Z) \mid \sum_{\sigma\in\Gamma}(\#\sigma)^2\left|\langle Z_\sigma,\xi\rangle\right|^2<\infty\right\}.$$

The operator N plays an important role in functional analysis of discrete-time normal noises. For example, $-N$ generates the Ornstein-Uhlenbeck semigroup of operators on $L^2(Z)$ [6]. In a recent paper [2], N is used to study the regularity of solutions to the stochastic Schrodinger equation. However, N is not defined on whole $L^2(Z)$, namely $DomN\neq L^2(Z)$, which may cause inconvenience in its application.

On the other hand, as is shown in [7], one can use the canonical orthonormal basis of $L^2(Z)$ to construct a nuclear space $S(Z)$ such that $S(Z)$ is densely contained in $L^2(Z)$. Thus, by identifying $L^2(Z)$ with its dual, one can get a Gel'fand triple

$$S(Z)\subset L^2(Z)\subset S^*(Z). \qquad (1.1)$$

where $S^*(Z)$ is the dual of $S(Z)$, which is endowed with the strong topology, which can not be induced by any norm [3]. As usual, $S(Z)$ is called the testing functional space of Z, while $S^*(Z)$ is called the generalized functional space of Z. It turns out [4] that the generalized functional space $S^*(Z)$ can accommodate many quantities of theoretical interest that can not be covered by $L^2(Z)$.

In this paper, we would like to extend the number operator N to generalized functionals of Z. More precisely, we will define the number operator on the generalized functional space $S^*(Z)$.

Throughout this paper, N designates the set of all nonnegative integers and Γ the finite power set of N, namely

$$\Gamma=\{\sigma \mid \sigma\subset N,\#(\sigma)<\infty\},$$

where $\#\sigma$ means the cardinality of σ as a finite set. If $k\in N$ and $\sigma\in\Gamma$, then we simply write $\sigma\cup k$ for $\sigma\cup\{k\}$. Similarly, we use $\sigma\setminus k$.

II. PRELIMINARY

In what follows, we always assume that (Ω,F,P) is a given probability space. We use E to mean the expectation with respect to P. As usual, $L^2(\Omega,F,P)$ denotes the Hilbert space of square integrable complex-valued measurable functions on (Ω,F,P). We use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to mean the inner product and norm of $L^2(\Omega,F,P)$, respectively. By convention, $\langle\cdot,\cdot\rangle$ is conjugate-linear in its first argument and linear in its second argument.

Define2.1A sequence $Z=(Z_n)_{n\in N}$ of integrable random variables on (Ω,F,P) is called a discrete-time normal noise if it satisfies:

(i) $E[Z_n \mid F_{n-1}]=0$ for $n\geq 0$;

(ii) $E[Z_n^2 \mid F_{n-1}]=1$ for $n\geq 0$.

Here $F_{-1}=\{\phi,\Omega\}$, $F_n=\sigma(Z_k;0\leq k\leq n)$ for $n\in N$ and $E[\cdot \mid F_n]$ means the conditional expectation given F_n.

For a discrete-time normal noise $Z=(Z_n)_{n\in N}$ on (Ω,F,P), one can construct a corresponding family $\{Z_\sigma \mid \sigma\in\Gamma\}$ of random variables on (Ω,F,P) in the following manner

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$Z_\phi = 1$ and $Z_\sigma = \prod_{i\in\sigma} Z_i$, $\sigma\in\Gamma$, $\sigma\neq\phi$.

We call $\{Z_\sigma \mid \sigma\in\Gamma\}$ the canonical functional system of Z.

Lemma 2.1[6] Let $Z = (Z_n)_{n\in N}$ be a discrete-time normal noise on (Ω, F, P). Then its canonical functional system $\{Z_\sigma \mid \sigma\in\Gamma\}$ forms a countable orthonormal system in $L^2(\Omega, F, P)$.

Let $F_\infty = \sigma(Z_n; n\in N)$ be the $\sigma-field$ over Ω generated by a discrete-time normal noise $Z = (Z_n)_{n\in N}$ on (Ω, F, P). Then the canonical functional system $\{Z_\sigma \mid \sigma\in\Gamma\}$ is also a countable orthonormal system in the space $L^2(\Omega, F_\infty, P)$ of square integrable complex-valued measurable functions on (Ω, F_∞, P).

In the literature, $F_\infty - measurable$ functions on Ω are also known as functionals of Z. Thus elements of $L^2(\Omega, F_\infty, P)$ are naturally called square integrable

Define 2.2 A discrete-time normal noise $Z = (Z_n)_{n\in N}$ on (Ω, F, P) is said to have the chaotic representation property if its canonical functional system $\{Z_\sigma \mid \sigma\in\Gamma\}$ is total in $L^2(\Omega, F_\infty, P)$, where $F_\infty = \sigma(Z_n; n\in N)$.

Thus, if a discrete-time normal noise Z has the chaotic representation property, then its canonical functional system $\{Z_\sigma \mid \sigma\in\Gamma\}$ is actually an orthonormal basis of $L^2(\Omega, F_\infty, P)$.

From now on, we always assume that $Z = (Z_n)_{n\in N}$ is a given discrete-time normal noise on (Ω, F, P) that has the chaotic representation property.

For brevity, we use $L^2(Z)$ to denote the space of square integrable functionals of Z, namely

$$L^2(Z) = L^2(\Omega, F_\infty, P),$$

where $F_\infty = \sigma(Z_n; n\in N)$. For $k\geq 0$, we denote by F_k the σ- $field$ generated by $(Z_j; 0\leq j\leq k)$, namely

$$F_k = \sigma(Z_j; 0\leq j\leq k).$$

We note that $L^2(Z)$ shares the same inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ with $L^2(\Omega, F, P)$, and moreover the canonical functional system $\{Z_\sigma \mid \sigma\in\Gamma\}$ of Z forms a countable orthonormal basis for $L^2(Z)$, which we call the canonical orthonormal basis of $L^2(Z)$.

Lemma 2.2[7] *Let $\sigma\mapsto\lambda_\sigma$ be the $N - valued$ function on Γ given by*

$$\lambda_\sigma = \begin{cases} \prod_{k\in\sigma}(k+1), \sigma\neq\phi, \sigma\in\Gamma; \\ 1, \sigma=\phi, \sigma\in\Gamma. \end{cases} \tag{2.1}$$

Then, for $p>1$, the positive term series $\sum_{\sigma\in\Gamma}\lambda_\sigma^{-p}$ converges and moreover

$$\sum_{\sigma\in\Gamma}\lambda_\sigma^{-p} \leq \exp\left[\sum_{k=1}^{\infty} k^{-p}\right] < \infty \tag{2.2}$$

Using the $N -$ valued function defined by (2.1), we can construct a chain of Hilbert spaces consisting of functionals of Z as follows. For $p\geq 0$, we put

$$S_p(Z) = \left\{\xi\in L^2(Z) \mid \sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}\left|\langle Z_\sigma, \xi\rangle\right|^2 < \infty\right\} \tag{2.3}$$

and define

$$\langle\xi,\eta\rangle_p = \sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}\overline{\langle Z_\sigma,\xi\rangle}\langle Z_\sigma,\eta\rangle, \quad \xi,\eta\in S_p(Z).$$

It is not hard to check that, with $\langle\cdot,\cdot\rangle_p$ as the inner product, $S_p(Z)$ becomes a Hilbert space. We write $\|\xi\|_p = \sqrt{\langle\xi,\xi\rangle_p}$ for $\xi\in S_p(Z)$. Clearly, it holds that

$$\|\xi\|_p^2 = \sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}\left|\langle Z_\sigma,\xi\rangle\right|^2, \quad \xi\in S_p(Z) \tag{2.4}$$

Lemma2.3[4,7] *]For $p\geq 0$, $\{Z_\sigma \mid \sigma\in\Gamma\}\subset S_p(Z)$ and moreover the system $\{\lambda_\sigma^{-p} Z_\sigma \mid \sigma\in\Gamma\}$ forms an orthonormal basis for $S_p(Z)$.*

It is easy to see that $\lambda_\sigma\geq 1$ for all $\sigma\in\Gamma$. This implies that $\|\cdot\|_p \leq \|\cdot\|_q$ and $S_q(Z)\subset S_p(Z)$ whenever $0\leq p\leq q$. Thus we actually get a chain of Hilbert spaces of functionals of Z:

$$\cdots\subset S_{p+1}(Z)\subset S_p(Z)\subset\cdots\subset S_1(Z)\subset S_0(Z) = L^2(Z)$$

We now put

$$S(Z) = \bigcap_{p=0}^{\infty} S_p(Z)$$

and endow it with the topology generated by the norm sequence $\{\|\cdot\|_p\}_{p\geq 0}$. Note that, for each $p\geq 0$, $S_p(Z)$ is just the completion of $S(Z)$ with respect to $\|\cdot\|_p$. Thus $S(Z)$ is a countably-Hilbert space [1,3]. The next lemma, however, shows that $S(Z)$ even has a much better property.

Lemma2.4[4,7] *The space $S(Z)$ is a nuclear space, namely for any $p\geq 0$, there exists $q>p$ such that the inclusion mapping $i_{pq}: S_q(Z)\to S_p(Z)$ defined by $i_{pq}(\xi)=\xi$ is a Hilbert-Schmidt operator.*

For $p\geq 0$, we denote by $S_p^*(Z)$ the dual of $S_p(Z)$ and $\|\cdot\|_{-p}$ the norm of $S_p^*(Z)$. Then $S_p^*(Z)\subset S_q^*(Z)$ and

$\|\cdot\|_{-p} \geq \|\cdot\|_{-q}$ whenever $0 \leq p \leq q$. The lemma below is then an immediate consequence of the general theory of countably-Hilbert spaces (see, e.g., [1] or [3]).

Lemma 2.5[4,7] *Let $S^*(Z)$ be the dual of $S(Z)$ and endow it with the strong topology. Then*

$$S^*(Z) = \bigcup_{p=0}^{\infty} S_p^*(Z)$$

and moreover the inductive limit topology over $S^*(Z)$ given by space sequence $\{S_p^*(Z)\}_{p\geq 0}$ coincides with the strong topology.

We mention that, by identifying $L^2(Z)$ with its dual, one comes to a Gel'fand triple

$$S(Z) \subset L^2(Z) \subset S^*(Z),$$

which we refer to as the Gel'fand triple associated with the discrete-time normal noise Z.

Lemma 2.6[4] *The system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is contained in $S(Z)$ and moreover it forms a basis for $S(Z)$ in the sense that*

$$\xi = \sum_{\sigma \in \Gamma} \langle Z_\sigma, \xi \rangle Z_\sigma, \qquad \xi \in S(Z)$$

where $\langle \cdot,\cdot \rangle$ is the inner product of $L^2(Z)$ and the series converges in the topology of $S(Z)$.

Define 2.3[4,7] Elements of $S^*(Z)$ are called generalized functionals of Z, while elements of $S(Z)$ are called testing functionals of Z.

Thus, $S^*(Z)$ and $S(Z)$ can be accordingly called the generalized functional space and the testing functional space of Z, respectively. It turns out [4] that $S^*(Z)$ can accommodate many quantities of theoretical interest that can not be covered by $L^2(Z)$.

In the following, we denote by $\langle\langle \cdot,\cdot \rangle\rangle$ the canonical bilinear form on $S^*(Z) \times S(Z)$ given by

$$\langle\langle \Phi, \xi \rangle\rangle = \Phi(\xi), \qquad \Phi \in S^*(Z), \xi \in S(Z).$$

Note that $\langle\langle \cdot,\cdot \rangle\rangle$ is different from the inner product $\langle \cdot,\cdot \rangle$ of $L^2(Z)$.

Define 2.4[4] For $\Phi \in S^*(Z)$, its Fock transform is the function $\overline{\Phi}$ on Γ given by

$$\overline{\Phi}(\sigma) = \langle\langle \Phi, Z_\sigma \rangle\rangle, \qquad \sigma \in \Gamma \qquad (2.5)$$

where $\langle\langle \cdot,\cdot \rangle\rangle$ is the canonical bilinear form.

It is easy to verify that, for $\Phi, \Psi \in S^*(Z)$, $\Phi = \Psi$ if and only if $\overline{\Phi} = \overline{\Psi}$. Thus a generalized functional of Z is completely determined by its Fock transform. The following theorem characterizes generalized functionals of Z through their Fock transforms.

Lemma 2.7[4] *Let F be a function on Γ. Then F is the Fock transform of an element Φ of $S^*(Z)$ if and only if it satisfies*

$$|F(\sigma)| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma \qquad (2.6)$$

(2.9)

for $q > p + \frac{1}{2}$, one has

$$\|\Phi\|_{-q} \leq C\left[\sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p)}\right]^{\frac{1}{2}} \qquad (2.7)$$

and in particular $\Phi \in S_q^*(Z)$.

Remark2.1 (2.10) There exists a continuous linear mapping $R : L^2(Z) \to S^*(Z)$ such that

$$\langle\langle R\eta, \xi \rangle\rangle = \langle \eta, \xi \rangle, \quad \eta \in L^2(Z), \xi \in S(Z). \qquad (2.8)$$

where $\langle\langle \cdot,\cdot \rangle\rangle$ is the canonical bilinear form on $S^*(Z) \times S(Z)$. (2.11) We call R the Riesz mapping

I. MAIN RESULT

In this section, we define our number operator on the generalized functional space $S^*(Z)$ and show its links with the number operator N in $L^2(Z)$. Our main tool is the Fock transforms of generalized functionals of Z.

Recall that $\#\sigma$ means the cardinality of σ as a finite set. The following lemma gives an inequality concerning $\#\sigma$, which will be used later.

Lemma 3.1*For all $\sigma \in \Gamma$, it holds that $\#\sigma \leq \lambda_\sigma$.*

Proof. Let $\sigma \in \Gamma$. Clearly, $\#\sigma \leq \lambda_\sigma$ holds for the case of $\sigma = \phi$. For the case of $\sigma \neq \phi$, by assuming $\sigma = \{k_1, k_2, \cdots, k_n\}$ with $k_1 < k_2 < \cdots k_n$ .we have

$$\lambda_\sigma = (k_1+1)(k_2+1)\cdots(k_n+1) \geq 1 \times 2 \times \cdots \times n \geq n = \#\sigma \qquad (2.12)$$

This completes the proof.

Proposition3.2There exists a linear operator $n : S^*(Z) \to S^*(Z)$ such that

$$\overline{n\Phi}(\sigma) = \#\sigma\overline{\Phi}(\sigma), \quad \Phi \in S^*(Z), \sigma \in \Gamma \qquad (3.1)$$

where $\overline{\Phi}$ and $\overline{n\Phi}$ are Fock transforms of Φ and $n\Phi$, respectively.

Proof. Let $\Phi \in S^*(Z)$. Then, by Lemma 2.7, there exist constants $C \geq 0$ and $p \geq 0$ such that

$$\left|\overline{\Phi}(\sigma)\right| \le C\lambda_\sigma^{p}, \qquad \sigma\in\Gamma .$$

which together with Lemma 3.1 gives

$$\left|\#\sigma\overline{\Phi}(\sigma)\right| \le C\lambda_\sigma^{p+1}, \quad \sigma\in\Gamma .$$

which together Lemma 2.7 implies that there exists a unique generalized functional $\Phi'\in S^*(Z)$ such that

$$\overline{\Phi'}(\sigma)=\#\sigma\overline{\Phi}(\sigma), \qquad \sigma\in\Gamma . \qquad (3.2)$$

Thus we have a mapping $\Phi\to\Phi'$ from $S^*(Z)$ into itself, which we denote by n, namely

$$n\Phi=\Phi',\ \Phi\in S^*(Z)$$

Obviously n satisfies (3.1). Now let $\Phi_1,\Phi_2\in S^*(Z)$ and α_1,α_2 be any complex number. Then, by letting $\Psi=\alpha_1\Phi_1+\alpha_2\Phi_2$, we have

$$\begin{aligned}\overline{n\Psi}(\sigma)&=\#\sigma\hat{\Psi}(\sigma)\\&=\alpha_1\#\sigma\overline{\Phi_1}(\sigma)+\alpha_2\#\sigma\overline{\Phi_2}(\sigma)\\&=\alpha_1\overline{n\Phi_1}(\sigma)+\alpha_2\overline{n\Phi_2}(\sigma), \qquad \sigma\in\Gamma\end{aligned}$$

which implies that $n\Psi=\alpha_1 n\Phi_1+\alpha_2 n\Phi_2$. Thus n is a linear operator on $S^*(Z)$.

Proposition 3.3*The operator* n *is continuous on* $S^*(Z)$.

Proof. We need only to show that the composition $n\circ j_p : S_p^*(Z)\to S^*(Z)$ is continuous for each $p\ge 0$, where $j_p : S_p^*(Z)\to S^*(Z)$ denotes the natural embedding.

Let $p\ge 0$ and take $q>p+\frac{3}{2}$. Then, for each $\Phi\in S_p^*(Z)$, it follows from the inequality $\#\sigma\le\lambda_\sigma,\sigma\in\Gamma$ that

$$\begin{aligned}\left|\overline{n\Phi}(\sigma)\right|&=\#\sigma\left|\overline{\Phi}(\sigma)\right|=\#\sigma\left|\langle\langle\Phi,Z_\sigma\rangle\rangle\right|\le\lambda_\sigma\|\Phi\|_{-p}\|Z_\sigma\|_p\\&=\|\Phi\|_{-p}\lambda_\sigma^{p+1}, \forall\sigma\in\Gamma .\end{aligned}$$

which together with the characterization theorem (Lemma2.7) implies that $n\Phi\in S_q^*(Z)$ and

$$\|n\Phi\|_{-q}\le\left[\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-p-1)}\right]^{\frac{1}{2}}\|\Phi\|_{-p}. \qquad (3.3)$$

Thus, $\forall\Phi\in S_p^*(Z)$, we have $n\circ j_p(\Phi)=n\Phi\in S_q^*(Z)$, and moreover

$$\|n\circ j_p(\Phi)\|_{-q}\le\left[\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-p-1)}\right]^{\frac{1}{2}}\|\Phi\|_{-p}. \qquad (3.4)$$

which means that $n\circ j_p$ is a continuous linear operator from $S_p^*(Z)$ to $S_p^*(Z)$, hence a continuous linear operator from $S_p^*(Z)$ to $S^*(Z)$.

From Proposition 3.2 and Proposition 3.3, we see that n is actually a continuous linear operator from $S^*(Z)$ to itself.

Recall that the number operator N in $L^2(Z)$ is defined by

$$N\xi=\sum_{\sigma\in\Gamma}\#\sigma\langle Z_\sigma,\xi\rangle Z_\sigma, \qquad \xi\in DomN$$

where $\{Z_\sigma\mid\sigma\in\Gamma\}$ is the canonical orthonormal basis of $L^2(Z)$, and means the domain of given by

$$DomN=\left\{\xi\in L^2(Z)\mid\sum_{\sigma\in\Gamma}(\#\sigma)^2\left|\langle Z_\sigma,\xi\rangle\right|^2<\infty\right\}.$$

The next proposition then shows the link between n and N.

Proposition3.4*Let* $R:L^2(Z)\to S^*(Z)$ *be the Riesz mapping. Then it holds that*

$$nR\xi=RN\xi, \qquad \forall\xi\in DomN . \qquad (3.5)$$

where N *is the number operator in* $L^2(Z)$.

Proof. Let $\xi\in DomN$. Then, $\forall\sigma\in\Gamma$, we have

$$\overline{nR\xi}(\sigma)=\overline{n(R\xi)}(\sigma)=\#\sigma\overline{R\xi}(\sigma)=\#\sigma\langle\langle R\xi,Z_\sigma\rangle\rangle$$
$$=\#\sigma\langle\xi,\ Z_\sigma\rangle$$

which together with

$$\overline{RN\xi}(\sigma)=\overline{R(N\xi)}(\sigma)=\langle\langle R(N\xi),Z_\sigma\rangle\rangle=\langle N\xi,Z_\sigma\rangle$$

$=\langle\xi,NZ_\sigma\rangle=\#\sigma\langle\xi,Z_\sigma\rangle$ gives $\overline{nR\xi}(\sigma)=\overline{RN\xi}(\sigma)$. Thus, $nR\xi=RN\xi$ follows from the arbitrariness of $\sigma\in\Gamma$.

Remark 3.1 In view of Proposition 3.3, we may think of n as the extension of the number operator N to the generalized functional space $S^*(Z)$ and call it the number operator on

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