Global structure of positive solutions for superliner first-order periodic boundary value problems

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***Abstract*—In this paper,we consider the nonlinear eigenvalue problems**

$$\begin{cases} u'(t)+a(t)u(t)=\lambda h(t)f(t,u(t)),\ t\in(0,1),\\ u(0)=u(1),\end{cases}$$

where $\lambda>0$, $a\in C((0,1),[0,\infty))$, $h\in C((0,1),[0,\infty))$, **and there exist** $t_0\in(0,1)$, **such that** $h(t_0)>0$. $f\in C([0,\infty),[0,\infty))$, $f(0)=0,\ f(s)>0, s>0$.

***Index Terms*—Periodic boundary value problem;Eigenvalue Positive solutions; Existence *MSC(2010):*—39A10, 39A12**

I. INTRODUCTION

The first-order periodic boundary value problems play a very important role in many aspects. such as economy, finance, insurance, population structure and so on. Therefore, the existence of positive solutions is widely concerned by many scholars at home and abroad, and many rich and profound results have been obtained[1-10]. For example, the behavior of animal blood red blood cells, the survival competition between the two populations and the frequency of the circuit signals can be depicted by the first order periodic boundary value problem

$$\begin{cases} u'(t)=-a(t)g(u(t))+\lambda b(t)f(u(t-\tau(t))),\ t\in R,\\ u(0)=u(t+\omega),\end{cases}\quad(1.1)$$

with parameters.

In particular, Peng[1] uses fixed point theorems on cone to study the following questions

$$\begin{cases} u'(t)+f(t,u(t))=0,\ t\in(0,T),\\ u(0)=u(T),\end{cases}\quad(1.2)$$

Where as $T>0$, nonlinear term $f\in C((0,T)\times R,R)$,

The main results are as follows:

Theorem A. Assume that there exists a positive number $M>0$,such that $Mx-f(t,x)\geq 0$ for $x\geq 0, t\in J$ .If

$(A1)$ $\liminf\limits_{u\to 0^+}\min\limits_{t\in(0,T)}\dfrac{f(t,u)}{u}>0$, $\limsup\limits_{u\to+\infty}\max\limits_{t\in(0,T)}\dfrac{f(t,u)}{u}<0$

$(A2)$ $\liminf\limits_{u\to+\infty}\min\limits_{t\in(0,T)}\dfrac{f(t,u)}{u}>0$, $\limsup\limits_{u\to 0^+}\max\limits_{t\in(0,T)}\dfrac{f(t,u)}{u}<0$

then, PBVP (1.2) has at least one positive solution.

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Tisdell[2] uses Leray-Schauder degree and fixed point theory to discuss the first order periodic differential system

$$\begin{cases} u'(t)+a(t)u(t)=f(t,u(t)),\ t\in(0,1),\\ u(0)=u(1),\end{cases}\quad(1.3)$$

Thus, we obtain some sufficient conditions for the existence of positive solutions of (1.3).

Inspired by the above literatures, we use the Dancer's bifurcation to study the global structure of positive solutions for following periodic boundary value problems

$$\begin{cases} u'(t)+a(t)u(t)=\lambda h(t)f(t,u(t)),\ t\in(0,1),\\ u(0)=u(1),\end{cases}\quad(1.4)$$

We make the following assumptions:

$(H1)$ $h\in C((0,1),[0,\infty))$ is continuous, and there exist $t_0\in(0,1)$, such that $h(t_0)>0$;

$(H2)$ $f\in C([0,\infty),[0,\infty))$, $f(0)=0,\ f(s)>0, s>0$;

$(H3)$ $a\in C((0,1),[0,+\infty))$;

$(H4)$ $f_0=\infty$, where $f_0=\lim\limits_{s\to 0^+}\dfrac{f(s)}{s}$;

$(H5)$ $f_\infty\in[0,\infty]$, where $f_\infty=\lim\limits_{s\to\infty}\dfrac{f(s)}{s}$

The main results of the present paper are as follows:

Theorem 1.1. Let $(H1)$ - $(H5)$ hold. Let $E=\{u\in C[0,1]\,|\,u(0)=u(1)\}$ .Let Σ be the closure of the set of positive solutions for (1.4) in E .

(a) If $f_\infty=0$,then there exists a sub-continuum ζ of Σ with $(0,0)\in\zeta$ and

$$\mathrm{Proj}_R\zeta=[0,\infty).$$

(b) If $f_\infty\in(0,\infty)$, then there exists a sub-continuum ζ of Σ with $(0,0)\in\zeta$ and

$$\mathrm{Proj}_R\zeta\supseteq[0,\frac{\lambda_1}{f_\infty}).$$

(c) If $f_\infty=\infty$, then there exists a sub-continuum ζ of Σ with $(0,0)\in\zeta$, $\mathrm{Proj}_R\zeta$ is a bounded closed interval, and ζ approaches $(0,\infty)$ as $\|u\|\to\infty$.

Theorem 1.2.Let $(H1)$ - $(H5)$ hold.

(d) If $f_\infty=0$,then (1.4) has at least one positive solution for $\lambda\in(0,\infty)$.

(e) If $f_\infty\in(0,\infty)$, then (1.4) has at least one positive solution for $\lambda \in (0, \frac{\lambda_1}{f_\infty})$.

(f) If $f_\infty = \infty$, then (1.4) has at least one positive solution for $\lambda \in (0, \lambda_*)$.

II. Superior Limit and Component

$Y = C[0,1]$ is a Banach space, $K = \{u \in Y \mid u(t) \geq 0, t \in [0,1]\}$. The norm in $C[0,1]$ is defined as follows

$$|u|_0 = \max_{t\in[0,1]} |u(t)|.$$

Define an operator $T : K \to Y$ by,

$$Tu(t) = \int_0^1 H(t,s)h(s)f(u(s))ds, \ t \in [0,1].$$

Where

$$H(t,s) = \begin{cases} \dfrac{e^{\int_s^t a(\theta)d\theta}}{e^{\int_0^1 a(\theta)d\theta} - 1}, & 0 \leq t \leq s \leq 1, \\[2ex] \dfrac{e^{\int_s^t a(\theta)d\theta}}{1 - e^{-\int_0^1 a(\theta)d\theta}}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Denote $\sigma = e^{-\int_0^1 a(\theta)d\theta}$, then

$$\frac{\sigma}{\sigma\text{-}1} \leq H(t,s) \leq \frac{1}{1-\sigma}, (t,s) \in (0,1)\times(0,1). \ (2.1)$$

Denote the cone P in Y by

$$P = \{u \in Y \mid u(t) \geq \sigma\|u\|, t \in (0,1)\}.$$

Define an operator $T_\lambda : P \to Y$ by,

$$T_\lambda u(t) = \lambda \int_0^1 H(t,s)h(s)f(u(s))ds, \ t \in [0,1]. \ (2.2)$$

Definition 2.1. Let Y be a Banach space and $\{C_n \mid n = 1,2,\ldots\}$ be a family of subsets of Y. Then the superior limit D of $\{C_n\}$ is defined by

$$D := \limsup_{n\to\infty} C_n = \{x \in Y \mid \exists \{n_i\} \subset N$$
$$and \ x_{n_i} \in C_{n_i}, such \ that \ x_{n_i} \to x\}.$$

Definition 2.2. A component of a set M is meant a maximal connected subset of M.

Lemma 2.3. Suppose that Y is a compact metric space, A and B are non-intersecting closed subsets of Y, and no component of intersects both A and B. Then there exist two disjoint compact subsets X_A and X_B, such that $Y = X_A \cup X_B, A \subset X_A, B \subset X_B$.

Lemma 2.4. Let Y be a Banach space, and let $\{C_n\}$ be a family of connected subsets of Y, Assume that

(i) there exist $z_n \in C_n, n = 1,2,\ldots$, and $z^* \in X$, such that $z_n \to z^*$;

(ii) $\lim_{n\to\infty} r_n = \infty$, where $r_n = \sup\{\|x\| \mid x \in C_n\}$;

(iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of Y,

where

$$B_R = \{x \in X \mid \|x\| \leq R\}.$$

Then there exists an unbounded component C in D and $z^* \in C$.

Lemma 2.5. Assume that $(H1)$ hold. Then $T_\lambda : P \to P$ is completely continuous.

Lemma2.6. Let $(H1)-(H2)$ hold. Let $\Omega_r = \{u \in K \mid \|u\| < r, r > 0\}$. If $u \in \partial\Omega_r, r > 0$, then

$$\|T_\lambda u\| \leq \lambda \hat{M}_r \int_0^1 G(s,s)h(s)ds. \ (2.3)$$

Where $\hat{M}_r = 1 + \max_{0\leq s\leq r}\{f(s,u(s))\}$.

Proof: since $\forall t \in [0,1]$, $f(u(t)) \leq \hat{M}_r$, it follows that

$$\|T_\lambda u\| \leq \lambda \int_0^1 G(s,s)h(s)f(s,u(s))ds$$
$$\leq \lambda \hat{M}_r \int_0^1 G(s,s)h(s)ds.$$

Lemma 2.7. Let $(H1)$ hold, and let $r(T)$ be the spectral radius of T. Then $r(T) > 0$, and $r(T)$ is a simple eigenvalue with an eigenfunction $\varphi \in \operatorname{int} K_e$ and there is no other eigenvalue with a positive eigenfunction.

Lemma 2.8. Let $(H1)$ hold, and let $r(T)$ be the spectral radius of T. Then $\lambda_1 := \frac{1}{r(T)}$ is a simple eigenvalue with an eigenfunction $\varphi \in \operatorname{int} K_e$ and there is the unique eigenvalue with an eigenfunction $\varphi \in \operatorname{int} K_e$ and there is no other eigenvalue with a positive eigenfunction.

III. Eigenvalue with a Positive Eigenfunction

Denote $e(t) = 1, t \in [0,1]$, and let

$Y_e = \bigcup_{\rho>0} \rho[-1,1], |x|_e = \inf\{\rho \mid \rho > 0, x \in \rho[-1,1]\}$.

Set

$K_e = Y_e \cap K = \{x \in K | x \leq \rho e \ for \ some \ \rho > 0\}$.

Then

$(a1)$ K_e is a normal cone of Y_e with nonempty interior;

$(a2)$ $(Y_e, |\cdot|_e)$ is a Banach space and continuously imbedding in $(Y, \|\cdot\|)$.

Notice also that an $x \in Y_e$ is in $\operatorname{int} K_e$ the interior of K_e in Y_e if and only if $x \geq \rho e$ for some $\rho > 0$.

IV. PROOF OF THE MAIN RESULT

To prove the main result , we define $f^{[n]}(s):[0,\infty)\to[0,\infty)$ by

$$f^{[n]}(s)=\begin{cases} f(s), & s>(\frac{1}{n},\infty),\\ nf(\frac{1}{n})s, & s\in[0,\frac{1}{n}].\end{cases}$$

Then $f^{[n]}(s)\in C([0,\infty),[0,\infty))$ with $f^{[n]}(s)>0$ for all $s\in(0,\infty)$ and $(f^{[n]})_0=nf(\frac{1}{n})>0$.

By $(H3)$, it follow that that $\lim_{n\to\infty}(f^{[n]})_0=\infty$.

and accordingly, (b) hold. (c) can be deduced directly from the Arzela-Ascoli Theorem and the definition of $g^{[n]}$. Therefore , the superior limit of $\{C_+^{[n]}\}$, i. e. D, contains an unbounded connected component C with $(0,0)\in C$.

(a) $f_\infty=0$. In the case, we show that $\mathrm{Proj}_R C=[0,\infty)$.

Assume on the contrary that $\sup\{\lambda \mid (\lambda,y)\in C\}<\infty$, then there exists a sequence $\{(\mu_k,y_k)\}\subset C$ such that

$$\lim_{k\to\infty}\|y_k\|=\infty,|\mu_k|\le C_0, (4.1)$$

for some positive constant C_0 depending not on k . From Lemma 2.5 , we have that $\lim_{k\to\infty}\|y_k\|=\infty$. This together with the fact

$$\min_{\sigma\le t\le 1-\sigma} y_k(t)\ge\sigma\|y_k\|,\text{for all } 0<\sigma<\min\{t_0,1-t_0\}\ (4.2)$$

implies that

$$\lim_{k\to\infty} y_k(t)=\infty,\text{uniformly for } t\in[\sigma,1-\sigma].(4.3)$$

Since $(\mu_k,y_k)\in C$, we have that

$$\begin{cases} y_k'(t)+a(t)y_k(t)=\mu_k h(t)g(y_k(t)), & t\in(0,1),\\ y_k(0)=y_k(1), \end{cases}\quad(4.4)$$

Set $v_k(t)=y_k(t)/\|y_k\|$. Then $\|v_k\|=1$.

Now , choosing a subsequence and relabelling if necessary, it follows that there exists $(\mu_*,v_*)\in[0,C_0]\times E$ with

$$\|v_*\|=1$$

such that

$$\lim_{k\to\infty}(\mu_k,v_k)=(\mu_*,v_*),\text{in } R\times E\ (4.5)$$

Moreover, using $(4.3)-(4.4)$ and the assumption $f_\infty=0$, it follows that

$$\begin{cases} v_*'(t)+a(t)v_*(t)=\mu_* h(t)\cdot 0, & t\in(0,1)\\ v_*(0)=v_*(1). \end{cases}$$

And subsequently , $v_*(t)\equiv 0$ for $t\in[0,1]$. This contradicts (4.5) . Therefore

$$\sup\{\lambda|(\lambda,y)\in C\}=\infty.$$

(b) $f_\infty\in(0,\infty)$ .In this case, we show that $\mathrm{Proj}_R C\subseteq[0,\frac{\lambda_1}{f_\infty})$.

Let us rewrite (1.4) to the form

$$\begin{cases} u'(t)+a(t)u(t)=\lambda h(t)g_\infty u+\lambda h(t)\xi(u(t)), & t\in(0,1)\\ u(0)=u(1). \end{cases}$$

where $\xi(s)=g(s)-g_\infty s$. Obviously $\lim_{|s|\to\infty}\xi(s)/s=0$.

Now by the same method used to prove [6,Theorem 5.1], we may prove that C joins $(0,0)$ with $(\frac{\lambda_1}{f_\infty},\infty)$.

(c) $f_\infty=\infty$ In this case , we show that C joins $(0,0)$ with $(0,\infty)$.

Let $\{(\mu_k,y_k)\}\subset C$ be such that $|\mu_k|+\|y_k\|\to\infty$ as $k\to\infty$. then

$$\begin{cases} y_k'(t)+a(t)y_k(t)=\mu_k h(t)g(y_k(t)), t\in(0,1)\\ y_k(0)=y_k(1). \end{cases}$$

If $\{\|y_k\|\}$ is bounded, say, $\|y_k\|\le M_1$ for some M_1 depending not on k , the we may assume that

$$\lim_{k\to\infty}\mu_k=\infty.(4.6)$$

Note that

$$\frac{g(y_k(t))}{y_k(t)}\ge\inf\{\frac{g(s)}{s}|0<s\le M_1\}>0.$$

By condition $(H1)$, there exist some $0<\alpha<\beta<1$ such that $h(t)>0$ for $t\in[\alpha,\beta]$ .So there exists a constant $M_2>0$, such that

$$h(t)\frac{g(y_k)}{y_k(t)}>M_2>0,\ t\in[\alpha,\beta].(4.7)$$

Combining (4.6) and (4.7) with the relation

$$y_k'(t)+a(t)y_k(t)=\mu_k h(t)\frac{g(y_k(t))}{y_k(t)}y_k(t), t\in(0,1)\ (4.8)$$

From[3,Theorem 6.1], we deduced that must change its sign on $[\alpha,\beta]$ if k is large enough. This is a contradiction. Hence $\{\|y_k\|\}$ is unbounded.

Now , taking $\{(\mu_k,y_k)\}\subset C$ be such that

$$\|y_k\|\to\infty\ as\ k\to\infty\ (4.9)$$

We show that $\lim_{k\to\infty}\mu_k=0$.

Suppose on the contrary that , choosing a subsequence and rebelling if necessary, $\mu_k\ge b_0$ for some constant $b_0>0$. Then we have from (4.9) $\|y_k\|\ \to\infty$

To apply the nonlinear Krein-Rutman Theorem, we extend f to an odd function $g:R\to R$ by

$$g(s)=\begin{cases} f(s) & if\quad s\ge 0,\\ -f(-s) & if\quad s<0. \end{cases}$$

Similarly we may extend $f^{[n]}$ to an odd function $g^{[n]}: R \to R$ for each $n \in N$.

Now let us consider the auxiliary family of the equations

$$\begin{cases} u'(t) + a(t)u(t) = \lambda h(t) g^{[n]} u, \ t \in (0,1), \\ u(0) = u(1). \end{cases}$$

Let $\zeta \in C(R)$ be such that

$$g^{[n]}(u) = (g^{[n]})_0 u + \zeta^{[n]}(u) = nf(\frac{1}{n})u + \zeta^{[n]}(u).$$

Note that

$$\lim_{|s| \to 0} \frac{\zeta^{[n]}(s)}{s} = 0.$$

Let us consider

$$Lu - \lambda h(t)(g^{[n]})_0 u = \lambda h(t) \zeta^{[n]}(u) \quad (4.10)$$

As a bifurcation problem from the trivial solution $u \equiv 0$.

Equation (4.1) can be converted to the equation

$$u(t) = \int_0^1 H(t,s)[\lambda h(s)(g^{[n]})_0 u(s) + \lambda h(s) \zeta^{[n]} u(s)] ds$$

$$:= (\lambda L^{-1}[h(\cdot)(g^{[n]})_0 u(\cdot)](t) + \lambda L^{-1}[h(\cdot)\zeta^{[n]}(u(\cdot))])(t)$$

Further we note that $\|L^{-1}[h(\cdot)\zeta^{[n]}(u(\cdot))]\| = o(\|u\|)$ for u near 0 in E.

By Lemma 2.7 and the fact $(g^{[n]})_0 > 0$, the results of nonlinear Krein-Rutman Theorem can be stated as follows: there exists a continuum $C_+^{[n]}$ of positive solutions of (4.1) joining to infinity in . Moreover, $(\lambda_1/(g^{[n]})_0, 0)$ is the only positive bifurcation point of (4.1) lying on trivial solutions line $u \equiv 0$.

Proof of Theorem 1.1 Let us verify that $\{C_+^{[n]}\}$ satisfies all of the conditions of Lemma 2.4. Since

$$\lim_{n\to\infty} \frac{\lambda_1}{(g^{[n]})_0} = \lim_{n\to\infty} \frac{\lambda_1}{nf(\frac{1}{n})} = 0,$$

Condition (a) in Lemma 2.4 is satisfied with $z^* = (0,0)$. Obviously

$$r_n = \sup\{|\lambda| + \|y\| \ |(\lambda, y) \in C_+^{[n]}\} = \infty,$$

as $k \to \infty$. This together with (4.3) and condition $(H1)$ imply that there exist constants α_1, β_1 with $\sigma < \alpha_1 < \beta_1 < 1 - \sigma$, such that

$h(t) > 0$, $\lim_{k\to\infty} \mu_k \frac{g(y_k(t))}{y_k(t)} = \infty$, for all $t \in [\alpha_1, \beta_1]$

for every fixed constant $0 < \sigma < \min\{t_0, 1 - t_0\}$. Thus, we have from (4.8) and [3, Theorem 6.1] that y_k must change its sign on $[\alpha_1, \beta_1]$ if k is large enough. This is a contradiction. Therefore $\lim_{k\leftarrow\infty} \mu_k = 0$.

Proof of Theorem 1.2 (a) and (b) are immediate consequences of Theorem $1.1(a)$ and (b), respectively.

To prove (c), we rewrite (1.4) to

$$u = \lambda \int_0^1 H(t,s)h(s)f(u(s))ds =: T_\lambda u(t).$$

By Lemma 2.6, for every $r > 0$ and $u \in \partial\Omega_r$,

$$\|T_\lambda u\| \le \lambda \hat{M}_r \int_0^1 G(s,s)h(s)ds,$$

where $\hat{M}_r = 1 + \max_{0\le s\le r}\{f(s)\}$.

Let $\lambda_r > 0$ be such that

$$\lambda_r \hat{M}_r \left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1 - \sum_{i=1}^{m-2}\alpha_i \eta_i}\right) \int_0^1 G(s,s)h(s)ds = r.$$

Then for $\lambda \in (0, \lambda_r)$ and $u \in \partial\Omega_r$, $\|T_\lambda u\| \le \|u\|$. This means that

$$\Sigma \cap \{(\lambda, u) \in (0,\infty) \times K | 0 < \lambda < \lambda_r, u \in K : \|u\| = r\} = \phi \quad (4.11)$$

By Lemma 2.5 and Theorem 1.1, it follows that is also an unbounded component joining $(0,0)$ and $[0,\infty)$ in $[0,\infty) \times Y$. Thus, (4.10) implies that for $\lambda \in (0, \lambda_r)$, (1.4) has at least two positive solutions.

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