Existence of extremal periodic solutions for first-order functional differential equations

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***Abstract*— In this paper, we study the existence of periodic extremal solutions for the first-order functional differential equations**

$$x'(t)=a(t)g(x(t))x(t)-f(t,x(t-\tau(t)))$$

Where $a\in C(R,[0,\infty))$ is ω- periodic function and $\tau(t)\in C(R,R)$ is a continuous ω- periodic function. $\int_t^{t+\omega}a(t)dt>0,\ f\in C(R^2,[0,\infty))$ and $f(t,x(t))>0$ for $x>0$, $g\in C(R,[0,\infty))$ is non-decreasing function. The proof of the main results are based on the upper and lower solutions and monotone iteration method.

***Index Terms*— Upper and lower solutions, Extremal periodic solutions, Monotone iterative method, Existence.**

***MSC(2010):*—39A10, 39A12**

I. INTRODUCTION

Functional differential equations with delays appear in many ecological, economic, control and physiological models. An important question is whether these equations can support periodic solutions. Many authors have studied it extensively and obtained some results [1–15].

In particular, in 2004, Wang [1] applied the fixed point index theory to study when the condition $(A1)-(A2)$ is satisfied, the existence result of the solution of functional differential equations

$$x'(t)=a(t)g(x(t))x(t)-\lambda b(t)f(x(t-\tau(t))).(1.1)$$

Where

$(A1)$ $a,b\in C(R,[0,\infty))$ is ω- periodic function, $\int_t^{t+\omega}a(t)dt>0,\ \int_t^{t+\omega}b(t)dt>0,\ \tau(t)$ is a continuous ω- periodic function;

$(A2)$ $f,g\in C([0,\infty),[0,\infty))$ is continuous, $\forall\, u>0$, $f(u)>0,\ 0<l<g(u)<L<\infty$, where l,L is a constant.

Denote $f_0=\lim\limits_{u\to 0^+}\dfrac{f(u)}{u},\ f_\infty=\lim\limits_{u\to\infty}\dfrac{f(u)}{u},\ \sigma=e^{-\int_0^\omega a(t)dt}$,

$$M(r)=\max_{t\in[0,r]}f(t),\ m(r)=\min_{t\in[\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}r,r]}f(t).$$

Let $i_0=$ {the number of elements that valued is zero in f_0, f_∞ },

$i_\infty=$ { the number of elements that valued is infinity in f_0, f_∞ }.

The main results are as follows:

Theorem A Assume that $(A1)-(A2)$ hold, then

(a) if $i_0=1$ or $i_0=2$, then for $\lambda>\dfrac{1-\sigma^L}{m(1)\sigma^L\int_0^\omega b(s)ds}>0$, problem (1.1) has i_0 positive ω- periodic function.

(b) if $i_\infty=1$ or $i_\infty=2$, then for $0<\lambda<\dfrac{1-\sigma^L}{M(1)\sigma^L\int_0^\omega b(s)ds}$, problem (1.1) has i_∞ positive ω- periodic function.

(c) if $i_0=0$ or $i_\infty=0$, then for λ large or small enough, problem (1.1) has no positive ω- periodic function.

In 2010 ,Kang [5] apply the upper and lower solutions obtained the existence of periodic extremal solutions for the first-order functional differential equations

$$y'(t)=-a(t)y(t)-f(t,y(t-\tau(t))).(1.2)$$

where $a(t)>0,\ f$ is T periodic function about t ,and is a non decreasing continuous function about y .

The main results are as follows:

Theorem B If problem (1.2) has a lower solution $v_0(t)$ and a upper solution $\omega_0(t)$,for $\forall\ t_0\in[t,t+\omega],\ v_0(t_0)\le\omega_0(t_0)$, problem (1.2) has a minimal solution $v^*(t)$ and a maximal solution $\omega^*(t)$ .and it solution $y(t)$ are all in the $M=\{y\in C: v^*(t)\le y(t)\le\omega^*(t)\}$.

Inspired by the above works , we will discuss the existence of periodic extremal solutions for a broader class of first-order functional differential equations

$$x'(t)=a(t)g(x(t))x(t)-f(t,x(t-\tau(t)))\quad(1.3)$$

We make the following assumptions:

$(H1)$ $a\in C(R,[0,\infty))$ is ω- periodic function , $\int_0^\omega a(t)dt>0$, and $\tau(t)\in C(R,R)$ is a continuous ω- periodic function;

$(H2)$ $f\in C(R^2,[0,\infty))$ and f is ω- periodic function about t for $\forall t\in R$, $g\in C(R,[0,\infty))$ is non-decreasing function.

The main results of the present paper are as follows:

Theorem 1.1. Let $(H1)$ - $(H2)$ hold. If problem (1.3) has a lower solution α and a upper solution β , such that

$$\alpha(t)\le\beta(t),\ t\in R,\ (1.4)$$

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and f satisfies

$$f(t,v)-f(t,\omega)\le L(v-\omega),\ v\le\omega \quad (1.5)$$

where $L>0,\ v,w\in D=\{x\in X:\alpha\le x\le\beta\}$, then problem (1.3) has a minimal solution $x_{\min}$ and a maximal solution $x_{\max}$.

II. PRELIMINARIES

Let $X:=\{x\in C(R,R),\ x(t+\omega)=x(t),t\in R\}$, which is the Banach space with the norm

$$\|x\|=\max_{t\in[0,\omega]}|x(t)|.$$

Define 2.1 If $\alpha\in X\cap C^1(R,R)$ satisfies:

$$\alpha'(t)\ge a(t)g(\alpha(t))\alpha(t)-f(t,\alpha(t-\tau(t))),$$

then α is the lower solution of problem (1.3).

Define 2.2 If $\beta\in X\cap C^1(R,R)$ satisfies:

$$\beta'(t)\le a(t)g(\beta(t))\beta(t)-b(t)f(t,\beta(t-\tau(t))),$$

then β is the upper solution of problem (1.3).

For $\forall\ x\in X$, α,β are the upper and lower solutions of the problem (1.3). define

$$p_{\alpha,\beta}(x(t))=\max\{\alpha(t),\min\{u(t),\beta(t)\}\},$$

$$g_{\alpha,\beta}=g(p_{\alpha,\beta}(x(t))),\ f_{\alpha,\beta}=f(t,p_{\alpha,\beta}(x(t))).$$

then

$$\begin{cases} p_{\alpha,\beta}(x(t))=\alpha(t),\ x(t)\le\alpha(t),\\ p_{\alpha,\beta}(x(t))=\beta(t),\ x(t)\ge\beta(t),\\ \alpha(t)\le p_{\alpha,\beta}(x(t))\le\beta(t),\ x(t)\in X.\end{cases}\quad (2.1)$$

where $g_{\alpha,\beta},\ f_{\alpha,\beta}:X\to X$, and $g_{\alpha,\beta},\ f_{\alpha,\beta}$ is bounded.

Consider auxiliary problem

$$\begin{cases} x'(t)=a(t)g_{\alpha,\beta}(x(t))x(t)-f_{\alpha,\beta}(t,x(t-\tau(t))),\\ x(t)=x(t+\omega).\end{cases}\quad (2.2)$$

define the operator $T_{\alpha,\beta}:X\to X$ as follow

$$T_{\alpha,\ \beta}x(t)=\int_t^{t+\omega}G_{\alpha,\beta}(t,s,x)f_{\alpha,\beta}(t,x(t-\tau(t)))dt.$$

where

$$G_{\alpha,\beta}(t,s,x)=\frac{e^{-\int_t^s a(\theta)g_{\alpha,\beta}(x(\theta))x(\theta)d\theta}}{1-e^{-\int_0^\omega a(\theta)g_{\alpha,\beta}(x(\theta))x(\theta)d\theta}}.$$

The method of $[10]$ can verify $T_{\alpha,\beta}:X\to X$ is continuous and the solution of the problem (1.3) is equivalent to the fixed point of $T_{\alpha,\beta}$ in X.

Lemma1 $\forall x_1,x_2\in C^1(R,R)$, if

$$x_1'(r)-a(t)g(x_1(r))x_1(r)$$
$$\le x_2'(r)-a(r)g(x_2(r))x_2(r),\ r\in[t,t+\omega],\ (2.3)$$

and $x_1(t+\omega)\ge x_2(t+\omega)$, then $\forall\ r\in[t,t+\omega],\ x_1(r)\ge x_2(r)$.

Proof. Suppose on the contrary that $\exists r_0\in[t,t+\omega)$, s.t.

$$\min_{r\in[t,t+\omega]}(x_1-x_2)=x_1(r_0)-x_2(r_0)<0.$$

if $r_0\in(t,t+\omega)$, then $x_1'(r_0)=x_2'(r_0)$, by (2.2) and g is non-decreasing function, we can know

$$0=x_1'(r_0)-x_2'(r_0)$$
$$\le a(r_0)g(x_1(r_0))x_1(r_0)-a(r_0)g(x_2(r_0))x_2(r_0)<0.$$

which is contradiction.

if $r_0=0$, then $\exists\ r_1\in(t,t+\omega]$, s.t.for $\forall\ r\in[0,r_1]$, $x_1(r)-x_2(r)\le 0$, and $x_1'(r)-x_2'(r)>0$, has

$$0<x_1'(r)-x_2'(r)$$
$$\le a(r)g(x_1(r))x_1(r)-a(r)g(x_2(r))x_2(r)<0.$$

which is contradiction. thus for $\forall\ r\in[t,t+\omega]$, has $x_1(r)\ge x_2(r)$. The conclusion is proved.

III. PROOF OF THE MAIN RESULT

Proof of Theorem 1 Through the monotone iterative method we construct $x_{\min}$, let $x_0=\alpha$, define the solution $x_{n+1}(n=0,1,2,\ldots)$ of the following problem

$$\begin{cases} x_{n+1}'(t)-a(t)g_{\alpha,\beta}(x_{n+1}(t))x_{n+1}(t)=-f_{\alpha,\beta}(t,x_n(t-\tau(t))),\\ x_{n+1}(t)=x_{n+1}(t+\omega).\end{cases}\quad (3.1)$$

We first prove:

$$\alpha(r)\le x_n(r)\le\beta(r)\ (n=0,1,2,\ldots),\ r\in[t,t+\omega].\ (3.2)$$

It is known from the inductive method , as $n=0$,

$$x_1'(r)-a(r)g_{\alpha,\beta}(x_1(r))x_1(r)$$
$$=-f_{\alpha,\beta}(r,x_0(r-\tau(r)))$$
$$=-f_{\alpha,\beta}(r,\alpha(r-\tau(r)))$$
$$\le\alpha'(r)-a(r)g_{\alpha,\beta}(\alpha(r))\alpha(r).$$

By Lemma 1 ,we can know that $x_1(r)\ge\alpha(r)$. suppose $\alpha(r)\le x_n(r)\le\beta(r)$, next to prove $\alpha(r)\le x_{n+1}(r)$ .By g is a non-decreasing function and (3.1)we can obtain that

$$x_{n+1}'(r)-a(r)g_{\alpha,\beta}(x_{n+1}(r))x_{n+1}(r)$$
$$=-f_{\alpha,\beta}(r,x_n(r-\tau(r)))$$
$$\le L(\alpha((r-\tau(r))-x_n((r-\tau(r)))-f_{\alpha,\beta}(r,\alpha(r-\tau(r)))$$
$$\le-f_{\alpha,\beta}(r,\alpha(r-\tau(r)))$$
$$\le\alpha'(r)-a(r)g_{\alpha,\beta}(\alpha(r))\alpha(r).$$

by lemma 1 we can know $\forall\ r\in[t,t+\omega],\ \alpha(r)\le x_{n+1}(r).$ Similarly $\forall\ r\in[t,t+\omega],\ x_{n+1}(r)\le\beta(r)$, so (3.2) has been proved.

Next to prove

$$x_n(r)\le x_{n+1}(r)\ \ (n=0.1,2,\ldots),\ r\in[t,t+\omega],\ (3.3)$$

By the inductive method, when $n=0$, (3.3) is clearly established. Suppose $x_{n-1}\le x_n$. by $(3.1)(3.2)$ we can obtain that

$$x_{n+1}'(r)-a(r)g_{\alpha,\beta}(x_{n+1}(r))x_{n+1}(r)=-f_{\alpha,\beta}(r,x_n(r-\tau(r)))$$
$$\le L(x_{n-1}((t-\tau(r))-x_n((r-\tau(r)))-f_{\alpha,\beta}(r,x_{n-1}(r-\tau(r)))$$
$$\le-f_{\alpha,\beta}(r,x_{n-1}(r-\tau(r)))$$
$$=x_n'(r)-a(r)g_{\alpha,\beta}(x_n(r))x_n(r).$$

By lemma 1, for any $r \in [t, t+\omega]$, (3.3) is established. By (3.2) and (3.3) we can obtain that

$$\alpha(r) = x_0(r) \le x_1(r) \le \dots\ x_n(r) \le x_{n+1}(r) \le \dots \le \beta\ (r), r \in [t, t+\omega].$$

Thus, as $r \in [t, t+\omega]$,

$$x_{\min}(r) := \lim_{n\to\infty} x_n(r) \qquad (3.4)$$

is existence. Because $\{x_n\}_{n\ge1}$ is bounded in D, by Arzela-Ascoli theorem we can get $x_{\min} \in D$, and by (3.4) and Dini theorem, $\{x_n\}$ can be uniformly converged to $x_{\min}$ in D.

Consider the following questions

$$\begin{cases} x'_{n+1}(t) - a(t) g_{\alpha,\beta}(x_{n+1}(t)) x_{n+1}(t) = -f_{\alpha,\beta}(t, x_{n+1}(t-\tau(t))), \\ x_{n+1}(t) = x_{n+1}(t+\omega). \end{cases} \qquad (3.5)$$

Define an operator as follows

$$T_{n,\alpha,\beta}(x(t)) := \int_t^{t+\omega} G(t,s,x) f_{\alpha,\beta}(s, x_n(s-\tau(s))) ds.$$

Because $T_{n,\alpha,\beta}(x_{n+1}) = x_{n+1}$, continuity of operator T and

$$f_{\alpha,\beta}(t, x_{n+1}(t-\tau(t))) \to f_{\alpha,\beta}(t, x_{\min}(t-\tau(t)))$$

we can get $T_{n,\alpha,\beta}(x_{n+1}) \to T_{\alpha,\beta}(x_{\min})$ in D. That is $T_{x_{\min}} = x_{\min}$,So $x_{\min} \in D$ is the solution of the problem (1.3).

Finally, suppose that another $x \in D$ is the solution of the problem (1.3), then

$$x_n(r) \le x(r)\ (n = 0,1,2 \dots), r \in [t, t+\omega], \qquad (3.6)$$

obviously when $n = 0$, (3.6) was set up, from the inductive method, we can assume that $x_{n+1}(r) \le x(r)$, and then prove $x_{n+1}(r) \le x(r)$, by (3.1) ,we can obtained that

$$\begin{aligned} & x'_{n+1}(r) - a(r) g_{\alpha,\beta}(x_{n+1}(r)) x_{n+1}(r) = -f_{\alpha,\beta}(r, x_n(r, \tau(r))) \\ & \le L(x((r-\tau(r)) - x_n((r-\tau(r))) - f_{\alpha,\beta}(r, x(r-\tau(r))) \\ & = x'(r) - a(r) g_{\alpha,\beta}(x(r)) x(r), \end{aligned}$$

by the lemma 1, (3.6) is set up.

So from (3.6), we can get $x_{\min} \le x$,that is $x_{\min} \in D$ is the minimal solution to the problem (1.3).

Similarly, let $y_0 = \beta$, by the existence of descending sequence $\{y_n\}$, we can obtain $x_{\max} \in D$ is the maximal solution to the problem (1.3) .Theorem 1 is proved.

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