Existence of Positive Solutions for fourth-order Differential Equations with indeterminate weights

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***Abstract*— In this paper, the existence of positive solutions of fourth-order differential equations with indeterminate weights is considered as follows**

$$\begin{cases} u^{(4)} + Mu = \lambda a(x) f(u), x \in (0,1) \\ u(0) = u(1), u''(0) = u'(1) = 0. \end{cases}$$

Where M is a constant, $a:(0,1) \to R$, $f: R^+ \to R$ is continuous, $f(0) > 0$, $\lambda > 0$ is a parameter. Our approach is based on the Leray-Schauder fixed point theorem.

***Index Terms*— Green function, Leray-Schauder fixed point theorem, Existence. *MSC(2010):*—39A10, 39A12**

I. INTRODUCTION

As we all know, four-order differential equations are widely used in elastic mechanics and Engineering Physics. They are mainly used to describe the deformation of elastic beams. For example, under the Lidstone boundary condition $u(0) = u(1) = u''(0) = u''(1)$, it can be used to simulate the deformation phenomenon of elastic beams at both ends simply supported. In recent years, the existence of positive solutions has attracted the attention of many scholars, and some results have been achieved. [1–13]. In particular, in 2015, R.Vrabel [1] applied the upper and lower solutions to study the existence of

$$\begin{cases} u^{(4)}(x) + (k_1 + k_2)u''(x) + k_1 k_2 u(x) = f(x, u(x)), x \in (0,1) \\ u(0) = u(1) = u''(0) = u'(1) = 0. \end{cases} \quad (1.1)$$

solutions when $k_1 < k_2 < 0$.

The main results are as follows:

Theorem A Assume that the problem (1.1) has a lower solution α and an upper solution β, such that $\alpha(x) \le \beta(x)$, $x \in [0,1]$. If $f:[0,1] \times [\alpha(x), \beta(x)] \to R$ is continuous, and for $\alpha(x) \le u_1 \le u_2 \le \beta(x)$, $x \in [0,1]$ satisfies

$$f(x, u_1) \le f(x, u_2)$$

then problem (1.1) has a solutions $y(x)$ satisfies

$$\alpha(x) \le y(x) \le \beta(x), 0 \le x \le 1.$$

In 2018, Ma [2] etc. extends the main results of [1]. In this paper, the existence of problem (1.1) solutions when $0 < k_1 < k_2 < x_1^2 \approx 4.11585$ is discussed by using the upper and lower solutions. Where x_1 is the first positive solution of the equation

The main results are as follows:

Theorem B Assume that the problem (1.1) has a lower solution α and an upper solution β, such that $\alpha(x) \le \beta(x)$, $x \in [0,1]$. Let $E := \{(x,u) \in R^2 : 0 \le x \le 1, \alpha(x) \le u \le \beta(x)\}$, If $f: E \to R$ is continuous, and for $\alpha(x) \le u_1 \le u_2 \le \beta(x)$, $x \in [0,1]$ satisfies

$$f(x, u_1) \le f(x, u_2)$$

then problem (1.1) has a solutions $y(x)$ satisfies

$$\alpha(x) \le y(x) \le \beta(x), 0 \le x \le 1.$$

It is worth noting that [1] consider the case of $k_1 < k_2 < 0$, [2] consider the case of $0 < k_1 < k_2$, However, we will discuss the situation $k_1 + k_2 = 0$. Motivated by the above works, we will apply the Leray-Schauder fixed point theorem to establish the existence of positive solutions to the following fourth-order periodic value problems

$$\begin{cases} u^{(4)} + Mu = \lambda a(x) f(u), x \in (0,1) \\ u(0) = u(1) = u''(0) = u'(1) = 0. \end{cases} \quad (1.2)$$

We make the following assumptions:

$(H1)$ $f: R^+ \to R$ is continuous, $f(0) > 0$;

$(H2)$ $a:[0,1] \to R$ is continuous and not identically zero, there exists a number $k > 1$ such that

$$\int_0^1 k(x,y) a^+(y) dy \ge k \int_0^1 k(x,y) a^-(y) dy$$

for every $x \in [0,1]$, where a^+ (resp. a^-) is the positive (resp.negative) part of a, $K(x,y)$ is the Green's function of

$$\begin{cases} u^{(4)}(x) + Mu(x) = 0, x \in (0,1) \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases}$$

and

$$K(x,y) = \begin{cases} [\csc(m) + \sin(m - mx)\sin(my) - \\ \operatorname{csc}h(m)\sinh(m - mx)\sinh(my)]/2m^3, \\ 0 \le y \le x \le 1, \\ [\csc(m) + \sin(m - my)\sin(mx) - \\ \operatorname{csc}h(m)\sinh(m - my)\sinh(mx)]/2m^3, \\ 0 \le x \le y \le 1. \end{cases}$$

The main results of the present paper are as follows:

Theorem 1.1. Let $(H1)$ - $(H2)$ hold. Then there exists a

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positive number λ^* such that (1.2) has a positive solution for $0<\lambda<\lambda^*$.

Remark The requirement of f in document [1] is monotonous increasing. In this paper, f is continuous and weight function are allowed to change sign, so the condition of this paper is weaker than that of [1].

II. Preliminaries

Throughout the paper, we assume that $f(u)=f(0)$ for $u\le 0$, $C[0,1]$ is a Banach space. The norm in $C[0,T]$ is defined as follows

$$\|u\|=\max_{t\in[0,1]}|u(t)|.$$

We first recall the following fixed point result of Leray-Schauder fixed point theorem in a space.

Lemma2.1.[15] Define

$$W:=\{u\in C^4([0,1]):u(0)=u(1)=u''(0)=u''(1)=0\},$$

linear operator $L_c:W\to C([0,1])$ and $L_c u=u^{(4)}+Mu$, $u\in W$. then

(i) L_M is strongly inverse-positive in $W_0 \Leftrightarrow -\pi^4<M\le 0$;

(ii) $-\frac{c_0}{4}\le M<-\pi^4 \Rightarrow L_M$ is strongly inverse-negative in W_0.

Here $c_0=4k_0^4\approx 950.8843$ and $k_0\approx 3.9266$ is the smallest positive solution of the equation $\tan k=\tanh k$.

Proof. For part (i) see [15], Chapter 2, Section 4.1.3.

We shall prove that if $\pi<m\le k_0$ then $K(x,y)<0$ for all $x,y\in(0,1)$. From the fact that $k_0<2\pi$, we have that $\csc(m)<0$, so since the Green's function $K(x,y)$ is symmetric and $\sinh(m)>0$, we only must show that for all $t,s\in(0,1)$.

$$\sin(mx)\sin(m(1-y))\sinh(m)-\sinh(m(1-y))\sinh(mx)>0$$

which making $\tau=1-y$ is equivalent to

$$\frac{\sin(m\tau)\sin(m\tau)}{\sin(m)}<\frac{\sinh(mx)\sinh(m\tau)}{\sinh m}$$

for all $x,\tau\in(0,1)$. Clearly it suffices to consider the case $\sin(m\tau)>0$ and $\sin(mt)<0$. Since $\sin(x)\sinh(x)$ for all $x>0$ it is enough to prove that

$$\frac{\sin(mx)}{\sinh(mx)}>\frac{\sin(m)}{\sinh(m)} \text{ for all } x\in(0,1). \quad (2.1)$$

But this inequality follows immediately from the derivative of $\frac{\sin(x)}{\sinh(x)}$ is strictly negative in $(0,k_0)$. Therefore since $mx<m\le k_0$ we have that (2.1) holds.

Lemma2.2. Let $0<\sigma<1$. Then there exists a positive number $\bar\lambda>0$ such that, for $0<\lambda<\bar\lambda$, the problem

$$\begin{cases} u^{(4)}+Mu=\lambda a^+(x)f(u), x\in(0,1) \\ u(0)=u(1)=u''(0)=u'(1)=0. \end{cases} \quad (2.2)$$

has a positive solution $\bar u_\lambda$ with $\|\tilde u_\lambda\|\to 0$ as $\lambda\to 0$, and

$$\tilde u_{\lambda(x)}\ge\lambda\sigma f(0)p(x), x\in(0,1),.$$

where $p(x)=\int_0^T K(x,y)a^+(y)dy.$

Proof. For each $u\in C[0,1]$, let

$$Au(x)=\lambda\int_0^1 K(x,y)a^+(y)f(u(y))dy, x\in[0,1]$$

Then A: $C[0,1]\to C[0,1]$ is completely continuous and fixed points of A are solutions of (2.2). We shall apply the Leray-Schauder fixed point theorem to prove that A has a fixed point for λ small. Let $\varepsilon>0$ be such that

$$f(x)\ge\sigma f(0) \text{ for } 0\le s\le\varepsilon.$$

suppose that $\lambda<\frac{\varepsilon}{2\|p\|\tilde f(\varepsilon)}$, where $\tilde f(t)=\max_{0\le s\le t} f(s)$.

Then there exists $A_\lambda\in(0,\varepsilon)$ such that

$$\frac{\tilde f(A_\lambda)}{A_\lambda}=\frac{1}{2\lambda\|p\|}.$$

Let $u\in C[0,1]$ and $\theta\in(0,1)$ be such that $u=\theta Au$. Then we have

$$\|u\|\le\lambda\|p\|\tilde f(\|u\|),$$

or

$$\frac{\tilde f(\|u\|)}{\|u\|}\ge\frac{1}{\lambda\|p\|}.$$

which implies that $\|u\|\ne A_\lambda$. Note that $A_\lambda\to 0$ as $\lambda\to 0$. By the Leray-Schauder fixed point theorem, A has a fixed point $\tilde u_\lambda$ with $\|\tilde u_\lambda\|\le A_\lambda\le\varepsilon$. Consequently, $\tilde u_\lambda(x)\ge\lambda\sigma f(0)p(x)$, $x\in[0,1]$, and the proof is complete.

III. Proof of the Main Result

Proof of Theorem 1.1 Let $q(x)=\int_0^1 a^-(y)dy$. By $(H2)$, there exist positive numbers $\alpha,\gamma\in(0,1)$ such that

$$q(x)|f(s)|\le\gamma p(x)f(0), \quad (3.1)$$

for $s\in[0,\alpha]$. Fix $\sigma\in(\gamma,1)$ and let $\lambda^*>0$ be such that

$$\|\tilde u_\lambda\|+\lambda\sigma f(0)\|p\|\le\alpha, \quad (3.2)$$

for $\lambda<\lambda^*$. where $\tilde u_\lambda$ is given by Lemma 2.2, and

$$|f(x)-f(y)|\le f(0)(\frac{\sigma-\gamma}{2}), \quad (3.3)$$

for $x,y\in[-\alpha,\alpha]$ with $|x-y|\le\lambda^*\sigma f(0)\|p\|$.

Let $\lambda < \lambda^*$. We look for a solution u_λ of (1.3) of the form $\vartheta_\lambda + v_\lambda$. Thus v_λ solves

$$\begin{cases} v_\lambda^{(4)}(x) + Mv_\lambda = \lambda a^+(x)(f(\vartheta_\lambda + v_\lambda) - f(\vartheta_\lambda)) \\ -\lambda a^-(x) f(\vartheta_\lambda + v_\lambda), x \in (0,1) \\ v_\lambda(0) = v_\lambda(1) = v_\lambda''(0) = v_\lambda''(1) = 0, \end{cases}$$

For each $\omega \in C[0,1]$, let $v = A\omega$ be the solution of

$$\begin{cases} v^{(4)}(x) + Mv = \lambda a^+(x)(f(\vartheta_\lambda + \omega) - f(\vartheta_\lambda)) \\ -\lambda a^-(x) f(\vartheta_\lambda + \omega), x \in (0,1) \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

Then $A: C[0,1] \to C[0,1]$ is completely continuous. Let $v \in C[0,1]$ and $\theta \in (0,1)$ be such that $v = \theta Av$. Then we have

$$v^{(4)}(x) + Mv = \lambda\theta a^+(x)(f(\vartheta_\lambda + v) - f(\vartheta_\lambda))$$
$$-\lambda\theta a^-(x) f(\vartheta_\lambda + v).$$

We claim that $\|v\| \neq \lambda\sigma f(0)\|p\|$, Suppose to the contrary that $\|v\| = \lambda\sigma f(0)\|p\|$. Then by (3.2) and (3.3), we obtain

$$\|\vartheta_\lambda + v\| \le \|\vartheta_\lambda\| + \|v\| \le \alpha,$$

and

$$\|f(\vartheta_\lambda + v) - f(\vartheta_\lambda)\| \le f(0)\frac{\sigma - \gamma}{2}.$$

which, together with (3.1), implies that

$$\begin{aligned} |v(x)| &\le \lambda\frac{\sigma - \gamma}{2} f(0)p(x) + \lambda\gamma f(0)p(x) \\ &= \lambda\frac{\sigma + \gamma}{2} f(0)p(x), \ x \in (0,1) \end{aligned} \tag{3.4}$$

In particular

$$\begin{aligned} \|v(x)\| &\le \lambda\frac{\sigma + \lambda}{2} f(0)\|p(x)\| \\ &< \lambda\sigma f(0)\|p\| \end{aligned}$$

a contraction, and the claim is proved. By the Leray-Schauder fixed point theorem, A has a fixed point v_λ with $\|v_\lambda\| \le \lambda\sigma f(0)\|p\|$. Hence v_λ satisfies (3.4) and, using Lemma 2.1, we obtain

$$\begin{aligned} u_\lambda(x) &\ge \vartheta_\lambda - v_\lambda(x) \\ &\ge \lambda\sigma f(0)p(x) - \lambda\frac{\sigma + \gamma}{2} f(0)p(x) \\ &= \lambda\frac{\sigma + \gamma}{2} f(0)p(x) \end{aligned}$$

i.e., u_λ is a positive solution of (1.2). This completes the proof of Theorem 1.1.

IV. Application

Example 4.1 Consider the following nonlinear second-order periodic boundary value problems

$$\begin{cases} u^{(4)}(x) + 16u(x) = \lambda a(x) f(u), x \in (0,1) \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{4.1}$$

where λ is a positive parameter, $a(x) = \ln x$, $f(u) = -u^2 + 1$, u>0 is continuous, $m = 2$ satisfies the assumption $(H1)$.

Since $a(x) = \ln x$ is continuous on $[0,1]$, and there exists a number $k > 1$ such that

$$\int_0^1 k(x,y)a^+(y)dy \ge k\int_0^1 k(x,y)a^-(y)dy$$

for every $x \in [0,1]$, where a^+ (resp. a^-) is the positive (resp. negative) part of a, $K(x, y)$ is the Green's function of

$$\begin{cases} u^{(4)}(x) + 16u(x) = 0, x \in (0,1) \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases}$$

and

$$K(x,y) = \begin{cases} [\csc(2) + \sin(2-2x)\sin(2y) - \\ \operatorname{csch}(2)\sinh(2-2x)\sinh(2y)]/16, \\ 0 \le y \le x \le 1, \\ [\csc(2) + \sin(2-2y)\sin(2x) - \\ \operatorname{csch}(2)\sinh(2-2y)\sinh(2x)]/16, \\ 0 \le x \le y \le 1. \end{cases}$$ whic

h satisfies the assumption $(H2)$. By Theorem 1.1, if $(H1) - (H2)$ hold, then there exists a positive number λ^* such that (4.1) has a positive solution for $0 < \lambda < \lambda^*$.

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