Positive Solutions of Periodic Boundary Value Problems for a Class of Second-order Ordinary Differential Equations

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***Abstract*— In this paper, we consider the existence of positive solutions to the second-order periodic boundary value problems**

$$\begin{cases} u''(x)+d^2u(x)=\lambda a(x)f(u), x\in(0,T) \\ u(0)=u(T), u'(0)=u'(T). \end{cases}$$

Where $f:R^+\to R$ is continuous, $f(0)>0$, d is a constant, $a:(0,T)\to R$ may change sign, and $\lambda>0$ is sufficiently small. Our approach is based on the Leray-Schauder fixed point theorem.

***Index Terms*— Leray-Schauder fixed point theorem, Periodic boundary value problems, Positive solutions, Existence.**
***MSC(2010):*—39A10, 39A12**

I. INTRODUCTION

Recently, periodic boundary value problems have been studied extensively [1–12]. [1] uses the cone fixed point theorem to study the existence of positive solutions of the second-order ω- periodic boundary value problem. [2] deal with periodic boundary value problems using the method of upper and lower solutions.

In particular, in 1998, Jiang [3] obtained the existence of positive solutions by using Krasnoselskiis fixed point theorem

$$\begin{cases} -u''+Mu=f(t,u), x\in(0,2\pi) \\ u(0)=u(2\pi), u'(0)=u'(2\pi). \end{cases} \quad (1.1)$$

The main results are as follows:

Theorem A Assume that $f(t,u):[0,2\pi]\times[0,\infty)\to[0,\infty)$ is continuous, Then the periodic boundary value problem (1.1) has a positive solutions, provided $M>0$ and one of the following conditions hold:

$(A1)\ \lim_{u\to0}\max_{t\in[0,2\pi]}\frac{f(t,u)}{u}=0$, and $\lim_{u\to0}\max_{t\in[0,2\pi]}\frac{f(t,u)}{u}=\infty$ or

$(A2)\ \lim_{u\to0}\min_{t\in[0,2\pi]}\frac{f(t,u)}{u}=\infty$, and $\lim_{u\to0}\min_{t\in[0,2\pi]}\frac{f(t,u)}{u}=0$.

In 2010, Hao [4] used the fixed point index theorem to discuss the existence, multiplicity and nonexistence of positive solutions for periodic boundary value problems

$$\begin{cases} u''+a(t)u=\lambda f(t,u), \quad x\in(0,2\pi), \\ u(0)=u(2\pi), u'(0)=u'(2\pi). \end{cases} \quad (1.2)$$

where $a\in L^1(0,2\pi)$, $\lambda>0$.

The main results are as follows:

Theorem B Assume that $f:[0,2\pi]\times[0,\infty)\to[0,\infty)$ is continuous, and

$$f_\infty=\lim_{x\to+\infty}\min_{t\in[0,2\pi]}\frac{f(t,x)}{x}=\infty$$

Then there exist $\lambda^*>0$, such that the periodic boundary value problem (1.2) has at least two positive solutions for $0<\lambda<\lambda^*$, at least one positive solution for $\lambda=\lambda^*$ and no positive solution for $\lambda>\lambda^*$.

It is worth noting that [3] and [4] consider the case of $f:[0,2\pi]\times[0,\infty)\to[0,\infty)$ is continuous, However, we will discuss the broader situation $f:[0,T]\times R^+\to R$ is continuous. And as far as we know, second-order periodic boundary value problems have not been studied by applying the Leray-Schauder fixed point theorem.

Motivated by the above works, we will apply the Leray-Schauder fixed point theorem to establish the existence of positive solutions to the following second-order periodic value problems

$$\begin{cases} u''(x)+d^2u(x)=\lambda a(x)f(u), x\in(0,T) \\ u(0)=u(T), u'(0)=u'(T). \end{cases} \quad (1.3)$$

We make the following assumptions:

$(H1)$ $f:R^+\to R$ is continuous, $f(0)>0$, $\lambda>0$, $d>0$ and $d^2<\frac{4}{T}$;

$(H2)$ a is a constant on $[0,T]$, and not identically zero, there exists a number $k>1$ such that

$$\int_0^T k(x,y)a^+(y)dy\ge k\int_0^T k(x,y)a^-(y)dy$$

for every $x\in[0,T]$,where a^+ (resp. a^-) is the positive (resp.negative) part of a, $K(x,y)$ is the Green's function of

$$\begin{cases} u''(x)+d^2u(x)=0, x\in(0,T) \\ u(0)=u(T), u'(0)=u'(T). \end{cases}$$

and

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$$K(x,y)=\begin{cases}\dfrac{\sin d(x-y)+\sin d(T-x+y)}{2d(1-\cos dT)}, 0\le x\le y\le T,\\ \dfrac{\sin d(y-x)+\sin d(T-y+x)}{2d(1-\cos dT)}, 0\le y\le x\le T.\end{cases}$$

The main results of the present paper are as follows:

Theorem 1.1. Let $(H1)$ - $(H2)$ hold. Then there exists a positive number λ^* such that (1.3) has a positive solution for $0<\lambda<\lambda^*$.

II. Preliminaries

Throughout the paper, we assume that $f(u)=f(0)$ for $u\le 0$, $C[0,T]$ is a Banach space. The norm in $C[0,T]$ is defined as follows

$$|u|_0=\max_{t\in[0,T]}|u(t)|.$$

We first recall the following fixed point result of Leray-Schauder fixed point theorem in a space.

Lemma2.1. Let $0<\sigma<1$. Then there exists a positive number $\bar{\lambda}>0$ such that, for $0<\lambda<\bar{\lambda}$, the problem

$$\begin{cases}u''(x)+d^2u(x)=\lambda a^+(x)f(u), x\in(0,T)\\ u(0)=u(T), u'(0)=u'(T).\end{cases}\qquad(2.1)$$

has a positive solution $\bar{u}_\lambda$ with $|\bar{u}_\lambda|_0\to 0$ as $\lambda\to 0$, and

$$\bar{u}_\lambda(x)\ge\lambda\sigma f(0)p(x), x\in(0,T).$$

where $p(x)=\int_0^T K(x,y)a^+(y)dy.$

Proof. For each $u\in C[0,T]$, let

$$Au(x)=\lambda\int_0^T K(x,y)a^+(y)f(u(y))dy, x\in[0,T].$$

Then $A:C[0,T]\to C[0,T]$ is completely continuous and fixed points of A are solutions of (2.1). We shall apply the Lemma 2.1 to prove that A has a fixed point for λ small. Let $\varepsilon>0$ be such that

$$f(x)\ge\sigma f(0) \text{ for } 0\le s\le\varepsilon.$$

suppose that $\lambda<\dfrac{\varepsilon}{2|p|_0\tilde{f}(\varepsilon)}$, where $\tilde{f}(t)=\max_{0\le s\le t}f(s)$.

Then there exists $A_\lambda\in(0,\varepsilon)$ such that

$$\frac{\tilde{f}(A_\lambda)}{A_\lambda}=\frac{1}{2\lambda|p|_0}.$$

Let $u\in C[0,T]$ and $\theta\in(0,1)$ be such that $u=\theta Au$. Then we have

$$|u|_0\le\lambda|p|_0\tilde{f}(|u|_0),$$

or

$$\frac{\tilde{f}(|u|_0)}{|u|_0}\ge\frac{1}{\lambda|p|_0}.$$

which implies that $|u|_0\ne A_\lambda$. Note that $A_\lambda\to 0$ as $\lambda\to 0$. By the Lemma 2.1, A has a fixed point $\tilde{u}_\lambda$ with $|\tilde{u}_\lambda|_0\le A_\lambda\le\varepsilon$. Consequently, $\tilde{u}_\lambda(x)\ge\lambda\sigma f(0)p(x)$, $x\in[0,T]$, and the proof is complete.

III. Proof of the Main Result

Proof of Theorem 1.1 Let $q(x)=\int_0^T K(x,y)a^-(y)dy$. By $(H2)$, there exist positive numbers $\alpha,\gamma\in(0,1)$ such that

$$q(x)|f(s)|\le\gamma p(x)f(0),\qquad(3.1)$$

for $s\in[0,\alpha]$. Fix $\sigma\in(\gamma,1)$ and let $\lambda^*>0$ be such that

$$|\tilde{u}_\lambda|_0+\lambda\sigma f(0)|p|_0\le\alpha,\qquad(3.2)$$

for $\lambda<\lambda^*$. where $\tilde{u}_\lambda$ is given by Lemma 2.2, and

$$|f(x)-f(y)|\le f(0)(\frac{\sigma-\gamma}{2}),\qquad(3.3)$$

for $x, y\in[-\alpha,\alpha]$ with $|x-y|\le\lambda^*\sigma f(0)|p|_0$.

Let $\lambda<\lambda^*$. We look for a solution u_λ of (1.3) of the form $\tilde{u}_\lambda+v_\lambda$. Thus v_λ solves

$$\begin{cases}v_\lambda''(x)+d^2v_\lambda=\lambda a^+(x)(f(\tilde{u}_\lambda+v_\lambda)-f(\tilde{u}_\lambda))\\ -\lambda a^-(x)f(\tilde{u}_\lambda+v_\lambda), x\in(0,T)\\ v_\lambda(0)=v_\lambda(T), v_\lambda'(0)=v_\lambda'(T),\end{cases}$$

For each $\omega\in C[0,T]$, let $v=A\omega$ be the solution of

$$\begin{cases}v''(x)+d^2v=\lambda a^+(x)(f(\tilde{u}_\lambda+\omega)-f(\tilde{u}_\lambda))\\ -\lambda a^-(x)f(\tilde{u}_\lambda+\omega), x\in(0,T)\\ v(0)=v(T), v'(0)=v'(T),\end{cases}$$

Then $A:C[0,T]\to C[0,T]$ is completely continuous. Let $v\in C[0,T]$ and $\theta\in(0,1)$ be such that $v=\theta Av$. Then we have

$$v''(x)+d^2v=\lambda\theta a^+(x)(f(\tilde{u}_\lambda+v)-f(\tilde{u}_\lambda))$$
$$-\lambda\theta a^-(x)f(\tilde{u}_\lambda+v).$$

We claim that $|v|_0\ne\lambda\sigma f(0)|p|_0$, Suppose to the contrary that $|v|_0=\lambda\sigma f(0)|p|_0$. Then by (3.2) and (3.3), we obtain

$$|\tilde{u}_\lambda+v|_0\le|\tilde{u}_\lambda|_0+|v_\lambda|_0\le\alpha,$$

and

$$|f(\tilde{u}_\lambda+v)-f(\tilde{u}_\lambda)|_0\le f(0)\frac{\sigma-\gamma}{2}.$$

which, together with (3.1), implies that

$$|v(x)| \le \lambda \frac{\sigma-\gamma}{2} f(0)p(x) + \lambda\gamma f(0)p(x) \qquad (3.4)$$
$$= \lambda \frac{\sigma+\gamma}{2} f(0)p(x), \ x \in (0,T)$$

In particular

$$|v(x)|_0 \le \lambda \frac{\sigma+\lambda}{2} f(0)|p(x)|_0$$
$$< \lambda\sigma f(0)|p|_0$$

a contraction, and the claim is proved. By the Leray-Schauder fixed point theorem, A has a fixed point v_λ with $|v_\lambda|_0 \le \lambda\sigma f(0)|p|_0$. Hence v_λ satisfies (3.4) and, using Lemma 2.2, we obtain

$$u_\lambda(x) \ge \vartheta_\lambda - v_\lambda(x)$$
$$\ge \lambda\sigma f(0)p(x) - \lambda\frac{\sigma+\gamma}{2} f(0)p(x)$$
$$= \lambda\frac{\sigma+\gamma}{2} f(0)p(x)$$

i.e., u_λ is a positive solution of (1.3). This completes the proof of Theorem 1.1.

IV. Application

Example 4.1 Consider the following nonlinear second-order periodic boundary value problems

$$\begin{cases} u''(x) + 4u(x) = \lambda a(x) f(u), x \in (0,T) \\ u(0) = u(T), u'(0) = u'(T). \end{cases} \qquad (4.1)$$

where λ is a positive parameter, $a(x) = \ln x$, $f(u) = -u^2 + 1$, u>0 is continuous, $d = 2$ satisfies the assumption $(H1)$.

Since $a(x) = \ln x$ is continuous on $[0,T]$, and there exists a number $k > 1$ such that

$$\int_0^T k(x,y)a^+(y)dy \ge k\int_0^T k(x,y)a^-(y)dy$$

for every $x \in [0,T]$, where a^+ (resp. a^-) is the positive (resp. negative) part of a , $K(x, y)$ is the Green's function of

$$\begin{cases} u''(x) + 4u(x) = 0, x \in (0,T) \\ u(0) = u(T), u'(0) = u'(T). \end{cases}$$

and

$$K(x,y) = \begin{cases} \dfrac{\sin 2(x-y) + \sin 2(T-x+y)}{4(1-\cos 2T)}, 0 \le x \le y \le T, \\ \dfrac{\sin 2(y-x) + \sin 2(T-y+x)}{4(1-\cos 2T)}, 0 \le y \le x \le T. \end{cases}$$

which satisfies the assumption $(H2)$.

By Theorem 1.1, if $(H1)-(H2)$ hold, then there exists a positive number λ^* such that (4.1) has a positive solution for $0 < \lambda < \lambda^*$.

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