Positive Solutions of Periodic Boundary Value Problems for a Class of Second-order Ordinary Differential Equations

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Abstract— In this paper, we consider the existence of positive solutions to the second-order periodic boundary value problems

\[ \begin{cases} u''(x) + d^2u(x) = \lambda a(x)f(u), x \in (0, T) \\ u(0) = u(T), u'(0) = u'(T). \end{cases} \]

where \( a \in L^1(0,2\pi), \lambda > 0 \).

The main results are as follows:

Theorem A Assume that \( f : [0,2\pi] \times [0,\infty) \rightarrow [0,\infty) \) is continuous. And as far as we know, second-order periodic boundary value problems have not been studied by applying the Leray-Schauder fixed point theorem.

Motivated by the above works, we will apply the Leray-Schauder fixed point theorem to establish the existence of positive solutions to the following second-order periodic boundary value problems

\[ \begin{cases} u''(x) + d^2u(x) = \lambda a(x)f(u), x \in (0, T) \\ u(0) = u(T), u'(0) = u'(T). \end{cases} \]

We make the following assumptions:

(H1) \( f : R^+ \rightarrow R \) is continuous, \( f(0) > 0 \), \( \lambda > 0, d > 0 \) and \( d^2 < \frac{4}{T} \);

(H2) \( a \) is a constant on \([0,T]\). and not identically zero, there exists a number \( k > 1 \) such that

\[ \int_0^T k(x,y) a^{-}(y)dy \geq k \int_0^T k(x,y) a^{-}(y)dy \]

for every \( x \in [0,T] \), where \( a^{-} \) (resp. \( a^{+} \)) is the positive (resp. negative) part of \( a \). \( K(x,y) \) is the Green’s function of

\[ \begin{cases} u''(x) + d^2u(x) = 0, x \in (0, T) \\ u(0) = u(T), u'(0) = u'(T). \end{cases} \]
which implies that $|u|_0 \neq A_\lambda$. Note that $A_\lambda \to 0$ as $\lambda \to 0$. By the Lemma 2.1, $A$ has a fixed point $\ell_\lambda$ with $|\ell_\lambda|_0 \leq A_\lambda \leq \epsilon$. Consequently, $\ell_\lambda(x) \geq \lambda \sigma f(0)p(x)$, x $\in [0, T]$, and the proof is complete.

III. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1 Let $q(x) = \int_0^x K(x, y) \kappa^-(y)dy$. By (H2), there exist positive numbers $\alpha$, $\gamma \in (0, 1)$ such that

$$q(x)|f(s)| \leq \gamma p(x)f(0),$$

(3.1)

for $s \in [0, \alpha]$. Fix $\sigma \in (\gamma, 1)$ and let $\lambda^* > 0$ be such that

$$|\ell_\lambda|_0 + \lambda \sigma f(0)|p|_0 \leq \alpha,$$

(3.2)

for $\lambda < \lambda^*$. Then, by Lemma 2.2, we obtain

$$\int f(x) - f(y) \leq f(0)\left(\frac{\sigma - \gamma}{2}\right),$$

(3.3)

for $x, y \in [-\alpha, \alpha]$ with $|x - y| \leq \lambda^* \sigma f(0)|p|_0$.

Let $\lambda < \lambda^*$. We look for a solution $u_\lambda$ of (1.3) of the form

$$u_\lambda = v_\lambda + v_{\lambda}.$$

Thus $v_{\lambda}$ solves

$$\begin{cases}
\int u''(x) + d^2u(x) = \lambda a^+(x)f(\ell_\lambda) + f(\ell_\lambda), & x \in (0, T) \\
u(0) = u(T), u'(0) = u'(T), &
\end{cases}
$$

(2.1)

for each $u \in C[0, T], \ let A : C[0, T] \to C[0, T]$ is completely continuous and fixed points of $A$ are solutions of (2.1). We shall apply the Lemma 2.1 to prove that $A$ has a fixed point for $\lambda$ small. Let $\epsilon > 0$ be such that

$$f(x) \geq \sigma f(0)\text{ for } 0 \leq s \leq \epsilon.$$

suppose that $\lambda < \frac{2\epsilon}{|p|_0 \ell(\epsilon)}$, where $\ell(t) = \max_{0 \leq s \leq t} f(s)$. Then there exists $A_\lambda \in (0, \epsilon)$ such that

$$\frac{\ell(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda |p|_0}.$$

(3.4)

Let $u \in C[0, T]$ and $\theta \in (0, 1)$ be such that $u = \theta Au$. Then we have

$$0 \leq \frac{\ell(|u|_0)}{|u|_0} \leq \frac{1}{2|p|_0}.$$

(3.5)

which is a contradiction with (3.1), implies that

$$\frac{\ell(|u|_0)}{|u|_0} \geq \frac{1}{2|p|_0}.$$
\[
|v(x)| \leq \lambda \frac{\sigma - \gamma}{2} f(0) p(x) + \lambda \gamma f(0) p(x)
\]
\[
= \lambda \frac{\sigma + \gamma}{2} f(0) p(x), \ x \in (0,T)
\]
In particular
\[
|v(x)|_0 \leq \lambda \frac{\sigma + \lambda}{2} f(0) p(x)_0
\]
\[
< \lambda \sigma f(0) p(0)
\]
a contraction, and the claim is proved. By the Leray-Schauder fixed point theorem, \( A \) has a fixed point \( \nu_x \) with
\[
|\nu_x|_0 \leq \lambda \sigma f(0) p(0)_0. \]
Hence \( \nu_x \) satisfies (3.4) and, using Lemma 2.2, we obtain
\[
u_j(x) \geq \omega_k^2 - \nu_j(x)
\]
\[
\geq \lambda \sigma f(0) p(x) - \lambda \frac{\sigma + \gamma}{2} f(0) p(x)
\]
\[
= \lambda \frac{\sigma + \gamma}{2} f(0) p(x)
\]
i.e., \( u_j \) is a positive solution of (1.3). This completes the proof of Theorem 1.1.

IV. APPLICATION

Example 4.1 Consider the following nonlinear second-order periodic boundary value problems
\[
\begin{align*}
u^*(x) + 4u(x) &= \lambda a(x) f(u), \ x \in (0, T) \\
u(0) &= u(T), u'(0) = u'(T)
\end{align*}
\]
(4.1)
where \( \lambda \) is a positive parameter, \( a(x) = \ln x \), \( f(u) = -u^2 + 1 \), \( u > 0 \) is continuous, \( d = 2 \) satisfies the assumption (H1).

Since \( a(x) = \ln x \) is continuous on \( [0, T] \), and there exists a number \( k > 1 \) such that
\[
\int_0^T k(x, y)a^+(y) dy \geq k \int_0^T k(x, y)a^-(y) dy
\]
for every \( x \in [0, T] \), where \( a^+ \) (resp. \( a^- \)) is the positive (resp. negative) part of \( a \), \( K(x, y) \) is the Green's function of
\[
\begin{align*}
u^*(x) + 4u(x) &= 0, \ x \in (0, T) \\
u(0) &= u(T), u'(0) = u'(T).
\end{align*}
\]
and
\[
K(x, y) =
\begin{cases}
\frac{\sin 2(x - y) + \sin 2(T - x + y)}{4(1 - \cos 2T)}, & 0 \leq x \leq y \leq T, \\
\frac{\sin 2(y - x) + \sin 2(T - y + x)}{4(1 - \cos 2T)}, & 0 \leq y \leq x \leq T.
\end{cases}
\]
which satisfies the assumption (H2).

By Theorem 1.1, if (H1)–(H2) hold, then there exists a positive number \( \lambda^* \) such that (4.1) has a positive solution for \( 0 < \lambda < \lambda^* \).

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REFERENCES


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