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Pseudo-umbilical time-like submanifolds in locally symmetric pseudo Riemann manifold

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Abstract— **In this paper, we studied the pseudo-umbilical time-like SubmanifoldsM ^{n} immersed in a locally symmetric pseudo Riemannian manifold N^{n+p}. when M^n is compact and with parallel mean curvature vector, the sufficient conditions for M^n to be total geodesic are obtained by using the Hopf maximum principle.**

Index Terms— **pseudo-Riemannian manifold; parallel mean curvature vector; time-like.**

I. INTRODUCTION

The studied of time-like submanifolds has been a long time, and at the same time it is also a research direction that is more popular in geometry. The development of the time-like submanifolds has been extended to the case of locally symmetric pseudo-Riemanns.

For pseudo-Riemannian manifolds, You-Jing hu and Yong-qiang Ji [1] studied the sufficient conditions for the compact maximal time-like submanifolds in the de Sitter space as sub-manifolds of all geodesics. Ying Li and Wei-dong Song [2] expand the outer space of the documentary [1] into a locally symmetric pseudo-Riemannian manifold. The studied of the compact maximum time-like submanifolds in a locally symmetric pseudo-Riemannian manifold the Sufficient conditions for submanifolds to become all geodesic, from which it can be considered whether it is possible to study the sufficient conditions for the formation of compact time-like submanifolds with parallel mean curvature vectors in a locally symmetric pseudo- Riemannian manifold as all geodesic submanifolds, reference to existing p documentary can be obtained from the documentary[3]

Theorem A Suppose N_p^{n+p} a ^{p} locally symmetric pseudo-Riemannian manifold, if M^n is Pseudo-umbilical time-like submanifolds with parallel mean curvature, if the square of the second basic form module length S in M n satisfied

$$S > [\frac{4}{3}(1-\delta)(P-2)n^{\frac{3}{2}} + \frac{2}{3}(1-\delta)(p-1)^{\frac{1}{2}}n^{\frac{3}{2}}H + n]\,(n-1) + pnH^2.$$

Then M^n is a n+1 dimensional all-umbilical hypersurface submanifold in N_p^{n+p}.

The section curvature K_N of N_p^{n+p} satisfies $0 < \delta \le$ KN $\le$ 1 and M^n is compact submanifolds.

In this paper, by improving the conditional conclusion of Theorem A, the following theorems and inferences are obtained.

Theorem 1 let N_p^{n+p} be a locally symmetric pseudo-Riemannian manifold,and its section curv ature K_N satisfies $0 < c_1 < K_N < c_2$, and M^n is N_p^{n+p} a compact pseudo-umbilical time-like submanifold with a parallel curvature vector field, if the square of the second basic form module length S is satisfied

$$S \ge p[nH^2 + nc_2 + \frac{4}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}(p-1) + \frac{2}{3}(c_2-c_1)n^{\frac{3}{2}}p^{\frac{1}{2}}] \qquad (1)$$

then M^n is a all geodesic.

Inference Let N_p^{n+p} be a locally symmetric pseudo-Riemannian space type, and the section curvature c > 0, when M^n is n dimension maximal time-like submanifolds in N_p^{n+p},if the square of the second basic form module length S is satisfied

$$S \ge npc$$

Then M n must be a all geodesic submanifold in N_p^{n+p}.

II. PRELIMINARIES

The scope of the various types of indicators agreed in this paper is:

$1 \le i, j, k \cdots \le n;\quad n+1 \le \alpha, \beta, \gamma \cdots \le n+p.$

N_p^{n+p} represent the indicator is p and section curvature K_N satisfies $0 \le c_1 \le K_N \le c_2$ Pseudo-Riemannian manifolds, M^n is a n dimension time-like submanifold that is immersed into the N_p^{n+p}. Select local pseudo-Riemann standard orthonormal frame field e1 · · · en+p in N_p^{n+p},When it is restricted to M^n, let ω1 · · · ωn+p for its dual frame.

Let N_p^{n+p} the pseudo-Riemannian metric d_{SN}^2 be

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$ds_N^2 = \sum_{i=1}^{n} \xi_i \omega_i^2 + \sum_{\alpha=n+1}^{n+p} \xi_\alpha {\omega_\alpha}^2,$ (inside $\xi i = -1, \xi\alpha = 1$)
if the pseudo-Riemannian induces M n is metric to be

$dS_M^2 = \sum_{i=1}^{n} \xi_i \omega_i^2$

M n is a time-like submanifold in N_p^{n+p} .

let the contact 1 form for ωAB of N_p^{n+p} , When limited to M^n,

$\omega_{i\alpha} = \sum_j \xi_i {h_{ij}}^\alpha \omega_j$, $h_{ij}^\alpha = h_{ji}^\alpha$.

h , $\overset{\varpi}{H}$, R_{ijkl} , $R_{\alpha\beta kl}$, K_{ABCD} respectively are the second basic form of M^n , mean curvature vector,curvature tensor,normal curvature tensor,

and normal curvature of tensor of N_p^{n+p} , then

$$h = \sum_{\alpha,i,j} \xi_i \xi_j {h_{ij}}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \ \overset{\varpi}{H} = \frac{1}{n} \sum_\alpha (\sum_i {h_{ii}}^\alpha) e_\alpha.$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha \xi_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i \xi_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta) , \tag{3}$$

$$R_{ij} = \sum_k K_{ikjk} - \sum_{\alpha,k} h_{ik}^\alpha h_{kj}^\alpha + \sum_{\alpha,k} h_{ij}^\alpha h_{kk}^\alpha. \tag{4}$$

h_{ijk}^α and h_{ijkl}^α respectively First-order and second-order covariant derivatives of ${h^\alpha}_{ij}$, then Codazzi and Ricci identity respectively

$$h_{ijk}^\alpha - h_{ikj}^\alpha = -K_{\alpha ijk} , \tag{5}$$

$$h_{ijkl}^\alpha - h_{ijkl}^\alpha = -\sum_m h_{mi}^\alpha R_{mjkl} - \sum_m h_{mj}^\alpha R_{mikl} - \sum_\beta h_{ij}^\beta R_{\beta\alpha kl}$$

The covariant derivative $K_{\alpha ijk,l}$ of the curvature tensor field $K_{\alpha ijk}$ is defined as

$$-\sum_l K_{\alpha ijk,l}\omega_l = dK_{\alpha ijk} - \sum_m K_{mijk}\omega_{m\alpha} + \sum_\beta K_{\alpha\beta jk}\omega_{\beta i} + \sum_\beta K_{\alpha i\beta k}\omega_{\beta j} + \sum_\beta K_{\alpha ij\beta}\omega_{\beta k} ,$$

When limited to M^n ,

$$-K_{\alpha ijk,l} = K_{\alpha ijkl} + \sum_m K_{mijk} h_{ml}^\alpha + \sum_\beta K_{\alpha\beta jk} h_{il}^\beta + \sum_\beta K_{\alpha i\beta k} h_{jl}^\beta + \sum_\beta K_{\alpha ij\beta} h_{kl}^\beta . \tag{7}$$

According to documentary[4],if N_p^{n+p} is locally symmetric, then $K_{\alpha ijk,l} = 0$,

$$K_{\alpha ijkl} = -\sum_m K_{mijk} h_{ml}^\alpha - \sum_\beta K_{\alpha\beta jk} h_{il}^\beta - \sum_\beta K_{\alpha i\beta k} h_{jl}^\beta - \sum_\beta K_{\alpha ij\beta} h_{kl}^\beta . \tag{8}$$

In order to complete the proof of the theorem,
Notes

$S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$, $H = |\overset{\varpi}{H}|^2 = \frac{1}{n}\sqrt{\sum_\alpha (\sum_i {h_{ii}}^\alpha)^2}$, $H_\alpha = (h_{ij}^\alpha)_{n\times n}$.

lemma 1[5] Set N_p^{n+p} is a n+p dimensional Pseudo Riemannian manifold,it's section curvatre K_N satisfies $0 \le c_1 \le K_N \le c_2$,then

$|K_{ABCD}| \le \frac{1}{2}(c_2 - c_1),$ each differs from the other in A and B;

$|K_{ABCD}| \le \frac{2}{3}(c_2 - c_1),$ each differs from the other in A,B,C and D.

III. PROOF OF THEOREM

Available from (2)-(8)

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + \sum_\alpha \sum_{i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \ge \sum_\alpha \sum_{i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= \sum_{\alpha,i,j,k} h_{ij}^\alpha h_{kkij}^\alpha + \sum_{\alpha,\beta} \sum_{i,j,k} h_{ij}^\alpha h_{kk}^\beta K_{\alpha ij\beta} + 2\sum_{\alpha,\beta}\sum_{i,j,k} h_{ij}^\alpha h_{kj}^\beta K_{\alpha\beta ik} \\ &+ \sum_{\alpha,\beta}\sum_{i,j,k} h_{ij}^\alpha h_{ij}^\beta K_{\alpha k\beta k} + \sum_{\alpha,\beta} [tr(H_\alpha H_\beta)]^2 + \\ &2\sum_{\alpha,\beta}[tr(H_\alpha^2 H_\beta^2) - tr(H_\alpha H_\beta)^2] - \sum_{\alpha,\beta} tr(H_\alpha^2 H_\beta) tr(H_\beta).\end{aligned} \tag{9}$$

field, en+p can be chosen to be parallel to $\overset{\omega}{H}$

$$\begin{cases} trH_\alpha = \sum_i h_{ii}^\alpha = 0 (\alpha \ne n+p), \\ trH_{n+p} = \sum_i h_{ii}^{n+p} = nH. \end{cases} \tag{10}$$

M^n is pseudo-umbilical, so

$h_{ij}^{n+p} = H\delta_{ij}$

The following will estimate each item in (9).

According to documentary[6], M^n has a parallel mean curvature vector field, then H is a constant,so $\sum_k h_{kkij}^\alpha = 0$,

then $\sum_{\alpha,i,j,k} h_{ij}^\alpha h_{kkij}^\alpha = \sum_{\alpha,i,j} h_{ij}^\alpha (\sum_k h_{kkij}^\alpha) = 0.$

A = (tr(HαHβ))p×p is real symmetric matrix, therefore, the standard frame field can be selected to diagonalize, so

$tr(H_\alpha H_\beta) = tr(H_\alpha^2)\delta_{\alpha\beta},$

$$\sum_{\alpha,\beta}[tr(H_\alpha H_\beta)]^2=\sum_{\alpha}[tr(H_\alpha^2)]^2\geq\frac{1}{p}S^2. \quad (12)$$

Fixed α, let $h_{ij}^{\alpha}=\lambda_i\delta_{ij}$, from Lemma 1

$$2\sum_{\beta}\sum_{ijk}h_{ij}^{\alpha}h_{jk}^{\beta}K_{\alpha\beta ki}=2\sum_{i,k,\beta}\lambda_i^{\alpha}h_{ik}^{\beta}K_{\alpha\beta ki}$$

$$\geq-2\sum_{\substack{i\neq k\\ \beta(\neq\alpha)}}\frac{2}{3}(c_2-c_1)|\lambda_i^{\alpha}||h_{ik}^{\beta}|\geq-\frac{1}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}\sum_{\beta\neq\alpha}trH_\beta^2$$

$$-\frac{1}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}(p-1)trH_\alpha^2,$$

And then

$$2\sum_{\alpha,\beta}\sum_{i,j,k}h_{ij}^{\alpha}h_{jk}^{\beta}K_{\alpha\beta ki}\geq-\frac{4}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}(p-1)S$$

(13)

because of the conditions assumed by the Theorem $0<c_1\leq K_N\leq c_2$ and diagonalization of the matrix,we can know

$$\left|\sum_{\alpha,\beta}\sum_{i,j,k}K_{\alpha k\beta k}h_{ij}^{\alpha}h_{ij}^{\beta}\right|=$$

$$\left|\sum_{\alpha,\beta(\neq\alpha)}\sum_{i,j,k}K_{\alpha k\beta k}h_{ij}^{\alpha}h_{ij}^{\beta}+\sum_{\alpha,i,j,k}K_{\alpha k\alpha k}(h_{ij}^{\alpha})^2\right|$$

$$\leq nc_2S$$

thereby $\sum_{\alpha,\beta}\sum_{i,j,k}K_{\alpha k\beta k}h_{ij}^{\alpha}h_{ij}^{\beta}\geq-nc_2S$

In addition,From (10) and Lemma 1.

$$\sum_{\alpha,\beta}\sum_{i,j,k}h_{ij}^{\alpha}h_{kk}^{\beta}K_{\alpha ij\beta}=\sum_{\alpha,\beta}\sum_{i,j}h_{ij}^{\alpha}K_{\alpha ij\beta}\sum_{k}h_{kk}^{\beta}$$

$$\geq-\frac{2}{3}(c_2-c_1)n|H|\sum_{\alpha,i,j}|h_{ij}^{\alpha}|\geq-\frac{2}{3}(c_2-c_1)n^2|H|S^{\frac{1}{2}}.$$

because of $S\geq nH^2$, further

$$\sum_{\alpha,\beta}\sum_{i,j,k}h_{ij}^{\alpha}h_{kk}^{\beta}K_{\alpha ij\beta}\geq-\frac{2}{3}(c_2-c_1)n^{\frac{2}{3}}p^{\frac{1}{2}}S. \quad (15)$$

Notice

$$\sum_{\alpha,\beta}[tr(H_\alpha^2H_\beta^2)-tr(H_\alpha H_\beta)^2]\geq0. \quad (16)$$

and by (10) and M n is pseudo-umbilical

$$-\sum_{\alpha,\beta}[tr(H_\alpha^2H_\beta)](trH_\beta)=-nH^2S \quad (17)$$

substitute the above estimates (11)-(17) into (9)

$$\frac{1}{2}\Delta S\geq S(-\frac{2}{3}(c_2-c_1)n^{\frac{3}{2}}p^{\frac{1}{2}}-\frac{4}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}$$

$$(p-1)-nH^2-nc_2+\frac{1}{p}S),$$

the conditions in the theorem are

$$S\geq p[nH^2+nc_2+\frac{4}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}(p-1)$$

$$+\frac{2}{3}(c_2-c_1)n^{\frac{3}{2}}p^{\frac{1}{2}}]$$

Guaranteed

$$-\frac{2}{3}(c_2-c_1)n^{\frac{3}{2}}p^{\frac{1}{2}}-\frac{4}{3}(c_2-c_1)(n-1)^{\frac{1}{2}}(p-1)$$

$$-nH^2-nc_2+\frac{1}{p}S\geq0.$$

it's not hard to verify $\frac{1}{2}\Delta S\geq0$.

M^n is compact, from the principle of Hopf maximum S is obtained as a constant, so $\frac{1}{2}\Delta S=0$ there must be S = 0 ,so M n is a all geodesic submanifolds.

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