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Improved of Approximating Function Li(x)

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***Abstract*— Let π(x) be the prime-counting function that gives the number of primes less than or equal to x, for any positive number x and let the approximating function Li(x) denote the off set integral logarithm of x. This function is a good approximation to the number of prime numbers less than x. We propose a simple modification of Li(x) gauss prediction function for reduces of π(x)-Li(x).**

***Index Terms*— Gauss prediction, Count Primes, Prime number, Prime race.**

I. INTRODUCTION

In the end of the 18 th century, Legendre and Gauss independently conjectured the prime number theorem [1]. The prime number theorem was first proved in 1896 by Jacques Hadamard and by Charles de la Vallée Poussin [2] independently, using properties of the Riemann zeta function introduced by Riemann in 1859 [3]. Proofs of the prime number theorem not using the zeta function or complex analysis were found around 1948 by Atle Selberg and by Paul Erdős [4].

II. DEFINITIONS AND PRELIMINARIES

Definition 1(Prime counting function).
In mathematics, the prime-counting function is the function counting the number of prime numbers less than or equal to some real number x[5]. It is denoted by $\pi(x)$

$\pi(x) = \#\{p \in N : p \leq x \text{ is a prime}\}$
For example,

$$\pi(6) = \#\{2,3,5\} = 3.$$

Some values of $\pi(x)$ are given in Table 1.

Table 1.

x	100	200	300	500	1000
$\pi(x)$	25	46	62	95	168

Theorem 1.
As x tends to infinity, the number of primes up to x is asymptotic to $x/\ln x$, "$a(x)$ is asymptotic to $b(x)$" and "$a(x) \sim b(x)$" both mean that the limit (as x approaches infinity) of the ratio $a(x)/b(x)$ is 1. This statement is the prime number theorem.[5]
The prime number theorem states that $\pi(x) \sim x/\ln x$, then in the sense that,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1. \qquad (1.1)$$

Some values of $\pi(x)$ and $x/\ln x$ for several $10^3 < x < 10^4$ are given in Table 2.

Table 2.

x	$\pi(x)$	$\frac{x}{lnx}$
1000	168	144,76
2000	303	263,12
3000	430	374,7
4000	550	482,27
5000	669	587,04
6000	783	689,69
7000	900	790,63
8000	1007	890,15
9000	1117	988,47
10000	1229	1085,73

Theorem 2.
Gauss was also studying prime tables and came up with a different estimate first published in 1863. [3]

$$\pi(x) \sim \int_2^x \frac{dt}{\ln t} = \mathrm{Li}(x)$$

Theorem 3.
In 1798 Legendre published the first significant conjecture on the approximate formula for $\pi(x)$ which appears in Legendre's Theoriedes Nombresis. [6]

$$\pi(x) \sim \frac{x}{\ln x - A}$$

Where A is a constant whose value Legendre gives as **1.08366.**

Some values of $\pi(x)$ with $\mathrm{Li}(x)$ and compare these estimates for several $10^3 < x < 10^{10}$ are given in Table 3.

Table 3.

x	$\pi(x)$	Gauss	Legendre
10^3	168	178	172
10^4	1229	1246	1231
10^5	9592	9630	9588
10^6	78498	78628	78534
10^7	664579	664918	665138
10^8	5761455	5762209	5760341
10^9	50847534	50849235	50917519
10^{10}	455052511	455055614	455743004

Some values difference of $\pi(x)$ and $\mathrm{Li}(x)$ are given in Table 4.

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Table 4.

x	$\pi(x)$	Li(x) - $\pi(x)$
10^8	5761455	753
10^9	50847534	1700
10^{10}	455052511	3103
10^{11}	118054813	11587
10^{12}	7607912018	38262
10^{13}	46065536839	108970
10^{14}	204941750802	314889
10^{15}	29844570422669	1052618
10^{16}	79238341033925	3214631
10^{17}	2623557157654233	7956588
10^{18}	24739954287740860	21949554
10^{19}	234057667276344607	99877774
10^{20}	2208196025609188840	222744643
10^{21}	21127269486018731928	597394253
10^{22}	201467286689315906290	1932355207

Theorem 4 (Littlewood):
There are arbitrarily large values of x for which $\pi(x) > \mathrm{Li}(x)$, that is, for which

$$\pi(x) > \int_2^x \frac{dt}{\ln t} \tag{1.4}$$

So what is the smallest x_1 for which $\pi(x_1) > \mathrm{Li}(x_1)$ Skewes obtained an upper bound for x_1 from Littlewood's proof [9], though not a particularly accessible bound. Skewes [10] proved in 1933 that

$$x_1 < 10^{10^{10^{34}}}$$

and to do even this he needed to make a significant assumption. Skewes assumed the truth of the "Riemann Hypothesis," a conjecture that we shall discuss a little later. For a long time, this "Skewes' number" was known as the largest number to which any "interesting" mathematical meaning could be ascribed. Skewes later gave an upper bound that did not depend on any unproved assumption, though at the cost of making the numerical estimate marginally more monstrous, and several improvements have been made since then[10],[11],[14].

- Skewes : $x_1 < 10^{10^{10^{10^{100}}}}$
- Lehman : $x_1 < 2 \times 10^{1165}$
- Te Riele : $x_1 < 6.658 \times 10^{370}$
- Lehman : $x_1 < 1.3982 \times 10^{316}$

III. IMPROVED LI(X)

$x \geq 5,$

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

We obtained through of $\mathrm{Li}(x)$ Gauss prediction function a new function H(x)

$$H(x) = \int_{\sqrt{Li(x)}}^x \frac{dt}{\ln t}$$

Some values of π (x), Li(x), Legendre and H(x) are given in Table 5.

Table 5.

x	π (x)	Li(x)	Legendre	H(x)
10^3	168	178	172	170
10^4	1229	1246	1231	1231
10^5	9592	9630	9588	9600
10^6	78498	78628	78534	78562
10^7	664579	664918	665138	664767
10^8	5761455	5762209	5769341	5761842
10^9	50847534	50849235	50917519	50848305
10^{10}	455052511	455055614	455743004	455053191

Comparisons of some values of π (x), Li(x), Legendre and H(x) are given in Table 6 and Table 7.

Table 6.

x	π(x)	Li(x)-$\pi(x)$	Leg.- $\pi(x)$	H(x)- $\pi(x)$
10^3	168	10	4	2
10^4	1229	17	2	2
10^5	9592	38	-4	8
10^6	78498	130	36	64
10^7	664579	339	559	188
10^8	5761455	754	7886	387
10^9	50847534	1701	69985	771
10^{10}	455052511	3103	690493	680

Table 7.

x	$\pi(x)$	Li(x) - $\pi(x)$	H(x)-$\pi(x)$
10^8	761455	753	387
10^9	50847534	1700	771
10^{10}	55052511	3103	680
10^{11}	118054813	11587	5126
10^{12}	7607912018	38262	20724
10^{13}	46065536839	108970	60680
10^{14}	204941750802	314889	180347
10^{15}	29844570422669	1052618	674050
10^{16}	79238341033925	3214631	2140425
10^{17}	2623557157654233	7956588	4886220
10^{18}	24739954287740860	21949554	13117711
10^{19}	234057667276344607	99877774	74330633
10^{20}	220819602560918840	222744643	148478468
10^{21}	21127269486018731928	597394253	380539444
10^{22}	201467286689315906290	1932355207	1296610851

IV. CONCLUSİON

We comparededed $\mathrm{H}(\mathrm{x}) - \pi(\mathrm{x})$ and $\mathrm{Li}(x) - \pi(x)$ for $5 \leq x \leq 10^{22}$ using maple computer program. We observation the values of $\mathrm{H}(\mathrm{x}) - \pi(\mathrm{x})$ less than $\mathrm{Li}(x) - \pi(x)$ in Table 6

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