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On modified GSI method for singular saddle point problems

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***Abstract*— Recently, Miao considered the GSI method for singular saddle point problems and studied the semi-convergence of the GSI method. In this paper, we prove the semi-convergence of Modified GSI method when it is applied to solve the singular saddle point problems.**

***Index Terms*— Singular saddle-point problems, Modifying Generalized stationary iteration method , Semi-convergence property, Moore-Penrose inverse.**

I. INTRODUCTION

We consider the singular saddle point problem of the form

$$\mathrm{Au} = \begin{pmatrix} \mathrm{A} & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} \mathrm{x} \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} = b, \qquad (1)$$

where $A \in \Re^{m\times m}$ is a symmetric positive definite (SPD) matrix, $B \in \Re^{m\times n}$ is a matrix of rank r, and $\mathrm{f} \in \Re^m$ and $\mathrm{g} \in \Re^n$ are given vectors , with $m \geq n$. When $\mathrm{r} = n$, note that the coefficient matrix A is the non-singular and the saddle point problem (1) has a unique solution. When $\mathrm{r} < n$, the coefficient matrix A is singular, in such case, we assume that the saddle point problem (1) is consistent [25] .

The saddle point problem(1) arises in a variety of scientific and engineering applications, such as constrained optimization, the finite element method to Stokes equations, fluid dynamics and weighted linear squares problem [7,15,19,22]. (1) is also termed as a Karsh-Kuhn-Tucker system, or an augmented system, or an equilibrium system [12–14].

Since matrix blocks A and B are large and sparse, (1) is suitable for being solved by the iterative methods. If $\mathrm{r} = n$, ie., the saddle point matrix A is non-singular, efficient iterative methods have been studied in many literatures, including Uzawa-type methods [6,3,8,10,11,23], preconditioned Krylov subspace iteration method [2,9], the generalized stationary iterative (GSI) method [21] and so on. If $\mathrm{r} < n$, (1) is a singular saddle point problem . Recently, several authors have presented semi-convergence analysis for solving the singular saddle point problem (1). Zheng et. al [25] studied semi-convergence of the PU (Parameterized Uzawa), Li et al. [18] examined semi-convergence of the GSSOR method , Zhang and Wang [17] studied semi-convergence of the GPIU method, Zhou and Zhang [20] stuided semi-convergence of the MGSSOR method and Miao [26] studied semi-convergence of GSI method. In this paper, the MGSI method for solving singular saddle point problem (1.1) is further investigated and semi-convergence conditions are proposed, Throughout this paper, for $A \in \Re^{n\times n}, B^T, \sigma(A)$ and $\rho(A)$ denote the transpose, the spectral set and the spectral radius of the matrix A, respectively. *In* is the identity matrix with order n. The remainder of this paper is organized as follows. In Section 2, we present the MGSI method for solving the singular saddle point problem (1), and give the corresponding semi-convergence analysis. in Section 3, the numerical experiments are performed to show the feasibility and effectiveness of the MGSI iteration method.

In the following, as one of these important iteration methods, the modified Generalized staionary iteration method be described:

Algorithm 11 (the MGSI Method) Let $A \in \Re^{m\times m}$ be a symmetric positive definite (SPD) matrix and $Q \in \Re^{n\times n}$ be a singular symmetric positive semi-definite (SPSD) approximate matrix of Schur complement $S = B^T A^{-1} B$ satisfying $N(Q) = N(S)$. or equivalently, $N(Q) = N(B)$, where$(\cdot)$ denotes the null space of matrix. Given the initial vectors $x^{(0)}, y^{(0)}$,and two nonzero relaxation parameters $\alpha, \beta = 0$. For $i = 1,2,\Lambda$, until the iteration sequence converges, compute

$$\begin{cases} \mathrm{x}^{(k+1)} = \left(1 - \dfrac{1}{\alpha}\right)x^{(k)} + \dfrac{1}{\alpha}\left(f - By^{(k)}\right), \\ y^{(k+1)} = Q^+\left\{Qy^{(k)} + \dfrac{1}{\alpha}B^T\left[\beta x^{(k+1)} + (1-\beta)x^{(k)}\right] - \dfrac{1}{\alpha}g\right]\right\}, \end{cases} \qquad (2)$$

where Q^+ is the Moore-Penrose inverse of matrix Q.

II. THE SEMI-CONVERGENCE OF THE MGSI METHOD

In this section, we analyze the convergence properties about the iteration scheme of MGSI method used for solving singular saddle-point problems (1).

The iteration scheme (2) can be induced from the splitting of matrix A :

$$A = M(\alpha,\beta) - N(\alpha,\beta),$$

Where

$$M(\alpha,\beta) = \begin{pmatrix} \alpha A & 0 \\ -\beta B^T & \alpha Q \end{pmatrix}, N(\alpha,\beta) = \begin{pmatrix} (\alpha-1)A & -B \\ (1-\beta)B^T & \alpha Q \end{pmatrix}. \qquad (3)$$

Then, it is easy to derive that the Moore-Penrose inverse of matrix $M(\alpha,\beta)$ has the form of

$$M(\alpha,\beta)^+ = \begin{pmatrix} \dfrac{1}{\alpha}A^{-1} & 0 \\ \dfrac{\beta}{\alpha^2}Q^+B^TA^{-1} & \dfrac{1}{\alpha}Q^+ \end{pmatrix}.$$

The iteration scheme (3) can be equivalently rewritten as

$$x_{k+1} = T(\alpha,\beta)x_k + M(\alpha,\beta)^+ b, \qquad (4)$$

where $T(\alpha,\beta) = M(\alpha,\beta)^+ N(\alpha,\beta)$.

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To study the convergence of iteration scheme (4),we have

Lemma 1 [4] Let $A=M-N$ be a proper splitting, which is provided that $N(A)=N(M)$ and $R(A)=R(M)$. Then the iteration scheme $x_{k+1}=Tx_k+M^{+}b$ $(k=0,1,2,\Lambda)$, with $T=M^{+}N$, converges to the solution $A^{+}b$ of linear systems $Ax=b$ for every initial vector x_0 if and only if the spectral radius of matrix T is less than 1,i,e., $\rho(T)<1$.

Lemma 2 Let matrices A and $M(\alpha,\beta)$ be defined in (1) and (3), respectively. Denote by Q be a singular SPSD approximate matrix of the Schur complement $S=B^TA^{-1}B$ satisfying $N(Q)=N(B)$. Then $A=M(\alpha,\beta)-N(\alpha,\beta)$ is a proper splitting.

Proof By Lemma 1, we only need to prove that the matrices A and $M(\alpha,\beta)$ be defined in (1) and (3), respectively, satisfy

$$N(A)=N(M(\alpha,\beta)) \text{ and } R(A)=R(M(\alpha,\beta)). \tag{5}$$

Firstly we prove $N(A)=N(M(\alpha,\beta))$. Let $(\xi_1^{\,T},\eta_1^{\,T})^T\in N(A)$, $\xi_1\in\Re^m,\mu_1\in\Re^n$, Then

$$\begin{cases} A\xi_1+B\eta_1=0,\\ B^T\xi_1=0. \end{cases} \tag{6}$$

Because of the nonsingularity of matrix A, we obtain $\xi_1=-A^{-1}B\eta_1$ from the first equation of (6) and substituting it into the second equation of (6) gives $B^TA^{-1}B\eta_1=0$, which means $\eta_1^{\,T}B^TA^{-1}B\eta_1=0$ or $(B\eta_1)^TA^{-1}(B\eta_1)=0$. Owing to the symmetric positive definiteness of A, we obtain $B\eta_1=0$. Taking it into the first equation of (6), we can get $\xi_1=0$. Hence, the null space of matrix A can be represented as

$$N(A)=\mathrm{span}(\Theta^T,\eta_1^{\,T})^T, with\ \Theta\in\Re^m,\eta_1\in N(B)\subset\Re^n. \tag{7}$$

Now, we consider the specific form of $N(M(\alpha,\beta))$, Let $(\xi_2^{\,T},\eta_2^{\,T})^T\in N(M(\alpha,\beta))$, $\xi_2\in\Re^m,\eta_2\in\Re^n$, then

$$\begin{cases} \alpha A\xi_2=0,\\ -\beta B^T\xi_2+\alpha Q\eta_2=0. \end{cases} \tag{8}$$

Since A is an SPD matrix, meanwhile $\alpha\neq 0$, we can obtain that $\xi_2=0$ and $Q\eta_2=0$ from (8).Using $N(Q)=N(B)$ gives $B\eta_2=0$, so

$$N(M(\alpha,\beta))=span(\Theta^T,\eta_2^{\,T})^T, with\ \Theta\in\Re^m,\eta_2\in N(B)\subset\Re^n. \tag{9}$$

From (7) and (9), the first equality of (7), i.e., $N(A)=N(M(\alpha,\beta))$ is obtained.

Inasmuch as $R(A)\oplus N(A^T)=\Re^{m+n}$ and $R(M(\alpha,\beta))\oplus N(M(\alpha,\beta)^T)=\Re^{m+n}$, the equality $R(A)=R(M(\alpha,\beta))$ is valid if we can obtain

$$N(A^T)=N(M(\alpha,\beta)^T). \tag{10}$$

Actually, the proof of (10) is totally similar to that of the first equality of (5), so we omit the detail. The proof is completed.

Lemma 3 Let $A\in\Re^{m\times m}$ be SPD matrix, $Q\in\Re^{n\times n}$ be a singular SPSD approximate matrix of the Schur complement $S=B^TA^{-1}B$ satisfying $N(Q)=N(S)$, or equivalently, $N(Q)=N(B)$. Then $\rho(T(\alpha,\beta))<1$ if the iteration parameters α and β satisfy

$$\alpha>\max\left(\frac{1}{2},\frac{\sqrt{\mu_{\max}}}{2}\right) \text{ and } 1-\frac{\alpha}{\mu_{\max}}<\beta<\frac{1}{2}+\frac{\alpha(2\alpha-1)}{\mu_{\max}}, \tag{11}$$

where $\mu_{\max}$ is the largest eigenvalues of the matrix respectively.

Proof After simple calculation, the iteration matrix of iteration scheme (4) can be written as

$$T(\alpha,\beta)=\begin{pmatrix} \left(1-\frac{1}{\alpha}\right)I_m & \frac{1}{\alpha}A^{-1}B\\ \frac{\alpha-\beta}{\alpha^2}Q^{+}B^T & I_n-\frac{\beta}{\alpha^2}Q^{+}B^TA^{-1}B \end{pmatrix}.$$

Let $B=U(B_r,0)V^*$ be the singular value decomposition of B, where $B_r=(\Sigma_r,0)\in\Re^{m\times r}$ with $\Sigma_r=diag(\sigma_1,\sigma_2,\Lambda,\sigma_r)$,$U,V$ are unitary matrices,Then

$$P=\begin{pmatrix} U & \\ & V \end{pmatrix},$$

is an $(m+n)\times(m+n)$ unitary matrix. Let $\hat{T}(\alpha,\beta)=P^*T(\alpha,\beta)P$. Then the matrix $T(\alpha,\beta)$ has the same eigenvalues with matrix $\hat{T}(\alpha,\beta)$. Here, we have used $(*)^*$ to denote the conjugate transpose of the corresponding complex matrix. Hence, we only need to demonstrate the semi-convergence of the matrix $\hat{T}(\alpha,\beta)$. Define matrices

$$\hat{A}=U^*AU,\hat{B}=U^*BV,\hat{Q}=V^*BV.$$

Then it holds that $\hat{B}=(B_r,0)$ and

$$\hat{Q}^{+}=V^*Q^{+}V=\begin{pmatrix} V_1^{\,*}Q^{+}V_1 & V_1^{\,*}Q^{+}V_2\\ V_2^{\,*}Q^{+}V_1 & V_2^{\,*}Q^{+}V_2 \end{pmatrix},$$

here we have partitioned the unitary matrix V into the block from $V=(V_1,V_2)$ confirmly to the partition of the matrix B. Through direct operations,we have

$$U^*A^{-1}BV=(U^*A^{-1}U)(U^*BV)=\hat{A}^{-1}(B_r,0)=(\hat{A}^{-1}B_r,0),$$

$$V^*Q^{+}B^*U=(V^*Q^{+}V)(V^*B^*U)=\begin{pmatrix} V_1^{\,*}Q^{+}V_1B_r^{\,T}\\ V_2^{\,*}Q^{+}V_1B_r^{\,T} \end{pmatrix},$$

and

$$\begin{aligned} V^*Q^{+}B^TA^{-1}BV &= (V^*Q^{+}V)(V^*B^TU)(U^*A^{-1}U)(U^*BV)\\ &= \hat{Q}^{+}(B_r0)^T\hat{A}^{-1}(B_r0)\\ &= \begin{pmatrix} V_1^{\,*}Q^{+}V_1B_r^{\,T}\hat{A}^{-1}B_r & 0\\ V_2^{\,*}Q^{+}V_1B_r^{\,T}\hat{A}^{-1}B_r & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \hat{T}(\alpha,\beta) &= P^*T(\alpha,\beta)P\\ &= \begin{pmatrix} \left(1-\frac{1}{\alpha}\right)I_m & \frac{1}{\alpha}U^*A^{-1}BV\\ \frac{\alpha-\beta}{\alpha^2}V^*Q^{+}B^*U & I_n-\frac{\beta}{\alpha^2}V^*Q^{+}B^TA^{-1}BV \end{pmatrix}\\ &= \begin{pmatrix} \hat{H}(\alpha,\beta) & 0\\ \hat{L}(\alpha,\beta) & I_{n-r} \end{pmatrix}, \end{aligned}$$

where

$$\hat{H}(\alpha,\beta)=\begin{pmatrix} \left(1-\frac{1}{\alpha}\right)I_m & \frac{1}{\alpha}\hat{A}^{-1}B_r\\ \frac{\alpha-\beta}{\alpha^2}V_2^{\,*}Q^{+}V_1B_r^{\,T} & I_r-\frac{\beta}{\alpha^2}V^{*}{}_1Q^{+}V_1B_r^{\,T}\hat{A}^{-1}B_r \end{pmatrix},$$

and

$$\hat{L}(\alpha,\beta)=\left(\frac{\alpha-\beta}{\alpha^2}V_2^*Q^+V_1B_r^T,\quad -\frac{\beta}{\alpha^2}V_2^*Q^+V_1B_r^T\hat{A}^{-1}B_r\right).$$

As $\hat{L}(\alpha,\beta)\neq 0$, from Lemma 2 [26] we know that the matrix $\hat{L}(\alpha,\beta)$ is semi-convergence if and only if $\rho(\hat{H}(\alpha,\beta))<1$. Hence, in what follows, we will give out the restrictions for the parameters α and β such that $\rho(\hat{H}(\alpha,\beta))<1$. Consider the following non-singular saddle point problem

$$\begin{pmatrix}\hat{A} & B_r\\ -B_r^T & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=\begin{pmatrix}\hat{b}\\ -\hat{q}\end{pmatrix}. \tag{12}$$

If the coefficient matrix of the non-singular saddle point problem (12) be splitted as

$$\begin{pmatrix}\hat{A} & B_r\\ -B_r^T & 0\end{pmatrix}=\begin{pmatrix}\alpha\hat{A} & 0\\ -\beta B_r^T & \alpha\hat{Q}_1\end{pmatrix}-\begin{pmatrix}(\alpha-1)\hat{A} & -B_r\\ (1-\beta)B_r^T & \alpha\hat{Q}_1\end{pmatrix},$$

with the preconditioning matrix $\hat{Q}_1=(V_1^*Q^+V_1)^+$, then the MGSI method for solving (12) can be well defined, and the iteration matrix is $\hat{H}(\alpha,\beta)$. From Theorem 2.4 in [21], we know that $\rho(\hat{H}(\alpha,\beta))<1$ if α and β satisfies

$$\alpha>\max\left(\frac{1}{2},\frac{\sqrt{\mu_{\max}}}{2}\right),\qquad \text{and}\qquad 1-\frac{\alpha}{\mu_{\max}}<\beta<\frac{1}{2}+\frac{\alpha(2\alpha-1)}{\mu_{\max}}.$$

here $\mu_{\max}$ is the largest eigenvalue of the matrix $Q_1^+B_r^T\hat{A}^{-1}B_r$. By (7), we can see that $\mu_{\max}$ is also the largest eigenvalue of the matrix $Q^+B^TA^{-1}B$.

Using Lemmas 1,2 and 3, we finally obtain the following convergence properties of the MGSI method.

Theorem 1 Let $A\in\Re^{m\times m}$ be SPD matrix, $Q\in\Re^{n\times n}$ be a singular SPSD approximate matrix of the Schur complement $S=B^TA^{-1}B$ satisfying $N(Q)=N(B)$. Then, the iteration sequence produced by the MGSI method converges to the solution A^+b of the singular saddle point problem (1), if parameters α and β satisfies

$$\alpha>\max\left(\frac{1}{2},\frac{\sqrt{\mu_{\max}}}{2}\right)\ \text{and}\ 1-\frac{\alpha}{\mu_{\max}}<\beta<\frac{1}{2}+\frac{\alpha(2\alpha-1)}{\mu_{\max}}, \tag{13}$$

where $\mu_{\max}$ is the largest eigenvalues of the matrix $Q^+B^TA^{-1}B$, respectively.

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