An efficient approximation method for the nonhomogeneous backward heat conduction problems

Jin Wen, Xiufen Cheng

Abstract— **This paper presents a new improved meshless numerical scheme to solve the nonhomogeneous backward heat conduction problems. Fundamental solutions of heat equations and radial basis functions of both and are employed to obtain a numerical ill-conditioned, the Tikhonov regularization method is adopted to solve the resulting system of linear equations. Therefore, generalized cross-validation (GCV) method is used to choose a regularization parameter. The accuracy and efficiency of our proposed method is illustrated by several typical 1-D and 2-D numerical examples.**

Index Terms— **nonhomogeneous backward heat conduction problems, Fundamental solutions,radial basis functions.**

I. INTRODUCTION

In many real application areas, it is sometimes impossible to specify the boundary conditions or the initial temperature while dealing with a heat conductor. For instance, in practice, one may have to investigate the temperature distribution and the heat flux history from the known data at a particular time. In other words, it may be possible to specify the temperature distribution at a particular time, say $t = t_f > 0$, and either

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temperature u or the heat flux $\frac{\partial u}{\partial x}$ on the boundary of the domain, and from this data the question arises as to whether the temperature distribution at any earlier time $t < t_f$ can be obtained. This is usually referred to as the backward heat conduction problem (BHCP), or the final boundary value problem, which is compared to initial boundary value problem.

In this paper, an improved approximation method for solving nonhomogeneous backward heat conduction problem (NBHCP) is investigated. The determination of the unknown initial temperature from observable scattered final temperature data is required in many practical situations. However, the solution process for BHCP is in nature "unstable", because the unknown solutions or parameters have to be determined from indirect observable data which contain measurement error. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix. Another difficulty to establish any numerical solution for the BHCP is due to the nature of its physical phenomena. Although heat conduction process is very smooth, the process is irreversible. In other words, the characteristic of the solution (for instance, the shape of the interior heat flow) is not affected by the observed final data.

For the NBHCP, we can refer to [1, 2]. In article [2], Trong et. al. Improved the quasi-boundary value method to regularize the nonlinear 1-D backward heat problem and to obtain some error estimates, and the approximation solution is calculated by the contraction principle. In [1], Dang and Nguyen investigated the inverse time problem for the nonhomogeneous heat equation, and they regularized this problem by the quasi-reversibility method and then obtained error estimates on the approximate solutions. Solutions were also calculated by the contraction principle.

The MFS was first introduced by Kupradze and Aleksidze [3] and it approximates the solution of the problem by linear combination of fundamental solutions of the governing differential operator. Therefore, it is an inherently meshless, integration free technique for solving partial differential equations and it has been used extensively for solving various types of partial differential equations. For instance, the solutions for potential problems by Mathon and Johnston [4], the exterior Dirichlet problem in acoustics by Kress and Mohsen [5], the biharmonic problems by Karageorghis and Fairweather [6]. More recently, the MFS has successfully been applied to approximate the solutions of nonhomogenous problems [7]. The details can be found in [8, 9]. These problems are all well-posed direct problems. In 2004, Hon and Wei [10] applied firstly the MFS to solve the inverse heat conduction problem (IHCP). Following their works, many researchers applied this method to solve many inverse problems: the BHCP [11]; the Cauchy problem associated with the Navier system in linear elasticity [12] and Helmholtz-type equations [13, 14], etc. As the authors mentioned in [15], the meshless methods require neither domain discretization as in the finite element method (FEM) and finite difference method (FDM) nor boundary discretization in the boundary element method (BEM), thus they improve the computational efficiency

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considerably, which makes them especially attractive for problems with complicated geometry in high dimensional cases. One possible disadvantage is the fact that the resulting system of linear equation is always ill-conditioned, even for a well-posed problem, see [9]. Therefore, special regularization methods are required in order to solve this system of algebraic equations.

The MFS can only solve some special kinds of nonhomogeneous heat equations, such as $f(t), f(x), e^{-\lambda t} f(x)$ and one can refer to [16, 17, 18, 19, 20, 21, 22] for more details.

Recently, some authors adopt the method of particular solution to solve various types of partial differential equations, such as Helmholtz-type operators [23], Poisson's equation [24], Laplace and biharmonic operators [25], elliptic partial differential equations [26], nonhomogeneous Cauchy problem of heat equation [27] and NBHCP [28], etc.

More recently, in [29], the authors proposed a new improved meshless method for solving inverse heat source problem. They combined the MFS and RBF methods to give the numerical solution about the inverse source problem. Motivated by their idea, we use the method to solve NBHCP. Because the homogenization of the nonhomogeneous heat equation in our paper is very difficult, and even impossible. So our proposed method can avoid the difficulty of homogenization.

The rest of this paper is organized as follows: in Section 2, we give the mathematical formulation of the proposed problem. Section 3 devotes to the existence and uniqueness of the proposed problem. Section 4 is devoted to solve the problem by using the method of fundamental solution-radial basis function (MFS-RBF). Tikhonov regularization method is given in Section 5. Section 6 presents several numerical experiments to verify the efficiency and accuracy of our method. And finally, Section 7 concludes.

II. Formulation of the Proposed Problem

Consider the following nonhomogeneous backward heat conduction problem:

$$u_t(x,t) = u_{xx}(x,t) + f(x,t), (x,t) \in (0,L)\times(0,t_f)\ , \tag{2.1}$$

with boundary conditions:

$$u(0,t) = g_1(t), 0 < t < t_f\ , \tag{2.2}$$

$$u(1,t) = g_2(t), 0 < t < t_f\ , \tag{2.3}$$

and final temperature distribution:

$$u(x,t_f) = u_f(x), 0 < x < L\ , \tag{2.4}$$

and the initial value

$$u(x,0) = u_0(x), 0 < x < L\ , \tag{2.5}$$

is to be determined.

In [11], for the special case when $f(x,t) \equiv 0$, Mera gave the MFS to solve the problem above. However, it is also a challenge to solve this problem by only MFS. Here, according to the method of [29], we combine the method of fundamental solutions and radial basis functions as an improved meshless method to solve it.

Remark 2.1. In this paper, our method can be recognized as the generalization of Mera's method. Meanwhile, we can give the comparison between our method and Mera's in Example 4 in Section 6.

III. Existence and Uniqueness

According to the definition of well-posedness given by Hadamard[30], our problem is ill-posed, since the backward heat conduction problem is unstable. However, the existence and uniqueness of NBHCP are still satisfied. Here, we give the results about existence and uniqueness of our problem as follows:

Theorem 3.1. *The solution to the NBHCP (2.1)-(2.5) is unique.*

Proof. If u_1 and u_2 are two different solutions to the problem (2.1)-(2.5), respectively. We can give a new function $\tilde{u} = u_1 - u_2$, and we can easily observe that $\tilde{u}$ satisfies the following homogeneous BHCP:

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t)\ , \tag{3.1}$$

with boundary conditions:

$$\tilde{u}(0,t) = 0, 0 < t < t_f\ , \tag{3.2}$$

$$\tilde{u}(1,t) = 0, 0 < t < t_f\ , \tag{3.3}$$

and final temperature distribution:

$$\tilde{u}(x,t_f) = 0, 0 < x < 1\ , \tag{3.4}$$

According to the classical homogeneous BHCP, such as the reference [31], the above problem (3.1)-(3.4) has a unique solution $\tilde{u} = 0$.

Therefore, our proposed problem (2.1)-(2.5) has a unique solution.

Remark 3.2. The above result can be also generalized to 2-D NBHCP.

IV. Method of Fundamental Solutions-Radial Basis Functions

When $f(x,t) \equiv 0$, Eq. (2.1) can be changed as the following

$$u_t(x,t) = u_{xx}(x,t)\ , \tag{4.1}$$

The fundamental solution for (4.1) is

$$F(x,t) = \frac{H(t)}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}\ , \tag{4.2}$$

where $H(t)$ is the Heaviside function. Assuming that $T > t_f$ is a constant, the following function

$$\phi(x,t) = F(x,t+T)\ , \tag{4.3}$$

is also a non-singular solution of (4.1) in the domain $[0,1]\times[0,t_f]$ According to the paper [32], Dou and Hon proposed a new method of source points has been presented for solving a backward time-fractional diffusion equation (BTFDE). In this paper, we also choose the source points $\Gamma = \{(x_i^*, t_i^*) : i = 1,2,\cdots,2M\}$ as follows:

$$\Gamma = \{(x_i^*, t_i^*) : -R < x < 1+R, t_i^* = -\delta t, i = 1,2,\cdots,2M\} \tag{4.4}$$

Where R and δt are fixed constants. Moreover, the collocation points $C = C_1 \cup C_2 \cup C_3 \cup C_4$ are chosen as follows:

$$C_1 = \{(x_q, t_p) : 0 \le x_q \le L, 0 \le t_p \le t_f, q = 1,\cdots,n, p = 1,\cdots,m\} \tag{4.5}$$

$$C_2=\{(x_j,t_j): x_j=0, 0\le t_j\le t_f, j=1,\cdots,n\}, \quad (4.6)$$

$$C_3=\{(x_k,t_k): x_k=L, 0\le t_k\le t_f, k=n+1,\cdots,n+m\} \quad (4.7)$$

$$C_4=\{(x_l,t_l): 0\le x_l\le L, t_l=t_f, l=n+m+1,\cdots,n+m+s\} \quad (4.8)$$

This section introduces the numerical scheme for solving the problem (2.1)−(2.5) using the hybrid meshless method, i.e., fundamental solutions and radial basis functions method. The purpose is to decide how radial basis functions and the fundamental solutions are applied to approximate the initial temperature of the nonhomogeneous backward heat conduction problem. The choice of the location of source points greatly affect the numerical results. So, in this paper, we choose the source points outside the domain $[0,L]\times[0,t_f]$. We suppose that the solution of (2.1)−(2.5) can be written as follows:

$$\tilde{u}(x,t)=\sum_{k=1}^{M}\lambda_k\phi_k(x,t)+\sum_{k=1+M}^{2M}\lambda_k\psi_k(x,t), \quad (4.9)$$

where $\phi_k(x,t)=f(x-x_k^*, t-t_k^*+T)$ in which $F(x,t)$ is the fundamental solution of the heat equation that is described in (4.2), and $\psi_k(x,t)=\psi(\|(x,t)-(x_k,t_k)\|_2)$ is a radial function that is defined in the Table 1, which can be found in Section 3.2 of [29]. We impose the approximate solution u^* to satisfy the given partial differential equation with the other conditions at any point $(x,t)\in C$ to rewrite the coefficients in (4.9) as follows:

$$A\lambda=b, \quad (4.10)$$

Where $\lambda=[\lambda_1,\lambda_2,\cdots,\lambda_{2M}]^T$, $b=[f,g_1,g_2,u_f]^T$ .Also, the $(n\times m+n+m+s)\times 2M$ matrix A can be subdivided into eight submatrices as follows:

$$A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\\ A_{31} & A_{32}\\ A_{41} & A_{42}\end{pmatrix}, \quad (4.11)$$

Where

$$A_{11}=\left[a_{lk}^{11}\right],$$

$$a_{lk}^{11}=0, l=1,\cdots,n\times m, k=1,\cdots,M; \quad (4.12)$$

$$A_{12}=\left[a_{lk}^{12}\right],$$

$$a_{lk}^{12}=\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\psi_k(x_l,t_l), l=1,\cdots,n\times m, k=M+1,\cdots,2M; \quad (4.13)$$

$$A_{21}=\left[a_{lk}^{21}\right],$$

$$a_{lk}^{21}=\phi_k(x_l,t_l), l=n\times m+1,\cdots,n\times m+n, k=1,\cdots,M; \quad (4.14)$$

$$A_{22}=\left[a_{lk}^{22}\right],$$

$$a_{lk}^{22}=\psi_k(x_l,t_l), l=n\times m+1,\cdots,n\times m+n, k=M+1,\cdots,2M; \quad (4.15)$$

$$A_{31}=\left[a_{lk}^{31}\right],$$

$$a_{lk}^{21}=\phi_k(x_l,t_l), l=n\times m+n+1,\cdots,n\times m+n+m, k=1,\cdots,M; \quad (4.16)$$

$$A_{32}=\left[a_{lk}^{32}\right],$$

$$a_{lk}^{32}=\psi_k(x_l,t_l), l=n\times m+n+1,\cdots,n\times m+n+m, k=M+1,\cdots,2M; \quad (4.17)$$

$$A_{41}=\left[a_{lk}^{41}\right],$$

$$a_{lk}^{41}=\phi_k(x_l,t_l), l=n\times m+n+m+1,\cdots,n\times m+n+m+s, k=1,\cdots,M; \quad (4.18)$$

$$A_{42}=\left[a_{lk}^{42}\right],$$

$$a_{lk}^{42}=\psi_k(x_l,t_l), l=n\times m+n+m+1,\cdots,n\times m+n+m+s, k=M=1,\cdots,2M; \quad (4.19)$$

V. Tikhonov Regularization

Because the NBHCP (2.1) to (2.5) is highly ill-posed, the matrix A in Eq.(4.10) is still ill-conditioned. That is to say most standard numerical methods cannot obtain good accuracy in solving the matrix equation (4.10) because of the bad condition number of the matrix A. As a matter of fact, the condition number of matrix A increases remarkably with reference to the total points. Several regularization methods have been developed for solving these illconditioned problems [33, 34].

In fact, the measurement data f, g_1, g_2 and u_f are usually contaminated by inherent measurement errors which leads to a noisy vector b^δ, and it can be given as $b_i^\delta=b_i(1+\delta(2rand(i)-1))$, here $rand(i)$ is a random number in [0, 1]. For this case we use the Tikhonov regularization technique [35] to solve the matrix equation (4.10) with noisy b^δ, i.e. solving the following minimization problem

$$\min_\lambda\{\|A\lambda-b^\delta\|^2+\alpha^2\|\lambda\|^2\}, \quad (5.1)$$

where $\|\cdot\|$ denotes the Euclidean norm and α is called a regularization parameter.

The choice of a suitable regularization parameter α is crucial to the accuracy of solution and is still under intensive research [35]. For the above problem, one can utilize the generalized cross-validation (GCV) criterion and L-curve method to choose the regularization parameter α. These two methods are both popular and successful [36]. We use generalized cross-validation (GCV) criterion to choose the regularization parameter α. The GCV criterion is a very popular and successful method for choosing the regularization parameter [36]. The GCV method determines the optimal regularization parameter by minimizing the following GCV function:

$$G(\alpha)=\frac{\|A\lambda^\alpha-b^\delta\|^2}{(trace(I-AA^I))^2}, \quad (5.2)$$

Where $A^I=(A^{tr}A+\alpha^2 I)^{-1}A^{tr}$ is a matrix which produces the regularized solution λ^{α^*} when multiplied with the right hand side b^δ, i.e., $\lambda^{\alpha^*}=A^I b^\delta$.

In the computation, we use the Matlab code developed by Hansen [37] for solving(4.10). Denote the regularized solution to (4.10) by λ^{α^*}. The approximating solution u^* to problem (2.1)–(2.5) is then given as

$$u^*(x,t)=\sum_{k=1}^{k=M}\lambda_k^{\alpha^*}\phi_k(x,t)+\sum_{k=1+M}^{k=2M}\lambda_k^{\alpha^*}\psi_k(x,t) \quad (5.3)$$

So, the approximate initial temperature is obtain as follows:

$$u_0^*(x) = \sum_{k=1}^{k=M} \lambda_k^{\alpha^*} \phi_k(x,0) + \sum_{k=1+M}^{k=2M} \lambda_k^{\alpha^*} \psi_k(x,0) \qquad (5.4)$$

VI. NUMERICAL VERIFICATION

In this section we test six examples to demonstrate the feasibility of our approach. The computations are performed using MATLAB 6.5 and we take $L=1$ and $t_f = 0.25$ or 1 without special specification. For the RBF, we choose Gaussian RBF in the following examples.

In order to check the accuracy of numerical computations, we compute the root mean square error (*RMSE*) and the relative root mean squares error (*RRMSE*) of $u_0(x)$ defined by

$$RMSE(u_0) = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(u_0^*(\bar{x}_i) - u_0(\bar{x}_i))^2}, \qquad (6.1)$$

$$RRMSE(u_0) = \frac{\sqrt{\sum_{i=1}^{N_t}(u_0^*(\bar{x}_i) - u_0(\bar{x}_i))^2}}{\sqrt{\sum_{i=1}^{N_t}(u_0(\bar{x}_i))^2}}, \qquad (6.2)$$

where $\{\bar{x}_i\}$ is a set of discrete points in the interval [0, 1].

Example 1: Take the exact solution to problem (2.1)–(2.5) as

$$u(x,t) = \cos x \exp(t), \qquad (6.3)$$

$$u(x,0) = \cos x, \qquad (6.4)$$

And

$$f(x,t) = 2\cos x \exp(t), \qquad (6.5)$$

Here, we choose $t_f = 0.25$. In Table 2 we show the *RRMSE*(*u*0) with various distance parameters R and δt when we fix $T=2, \delta = 0.01$. We can observe that the numerical results are stable with respect to parameters R and δt within a wide range. Thus, for our proposed method, the accuracy of approximate solution is relatively independent of the parameters R and δt. Fig. 1 shows a comparison between the exact solution of $u_0(x)$ and the approximate solutions $u_0^*(x)$ of $u_0(x)$ for $R=5, \delta t = 0.25, T=10, C=0.1, n=m=s=30,$ $M=100$ and various levels of noise added into the data using MFS-RBF. It is observed that as the noise level increases, the approximated function has an acceptable accuracy. Fig. 2 illustrates the relationship between *RMSE*(*u*0) (*RRMSE*(*u*0)) and the parameter *T*, and it can be seen that we can obtain better accuracy when $T>10$.

Fig. 1 displays the comparison between exact and approximate solutions when we choose $R=5, \delta t = 0.25, T=10, C=0.1, n=m=s=30, M=100$ for Example 1. It can be observed from this figure that our method works very well even for $\delta = 0.1$.

Table 1. Some well-known radial basis functions.

Infinitely smooth RBFs	$\psi(r)$
Gaussian	e^{-cr^2}
Multiquadrics (MQ)	$(\sqrt{c^2+r^2})^{-1}$
Inverse multiquadrics (IMQ)	$\sqrt{c^2+r^2}$
Inverse quadric (IQ)	$(c^2+r^2)^{-1}$

Table 2. The values of $RRMSE(u_0)$ for various values of R and δt in Example 1 when $\delta = 0.01, C=0.1, T=2$.

δt \ R	R=0	R=1	R=2	R=3	R=4	R=5
0.1	0.0503	0.0043	0.0218	0.0382	0.0741	0.0963
1.1	0.0897	0.1029	0.1247	0.1216	0.1309	0.1480
2.1	0.0525	0.0923	0.0627	0.0433	0.0352	0.0232
3.1	0.1050	0.1112	0.0842	0.0604	0.0365	0.1167
4.1	0.1372	0.1089	0.0908	0.0759	0.0545	0.0412
5.1	0.0405	0.0440	0.0832	0.0619	0.1170	0.1031

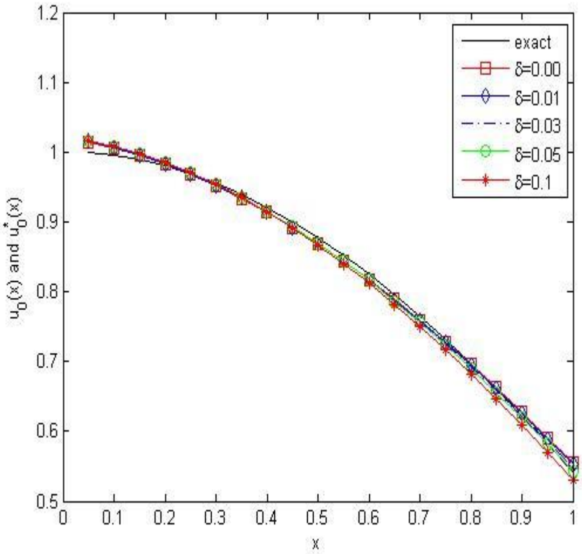


Figure 1. The relationship between exact and approximate solutions

where $R=5, \delta t = 0.25, T=10, C=0.1, n=m=s=30, M=100$ for Example 1. CPUtime=2.018224s.

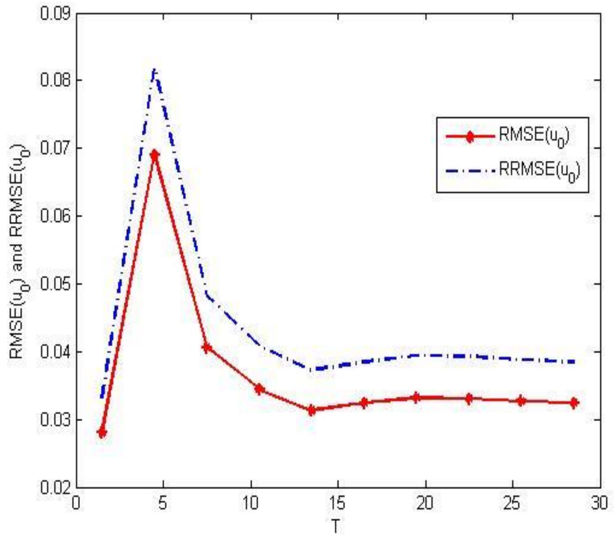


Figure 2. The relationship between *RRMSE*(u0) and T with R= 5, $\delta t = 0.1$, $C = 0.1$, $n = m = s = 30$ and $M = 60$ for Example 1. CPUtime=2.573632s.

Example 2: Consider the exact solution as follows:

$$u(x,t) = x^3 + 2t^2, \quad (6.6)$$

$$u(x,0) = x^3, \quad (6.7)$$

and $f(x,t) = 4t - 6x$. (6.8)

In this example, we fix $t_f = 0.25$ and our numerical results are displayed in Fig. 3 when we choose the same parameters as Fig. 3 except for $R = 2$. It is clear that the results about initial values are stable. The relationships between the accuracy and the parameter R is illustrated in Fig. 4. We can see from this figure that the accuracy of numerical solutions does not change too much as the parameter R increases up to 30. Thus, it is concluded that in order to obtain higher numerical accuracy, we can choose parameter R from this figure. Fig. 5 illustrates the relationships between the accuracy and the parameter M . It is observed that the accuracy of numerical results is relatively stable when the parameter M becomes larger and larger.

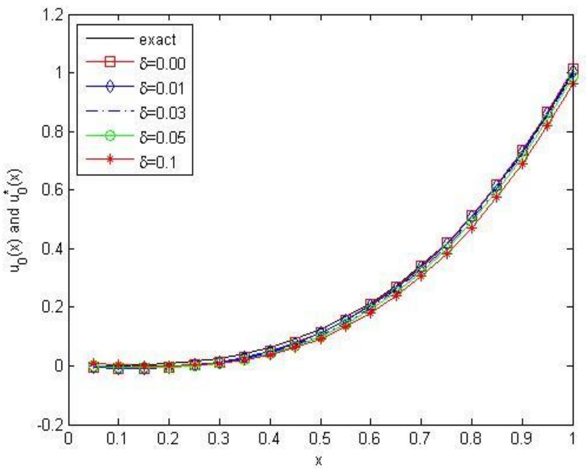


Figure 3. The relationship between exact and approximate solutions,

When $R = 2, \delta t = 0.25, T = 10, C = 0.1, n = m = s = 30, M = 100$ for Example 2. CPUtime=2.048593s.

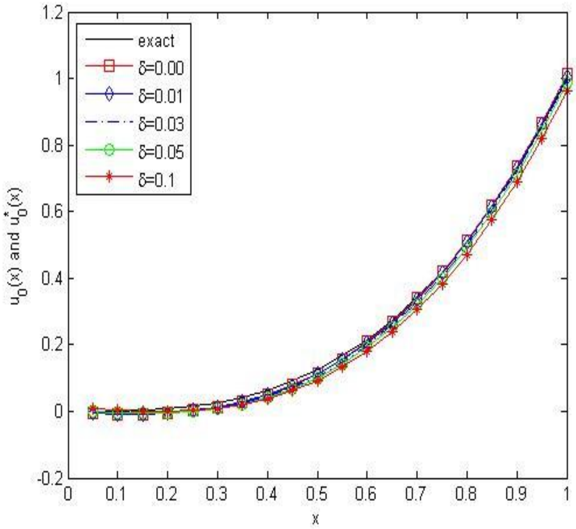


Figure 4. The relationship between *RRMSE*(u0) and R with $\delta t = 0.1, T = 5, C = 0.1, n = m = s = 30, M = 30 and \delta = 0.01$ for Example 2. CPUtime=2.048661s.

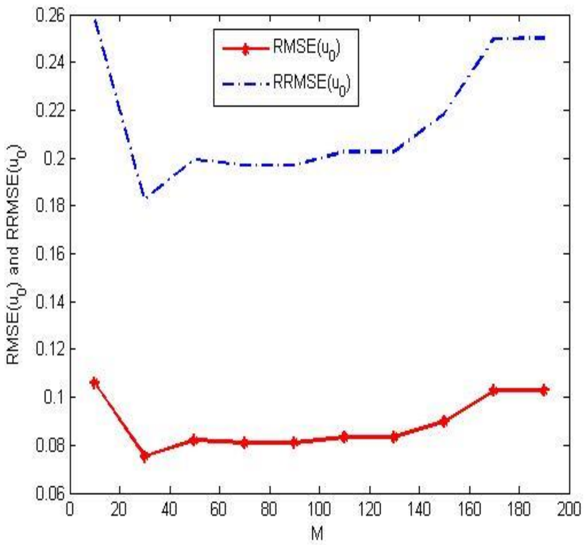


Figure 5. The relationship between *RRMSE*(u0) and M with $R = 2, \delta t = 0.25, T = 10, C = 5, n = m = s = 30, and \delta = 0.01$ for Example 2. CPUtime=4.251142s.

Example 3: We give a nonsmooth example as follows:

$$u(x,t) = |x^3 - 0.125| + t^3, \quad (6.9)$$

$$u(x,0) = |x^3 - 0.125|, \quad (6.10)$$

and $$f(x,t) = \begin{cases} 3t^2 + 6x, 0 < x < 0.5. \\ 3t^2 - 6x, 0.5 < x < 1. \end{cases} \quad (6.11)$$

In this example, we choose $t_f = 1$, and we can see that the heat source is more complicated than the previous ones, since it is nonsmooth at the point $x = 0.5$. As shown in Fig. 6, our method solves the model problem is quite satisfactorily even when the noise added to the data is up to 0.1. From this figure, we can see again that our method works well in computing the initial temperature.

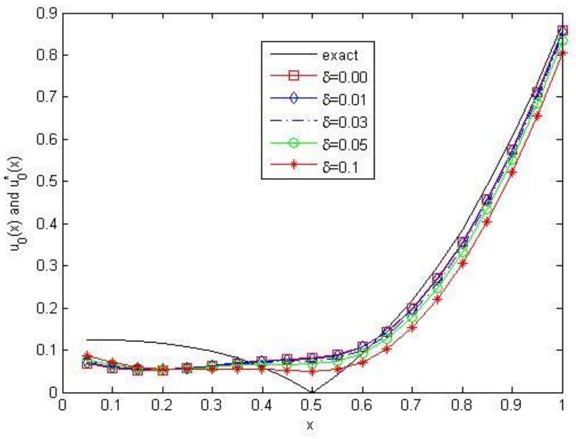


Figure 6. The exact and approximate solutions where R = 2, δt =0.5,C = 0.5, n =m = s = 30 and M = 150 for Example3 .CPUtime=3.198621s.

Example 4:(Compared with Mera's method) The exact solution is as follows [11]:

$$u(x,t) = \sin \pi x \exp(-\pi^2 t)\,, \qquad (6.12)$$

and the initial value is

$$u(x,0) = \sin(\pi x)\,, \qquad (6.13)$$

Here, we choose $t_f = 0.25$ which is the same as [11].

Fig. 7 shows the exact and approximated solutions by MFS-RBF and MFS respectively, when we choose $R = 20, \delta t = 0.25, T = 2, C = 0.5, n = m = s = 60, M = 30$ .It can be observed that the accuracy of our proposed method is better than the Mera's MFS method, while the CPUtime is near to each other.

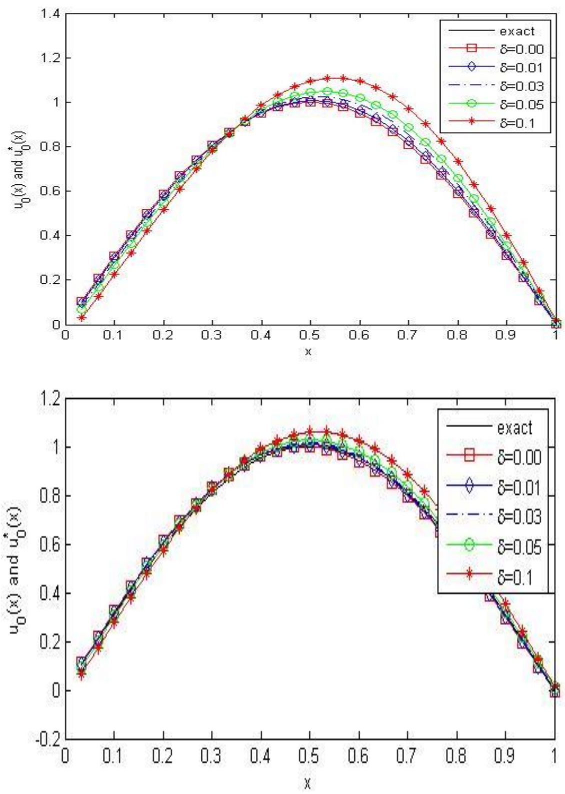


Figure 7. The exact and approximated solutions by MFS-RBF and MFS respectively for Example 4 when R = 20, δt = 0.25, C = 0.5, T =2, n = m = s = 60 and M = 30. CPUtime=6.425021 and 6.404281.

In the following example, we introduce a more complicated case to verify our proposed method.

Example 5:

$$u(x,0) = \begin{cases} 2x, 0 < x < 0.5. \\ -2x + 2, 0.5 < x < 1. \end{cases} \qquad (6.14)$$

In this example, we choose $t_f = 1$. We solve this direct problem by Crank-Nicolson scheme. The numerical results for $u(x,t)$ and $u_f(x)$ are shown in Fig.8. Fig. 9 displays the comparison between the exact solution and the numerical results. From this figure, we can see that the numerical approximation is not as good as in the previous examples, but it is in reasonable agreement with (6.14).

In order to show that our method is also used to solve 2-D NBHCP, we give the following example:

Example 6:

$$u(x,y,t) = \exp(t)(\sin x + \cos y)\,. \qquad (6.15)$$

Figs. 10(a) and (b) display the exact solution and approximate solution when $\delta = 0.01$. It can be observed that the accuracy of the proposed method is stable when we adopt the noisy data to solve this problem. The difference between exact and approximate solutions is illustrated in Fig. 10(c) when we added the noise level $\delta = 0.01$. We can see from this figure that our method is also useful for solving 2-D NBHCP.

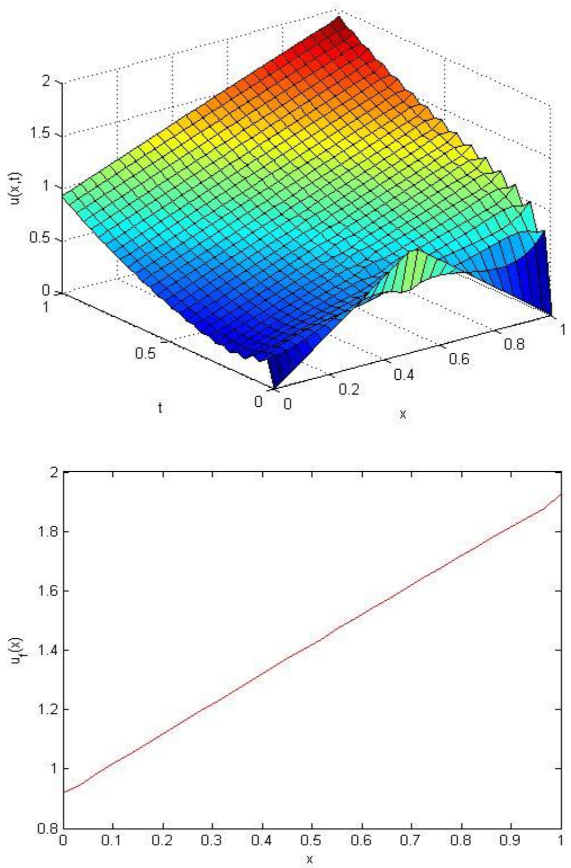


Figure 8. The function $u(x,t)$ and $u_f(x)$ obtained from the direct problem for Example 5.

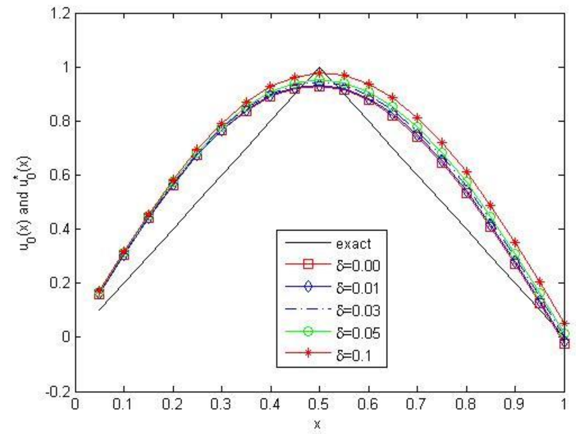
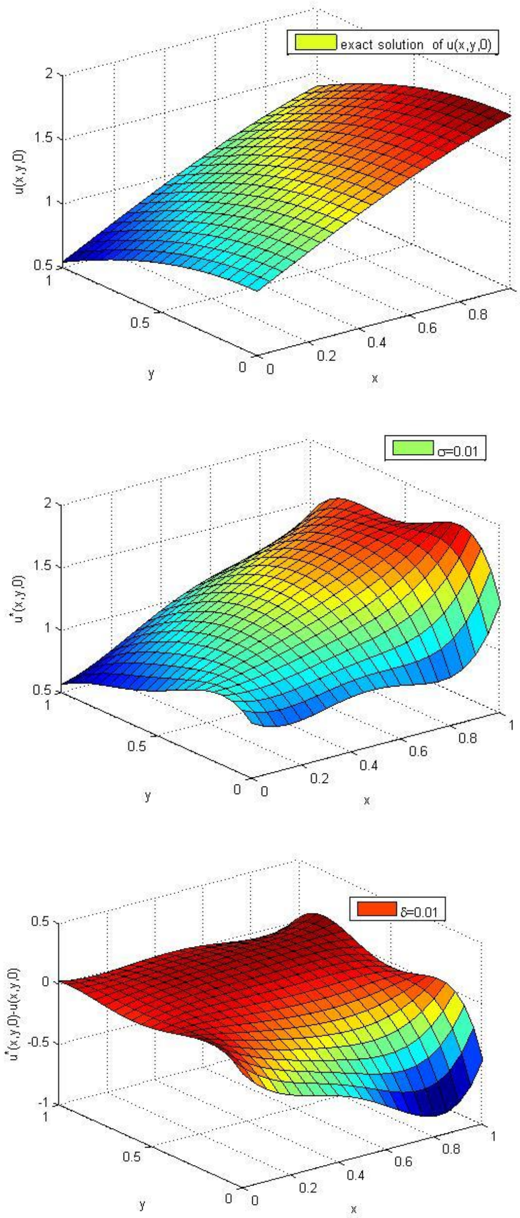


Figure 9. The exact and approximate solutions where R = 3, δt = 1.4, $n = m = s$ = 33, C = 0.5, T = 3 and M = 20 for Example 5.



(a)

(b)

(c)

Figure 10. (a) Exact solution of $u(x, y, 0)$; (b) The approximate
solution of $u(x, y, 0)$ with R = 2, T = 2, C = 0.2, and δt = 0.5 where δ = 0.01 ; (c) The difference between exact and approximate
solutions where R = 2, T = 2, C = 0.2, and δt = 0.5 for Example 6 when δ = 0.01.

VII. Conclusion

This paper is devoted to a new approach based on the fundamental solutions of the heat equation and the radial basis functions with the Tikhonov regularization method to solve the nonhomogeneous backward heat conduction problem. The numerical examples for various initial values are investigated. The numerical results show that our proposed method is reasonable, feasible and stable even for the 2-D case. For our work in the future, we will use this powerful method to solve more complicated problems, such as inverse boundary problem, inverse source problem, and so on.

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