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On preconditioned SSOR methods for the linear complementarity problem

Dan Zhang

***Abstract*— In this paper, we consider the preconditioned iterative methods for solving the linear complementarity problem associated with an M-matrix. Two preconditioned SSOR methods for solving the linear complementarity problem are proposed. The convergence of the proposed methods are analyzed, and the comparison results are derived. The comparison results show that the proposed preconditioned SSOR methods accelerate the convergent rate of the SSOR method. Numerical experiments verify the theory results.**

***Index Terms*— Linear complementarity problems, SSOR method, Preconditioner, Comparison theorem, M-matrix.**

I. INTRODUCTION

For a given matrix $A \in R^{n \times n}$ and a given vector $f \in R^n$, the linear complementarity problem, abbreviated as LCP, consists of finding a vector $x \in R^n$ such that

$$x \geq 0, \quad r = Ax - f \geq 0, \quad x^T r = 0. \qquad (1.1)$$

Here, the notation "$\geq$" denotes the component wise defined partial ordering between two vectors, and the superscript T denotes the transpose of a given vector.

The LCP of the form (1.1) arising in many scientific computing and engineering applications, for example,

contact problems with friction, free boundary value problems of fluid mechanics, the solution of optimization and behavioral models in biology and molecular biology, see [5, 6, 9]. The LCP (1.1) possesses a unique solution if and only if $A \in R^{n \times n}$ is a P-matrix, namely, a matrix whose all principal submatrices have positive determinants, see [5,6,17]. A positive diagonal M-matrix (see Section 2) is a P-matrix, and the LCP (1.1) with an M-matrix has the unique solution [4].

Numerical methods for LCP (1.1) have attracted much attentions.There are three main classes of iterative methods for the solution of the LCP (1.1): the projected methods [11, 12,16], the modulus algorithms [13] and the modulus-based matrix splitting iterative methods [3,7,21,22], see [12] for a survey of the solvers for LCP (1.1). We pay our attention in the present work to the SSOR method [8], which is a special projected method, for solving the LCP (1.1) with an M-matrix. For accelerating the convergent rate of the SSOR method [8], preconditioning techniques is often used [5, 20]. Preconditioning techniques for solving the large sparse linear algebraic equations $Ay = b$ have been investigated in depth, a number of preconditioners for the classical iterative methods were proposed [10, 14, 19]. In [10], the preconditioner

$$P_1 = I + S_1 = \begin{bmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

is proposed for accelerating the convergence rate of classical iterative method for the linear system with L-matrices. The preconditioner P_1 is generalized in [14] as

$$P_2 = I + S_1(\alpha) = \begin{bmatrix} 1 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1,n} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $\alpha_1, \alpha_2 \cdots \alpha_{n-1}$ are real constants, for accelerating the convergent rate of
the Gauss-Seidel method for the linear system with an M-matrix. To provide the
preconditioning effect on the last row and based on the preconditioner P_1, Niki etal. [19] proposed the preconditioner

$$P_1 = I + S_1 = \begin{bmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ -a_{n,1} & 0 & 0 & \cdots & 1 \end{bmatrix},$$

Following the same idea and based on the preconditioner P_2, we can propose the
preconditioner

$$P = (p_{ij}) = I + S(\alpha) = \begin{bmatrix} 1 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n} \\ \alpha_n a_{n,1} & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad (1.2)$$

with positive constants $\alpha_i (i = 1,2, \cdots n)$

In this paper, the preconditioner P in (1.2) is used to accelerate the convergent rate of the SSOR method [8] for

Dan Zhang, College of Mathematics and Statistics , Northwest Normal University, Lanzhou 730070,China

solving the LCP of the form (1.1). Two precondi-tioned SSOR methods are proposed, and its convergence are studied. The remainder of the paper are organized as follows.

In Section 2, some preliminaries are given. The projected method for solving LCP is recalled, and two preconditioned SSOR methods are proposed. In Section 3, the convergence of the preconditioned SSOR methods are studied. The comparison results about the convergent rates between the proposed preconditioned SSOR methods with the SSOR method [8] for LCP (1.1) with an M-matrix are given in Section 4. Numerical examples are given to demonstrate our theoretical results in Section 5. Finally, a brief conclusion is drawn in Section 6.

II. Preliminaries

Let us firstly summarize some notations. In reference to R^n and $R^{n\times n}$, the relation $\geq$ denotes partial ordering. In addition, for $x, y \in R^n$ we write $x > y$ (or $x \geq y$) if $x_i > y_i$ (or $x_i \geq y_i$) hold for $i = 1, \Lambda, n$. A nonsingular matrix $A = (a_{ij}) \in R^{n\times n}$ is termed an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. It's comparison matrix $\langle A \rangle = a_{ij}$ is defined by $a_{ii} = |a_{ii}|$, $a_{ij} = -|a_{ij}|$ (i $\neq$ j) for $i, j = 1, \Lambda, n$. A is said to be an H-matrix if $\langle A \rangle$ is an M-matrix. For simplicity, we may assume that $a_{ii} = 1$ for $i = 1, \Lambda, n$.

Secondly, we present some definitions and results about the splitting of matrix.

Definition 2.1 [20] Let $A \in R^{n\times n}$. The representation $A = M - N$ is called a splitting of A if M is nonsingular. Then $A = M - N$ is called

1 . convergent if $\rho(M^{-1}N) < 1$;

2 . regular if $M^{-1} \geq 0$, $N \geq 0$;

3 . weak regular if $M^{-1} \geq 0$, $M^{-1}N \geq$ 0;

4 . an M-splitting of A if M is an M-matrix and $N \geq 0$.

Lemma 2.1 [5] Let $A = M - N$ is an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.

Lemma 2.2 [2] A is monotone if and only if A is nonsingular with $A^{-1} \geq 0$.

Lemma 2.3 [15] Let A be an M-matrix, and x be a solution of LCP (1.1). If f_i > 0, then x_i > 0 and therefore $\sum_{j=1}^{n} a_{ij} x_j - f_i = 0$. Moreover, if $f \leq 0$, then $x = 0$ is the solution of LCP (1.1).

Lemma 2.4 [5] Let A be a Z-matrix. Then the following statements are equivalent:

(1) A is a nonsingular M-matrix.

(2) There exists a positive vector $v > 0$ such that $Av > 0$.

(3) Any weak regular splitting is convergent.

Lemma 2.5 [18] Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices A_1 and A_2, respectively, such that $M_1^{-1} \leq M_2^{-1}$, If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$, then for the monotonic norm associated with x, $\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x$. In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then $\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1)$.

Thirdly, we give the project methods, especially the SSOR method [8], for the LCP (1.1).

Definition 2.2 For $x \in R^n$, vector x_+ is defined such that $(x_+)_j = \max\{0, x_j\}$, $j = 1, \Lambda, n$. Then, for any $x, y \in R^n$. The following facts hold:

$(x + y)_+ \leq x_+ + y_+$;

$x_+ - y_+ \leq (x - y)_+$

$|x_+| = x_+ + (-x)_+$; and

$x \leq y$ implies that $x_+ \leq y_+$.

Following the Definition 2.2, the LCP (1.1) is equivalent to [1]

$$z = (z - \alpha\Omega(Az + f))_+ \qquad (2.1)$$

where α is a positive constant and the matrix Ω is positive diagonal. Let $0 < w < 2$ and $A = D - L - U$, where D, L and U are diagonal, strictly lower and upper triangular parts of A, respectively. Then (E, F) is called the SSOR splitting of A [8] if (E, F) is a splitting of A, and

$$E = 1/(w(w-2))(D - wL)D^{-1}(D - wU)$$

and

$$F = 1/(w(2-w))((1-w)D + wL)D^{-1}((1-w)D + wU)$$

From (2.1) and the SSOR splitting of A, two SSOR methods for solving the LCP(1.1) are defined as follows (see [8]):

Method 2.1 (SSOR method I);

Choose an initial vector $z^0 \in R^n$, a positive parameter w and set $k = 0$;

Compute

$$z^{k+1} = (z^k - D^{-1}[-wUz^{k+1} + (w(2-w)A + wU)z^k - w(2-w)f])_+$$

If $z^{k+1} = z^k$, then stop, otherwise set $k = k + 1$ and return to Step (2).

Method 2.2 (SSOR method II)

Choose an initial vector $z^0 \in R^n$, a positive parameter w and set $k = 0$;

Compute

$$z^{k+1} = (z^k - D^{-1}[-wLz^{k+1} + (w(2-w)A + wL)z^k - w(2-w)f])_+$$

If $z^{k+1} = z^k$, then stop, otherwise set $k = k + 1$ and return to Step (2)

Let

$$B_1 = I - wD^{-1}|L|, \quad C_1 = |I - D^{-1}[w(2-w)A + wL]| \qquad (2.2)$$

and

$$B_2 = I - wD^{-1}|U|, \quad C_2 = |I - D^{-1}[w(2-w)A + wU]| \qquad (2.3)$$

Then the convergence of the SSOR method I and SSOR method II are presented in
the following lemma [8, Theorem 2.1]

Lemma 2.6 [8] Let $A = (a_{ij}) \in R^{n\times n}$ be an H-matrix with positive diagonal elements. If $0 < w < 2$, then for any initial vector $z^0 \in R^n$, the iterative sequences z^k generated by the SSOR methods I and II converge to the unique solution z^* of the LCP (1.1) and it holds that $\rho(B_1^{-1}C_1)<1$ and $\rho(B_2^{-1}C_2)<1$.

Finally, we present the preconditioned SSOR methods..Let $\tilde{A}=PA, \tilde{f}=Pf$ and denote $\tilde{A}=\tilde{D}-\tilde{L}-\tilde{U}$, where $\tilde{D}$, $\tilde{U}$ and $\tilde{L}$ are diagonal, strictly lower and upper triangular matrices, respectively. Then the preconditioned SSOR methods for the LCP (1.1) are defined as follows:

Method 2.3 (Preconditioned SSOR method I):

Choose an initial vector $z^0 \in R^n$, a positive parameter w and set $k=0$;

Compute

$$z^{k+1}=(z^k-\tilde{D}^{-1}[-w\tilde{U}z^{k+1}+(w(2-w)\tilde{A}+w\tilde{U})z^k-w(2-w)\tilde{f}])_+$$

If $z^{k+1}=z^k$, then stop, otherwise set k := k + 1 and return to step (2)

Method 2.4 (Preconditioned SSOR method II):

Choose an initial vector $z^0 \in R^n$, a positive parameter w and set k = 0;

Compute

$$z^{k+1}=(z^k-\tilde{D}^{-1}[-w\tilde{L}z^{k+1}+(w(2-w)\tilde{A}+w\tilde{L})z^k-w(2-w)\tilde{f}])_+$$

If $z^{k+1}=z^k$, then stop, otherwise set $k=k+1$ and return to Step (2)

As the preconditioner P is defines as in (1.2), the elements $\tilde{a}_{ij}$ of $\tilde{A}$ satisfy

$$\tilde{a}_{ij}=\begin{cases} a_{ij}-\alpha_i a_{i,i+1}a_{i+1,j}, i\neq j, j=1,2\Lambda\ n, \\ a_{nj}-\alpha_n a_{n,1}a_{1,j}, i=j, j=1,2\Lambda\ n, \end{cases} \quad (2.4)$$

and the elements $\tilde{f}_i$ of $\tilde{f}$ satisfy

$$\tilde{f}_i=\begin{cases} f_i-\alpha_i a_{i,i+1}f_{i+1}, i\neq n \\ f_n-\alpha_n a_{n,1}f_1, i\neq n \end{cases} \quad (2.5)$$

III. CONVERGENCE ANALYSIS

In this section, we will consider the convergence of the preconditioned SSOR methods I and II for solving the LCP (1.1). From Lemma 2.3, if the problem LCP (1.1) has a nonzero solution, there is at least one index i such that f_i > 0.Without loss of generality, let us assume that f_1 > 0 and f_{i+1} > 0.

Theorem 3.1 Let $\tilde{A}=PA\equiv[\tilde{a}_{ij}]$, $\tilde{f}=Pf\equiv\tilde{f}_i$ If $f_1>0$ and $f_{i+1}>0$, then LCP(1.1) is equivalent to the linear complementarity problem

$$x\geq 0,\ \ \tilde{r}=\tilde{A}x-\tilde{f}\geq 0,\ \ x^T\tilde{r}=0. \quad (3.1)$$

Proof. Suppose that x is the solution to LCP (1.1). Because $f_1>0$ and $f_{i+1}>0$ from Lemma 2.3 we have that $x_1>0$, $\sum_{j=1}^n a_{1j}x_j-f_1=0$ and $x_{i+1}>0$, $\sum_{j=1}^n a_{i+1,j}x_j-f_{i+1}=0$.

If $i=n$, then we have

$$\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i=\sum_{j=1}^n (a_{nj}-\alpha_n a_{n1}a_{1j})x_j-(f_n-\alpha_n a_{n1}f_1)$$

$$=\sum_{j=1}^n a_{nj}x_j-f_n-\alpha_n a_{n1}(\sum_{j=1}^n a_{1j}x_j-f_1)$$

$$=\sum_{j=1}^n a_{ij}x_j-f_i \quad (3.2)$$

If $i\neq n$, then we get

$$\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i=\sum_{j=1}^n (a_{ij}-\alpha_i a_{i,i+1}a_{i+1,j})x_j-(f_i-\alpha_i a_{i,i+1}f_{i+1})$$

$$=\sum_{j=1}^n a_{ij}x_j-f_i-\alpha_i a_{i,i+1}(\sum_{j=1}^n a_{i+1,j}x_j-f_{i+1})$$

$$=\sum_{j=1}^n a_{ij}x_j-f_i \quad (3.3)$$

From (3.2) and (3.3), it can be seen that x is the solution of the LCP (3.1). Conversely, suppose that x is the solution of the LCP (3.1). It follows from Lemma 2.3 that $x_1>0$, $\sum_{j=1}^n \tilde{a}_{1j}x_j-\tilde{f}_1=0$ and $x_{i+1}>0$, $\sum_{j=1}^n \tilde{a}_{i+1,j}x_j-\tilde{f}_{i+1}=0$.

This together with (3.2) and (3.3) give $\sum_{j=1}^n a_{1j}x_j-f_1=0$ and $\sum_{j=1}^n a_{i+1,j}x_j-f_{i+1}=0$.

Thus for $i=n$ we have

$$\sum_{j=1}^n a_{ij}x_j-f_i=\sum_{j=1}^n a_{nj}x_j-f_n$$

$$=\sum_{j=1}^n (\tilde{a}_{ij}+\alpha_n a_{n1}a_{1j})x_j-(\tilde{f}_i+\alpha_n a_{n1}f_1)$$

$$=\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i+\alpha_n a_{n,1}(\sum_{j=1}^n a_{1,j}x_j-f_1)$$

$$=\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i$$

And for $i\neq n$, we can deduce that

$$\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i=\sum_{j=1}^n (\tilde{a}_{ij}+\alpha_i a_{i,i+1}a_{i+1,j})x_j-\sum_{j=1}^n (\tilde{f}_i+\alpha_i a_{i,i+1}f_{i+1})$$

$$=\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i+\alpha_i a_{i,i+1}(\sum_{j=1}^n a_{i+1,j}x_j-f_{i+1})$$

$$=\sum_{j=1}^n \tilde{a}_{ij}x_j-\tilde{f}_i$$

Hence, x is the solution of the LCP (1.1).

In what follows, we assume that the conditions

(H1) $0\leq\alpha_i\leq 1$ for $i=1,\Lambda\ ,n$,

(H2) $0\leq\alpha_i a_{i,i+1}a_{i+1,j}$, for $i=1,\Lambda\ ,n$.

Theorem 3.2 If A is an M-matrix, (H1)-(H2) hold, then $\tilde{A}=PA$ is an M-matrix

Proof. If A is an M-matrix, then $a_{ij}\leq 0$ for $i\neq j$. Now from (2.4) and the assumptions, we have

$$\tilde{a}_{ii}=a_{ii}-\alpha_i a_{i,i+1}a_{i+1,i}>0 \quad i\neq n, i=j;$$

$$\tilde{a}_{ij}=a_{ij}-\alpha_i a_{i,i+1}a_{i+1,j}<0 \quad i\neq n, i\neq j;$$

$$\tilde{a}_{nn}=a_{nn}-\alpha_n a_{n,1}a_{1,n}>0 \quad i=j=n;$$

$$\tilde{a}_{nj}=a_{nj}-\alpha_n a_{n,1}a_{1,j}<0 \quad i=n, i\neq j;$$

From Lemma 2.4 there exists a positive vector $y>0$ such that $Ay>0$. Note that $P\geq 0$, thus $\tilde{A}y=PAy>0$, and from Lemma 2.4 $\tilde{A}$ is an M-matrix.

From Theorems 3.1 and Theorems 3.2, we can establish the following convergence theorem for the preconditioned SSOR methods I and II for solving the LCP (1.1)

Theorem 3.3 Let $A=(a_{ij})\in R^{n\times n}$ be a nonsingular M-matrix. If P given in (1.2) satisfies the conditions of Theorem 3.2, then for $0<w<2$, the iterative sequences of the preconditioned SSOR methods I and II converge to the unique solution x^* of the LCP (1.1), where for the given vector f, its components $f_1>0$ and $f_{i+1}>0$.

Proof. Since A is a nonsingular M-matrix, by Theorem 3.2 $\tilde{A}$ is also an M-matrix, then $\tilde{A}$ is an H-matrix with positive diagonals. Hence, according to Lemma 2.6, the iterative sequences of the preconditioned SSOR methods I and II converge to the unique solution x^* of the LCP (3.1), or equivalently, the unique solution x^* of the LCP (1.1) by Theorem 3.1.

IV. Comparison Results

In this section, we will consider comparison theorems, which show that the PSSOR methods can increase the convergence of corresponding SSOR methods for the LCPs of M-matrices.

Let us consider the problem (1.1) with the splitting

$$A=D-L-U \qquad (4.1)$$

where D, L and U are diagonal, strictly lower and strictly upper triangular parts of A, respectively. We assume that

$$\tilde{A}=PA=(\tilde{a}_{ij}) \quad \tilde{f}=Pf \qquad (4.2)$$

where P satisfies Theorem 3.3 and

$$\tilde{a}_{ij}=\begin{cases} a_{ij}-\alpha_i a_{i,i+1}a_{i+1,j}, & i\neq n,\ j=1,2\Lambda\ n,\\ a_{nj}-\alpha_n a_{n,1}a_{1,j}, & i=n,\ j=1,2\Lambda\ n,\end{cases}$$

We split $\tilde{A}$ in(4.2) as

$$\tilde{A}=\tilde{D}-\tilde{L}-\tilde{U} \qquad (4.3)$$

where $\tilde{D},\tilde{L},$ and $\tilde{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\tilde{A}$, respectively. Apparently, it follows that $\tilde{D}=(d_{ii})$ with

$$d_{ij}=\begin{cases} a_{ii}-\alpha_i a_{i,i+1}a_{i+1,i}, & i\neq n,\\ a_{nn}-\alpha_n a_{n,1}a_{1,n}, & i=n,\end{cases}$$

$\tilde{L}=(l_{ii})$ with

$$l_{ij}=\begin{cases} a_{ij}-\alpha_i a_{i,i+1}a_{i+1,j}, & i\neq n, i>j,\\ a_{nj}-\alpha_n a_{n,1}a_{1,j}, & i=n,\end{cases}$$

$\tilde{U}=(u_{ii})$ with $u_{ij}=a_{ij}-\alpha_i a_{i,i+1}a_{i+1,j} \quad i<j$, respectively.

In what follows, we give some useful auxiliary results that are important for us
to provide comparison theorems.

Lemma 4.1 Let $A=(a_{ij})\in R^{n\times n}$ be an M-matrix. Assume that A is written as the splitting (4.1) and $D,L,U,\tilde{D},\tilde{L}$ and $\tilde{U}$ are given by (4.1)-(4.3). Then

$$D^{-1}|L|\le\tilde{D}^{-1}|\tilde{L}|,\quad D^{-1}|U|\le\tilde{D}^{-1}|\tilde{U}|,$$

Proof. Since $\tilde{A}$ is an M-matrix, naturally, an H-matrix with positive diagonals

$$\begin{cases} a_{ii}-\alpha_i a_{i,i+1}a_{i+1,i}>0, & i\neq n,\\ a_{nn}-\alpha_n a_{n,1}a_{1,n}>0, & i=n,\end{cases} \qquad (4.4)$$

Let us denote $D^{-1}|L|=(\bar{l}_{ij})$ $\tilde{D}^{-1}|\tilde{L}|=(\tilde{l}_{ij})$ Then we have

$$\bar{l}_{ij}=\begin{cases} \dfrac{1}{a_{ii}}|a_{ij}|, & i>j,\\ 0, & other,\end{cases}$$

And

$$\tilde{l}_{ij}=\begin{cases} \dfrac{1}{a_{ii}-\alpha_i a_{i,i+1}a_{i+1,i}}(|a_{i,j}|+\alpha_i a_{i,i+1}a_{i+1,j}), & i>j, i\neq n,\\ \dfrac{1}{a_{nn}-\alpha_n a_{n,1}a_{1,n}}(|a_{n,j}|+\alpha_n a_{n,1}a_{1,j}), & i=n,\end{cases}$$

On the one hand, from (4.4), $p_{ii}\ge 0$ and the fact that A is an M-matrix, we have

$$\frac{1}{a_{ii}}\le\frac{1}{a_{ii}-\alpha_i a_{i,i+1}a_{i+1,i}} \text{ and } |a_{ij}|\le(|a_{ij}|+\alpha_i a_{i,i+1}a_{i+1,j})$$

$$\frac{1}{a_{nn}}\le\frac{1}{a_{nn}-\alpha_n a_{n,1}a_{1,n}} \text{ and } |a_{nj}|\le(|a_{nj}|+\alpha_n a_{n,1}a_{1,j})$$

Therefore, we obtain that $\bar{l}_{ij}\le\tilde{l}_{ij}, i,j\in N$. In other words, $D^{-1}|L|\le\tilde{D}^{-1}|\tilde{L}|$.

Similarly, one can achieve that Similarly, one can achieve that $D^{-1}|U|\le\tilde{D}^{-1}|\tilde{U}|$.

Let

$$\tilde{B}_1=I-w\tilde{D}^{-1}|\tilde{L}|,\quad \tilde{C}_1=|I-\tilde{D}^{-1}[w(2-w)\tilde{A}+w\tilde{L}]| \qquad (4.5)$$

$$\tilde{B}_2=I-w\tilde{D}^{-1}|\tilde{U}|,\quad \tilde{C}_2=|I-\tilde{D}^{-1}[w(2-w)\tilde{A}+w\tilde{U}]| \qquad (4.6)$$

Lemma 4.2 Let $A=(a_{ij})\in R^{n\times n}$ be an M-matrix. Suppose that $\tilde{A}$ and $\tilde{f}$ are given by (4.2) and $\tilde{B}_1,\tilde{C}_1$ and $\tilde{B}_2,\tilde{C}_2$,are defined by (4.5) and (4.6), respectively. If $0<w<2$, then for any initial vector $x_0\in R^n$,the iterative sequences x^k generated by the PSSOR methods I and II converge to the unique solution x^* of the LCP (1.1) and it follows that $\rho(\tilde{B}_1^{-1}\tilde{C}_1)<1$ and $\rho(\tilde{B}_2^{-1}\tilde{C}_2)<1$.

Proof. By (4.2), $\tilde{A}$ is an H-matrix with positive diagonals. Hence, by Theorem 3.3, for any initial vector $x_0\in R^n$ the iterative sequences x^k of the PSSOR methods I and II converge to the unique solution of the LCP(1.1), and from Lemma 2.6 and the fact that $\tilde{A}$ is an H-matrix with positive diagonal entries, it follows that $\tilde{B}_1^{-1}\tilde{C}_1<1$ and $\tilde{B}_2^{-1}\tilde{C}_2<1$.

Theorem 4.1 Assume that A is a nonsingular M-matrix and A and $\tilde{A}$ have the splitting (4.1) and (4.3), respectively. Let B_1,C_1 and , $\tilde{B}_1,\tilde{C}_1$ be given as in (2.2) and (4.5), respectively. Then for the matrices $B_1^{-1}C_1$ for SSOR I and $\tilde{B}_1^{-1}\tilde{C}_1$ for PSSORI with respect to the LCPs, we have $\rho(\tilde{B}_1^{-1}\tilde{C}_1)\le\rho(B_1^{-1}C_1)<1$.

Proof. By Lemma 2.6 and the fact that A is an M-matrix, for any initial vector $x_0\in R^n$ the iterative sequence x^k

generated by SSOR I converges to the unique solution x^* of the LCP (1.1) and

$$\rho(B_1^{-1}C_1)<1 \tag{4.7}$$

Analogously, by Lemma 4.2 and the fact that $\tilde{A}$ is an H-matrix with positive diagonals, for any initial vector $y_0 \in R^n$ the iterative sequence y^k generated by PSSORI converges to the unique solution x^* of the LCP (1.1) and

$$\rho(\tilde{B}_1^{-1}\tilde{C}_1)<1 \tag{4.8}$$

Let us now consider the result $\rho(\tilde{B}_1^{-1}\tilde{C}_1)\le\rho(B_1^{-1}C_1)$ .In terms of Lemma 4.1, we have that $D^{-1}|L|\le\tilde{D}^{-1}|\tilde{L}|$,which is equivalent to $I-w\tilde{D}^{-1}|\tilde{L}|\le I-wD^{-1}|L|$

that is, $\tilde{B}_1\le B_1$,Notice that $\tilde{B}_1$ and B_1 are M-matrices, this implies that $0\le B^{-1}{}_1\le\tilde{B}^{-1}{}_1$. Let us denote $Q_1=B_1-C_1$ and $Q_2=\tilde{B}_1-\tilde{C}_1$,Observe that $\tilde{B}_1$ and B_1 are M-matrices and $\tilde{C}_1$ and C_1 are nonnegative, it holds that B_1-C_1 and $\tilde{B}_1-\tilde{C}_1$ are M-splittings of Q_1 and Q_2, respectively. It means from (4.7),(4.8) and Lemma 2.2 that Q_1 and Q_2 are M-matrices. Therefore, $Q_1^{-1}\ge0$ and $Q_2^{-1}\ge0$ which show by Lemma 2.2 that Q_1 and Q_2 are monotone. From the fact that an M-splitting is an regular splitting, it can be derived that B_1-C_1 and $\tilde{B}_1-\tilde{C}_1$ are regular splittings of the monotone matrices Q_1 and Q_2, respectively.

Note that A is an irreducible matrix, taking into account that $B_1^{-1}C_1=(I-wD^{-1}|L|)^{-1}|I-D^{-1}[w(2-w)A+wL]|$

this implies that the matrix $B_1^{-1}C_1$ is a nonnegative irreducible matrix. Thus, by means of Perron-Frobenius theorem (see Theorem 2.7 of [4]), $B_1^{-1}C_1$ has a positive Perron vector. By Lemma 2.5, as a result, we have $\rho(\tilde{B}_1^{-1}\tilde{C}_1)\le\rho(B_1^{-1}C_1)$. This completes the proof.

Similarly, we can obtain the following corollary.

Corollary 4.1

Assume that A is a nonsingular M-matrix and A and $\tilde{A}$ have the splitting (4.1) and (4.2), respectively. Let B_2, C_2 and $\tilde{B}_2,\tilde{C}_2$ be given as in (2.3) and (4.6), respectively. Then for the matrices $B_2^{-1}C_2$ for SSORII and $\tilde{B}_2^{-1}\tilde{C}_2$ for PSSORII with respect to the LCP (1.1), it holds that $\rho(\tilde{B}_2^{-1}\tilde{C}_2)\le\rho(B_2^{-1}C_2)<1$.

V. NUMERICAL EXAMPLES

In this section, an example is given for verifying the theoretical result.

Example 5.1 Consider the LCP with the system matrix $A\in R^{n\times n}$ and the Vector $f\in R^n$,

$$A=\begin{bmatrix} S & -I & -I & & & \\ & S & -I & O & & \\ & & S & O & -I & \\ & & & O & -I & \\ & & & & S & \end{bmatrix}\in R^{n\times n},\ f=\begin{bmatrix}-1\\1\\-1\\ \vdots \\(-1)^{n-1}\\(-1)^n\end{bmatrix}\in R^n$$

where $S=tridiag(-1,8,-1)\in R^{\bar{n}\times\bar{n}}$ and $I\in R^{\bar{n}\times\bar{n}}$ is the identity matrix and $\bar{n}^2=n$ It is easy to check that A is an M-matrix. So, the LCP has a unique solution. Taking into account that $f_2>0, f_4>0,\ldots$ hence $k_m\in\{2,4,6\ldots\}$.

The results are summarized in Table 1,2. By Table 1,2, we compare spectral radii of two PSSOR methods with those of corresponding two SSOR methods for w = 0.2 and w = 0.9 when $n=100,400,900$ and 1600 respectively. It is observed from Table 1, 2 that two preconditioned SSOR methods improve considerably convergence rate of corresponding two SSOR methods for the LCP(A,f), which confirm our theoretical results.

Table 1,2 list $\rho(B^{-1}C)$ and $\rho(\tilde{B}^{-1}\tilde{C})$ with different α and w for Example 5.1.

Table 1: $\rho(B^{-1}C)$ and $\rho(\tilde{B}^{-1}\tilde{C})$ with $(\alpha_1,\ldots,\alpha_{n-1},\alpha_n)^T=(0.1,\ldots,0.1,\frac{2}{3})^T$ $w=0.2$ for Example 5.1

N	SSOR1	PSSOR1	SSOR11	PSSOR11
100	0.1150	0.0564	0.1150	0.0772
400	0.1193	0.0587	0.1193	0.0812
900	0.1202	0.0591	0.1202	0.0820
1600	0.1205	0.0593	0.1205	0.0824

VI. CONCLUDING REMARKS

In this paper, for the LCPs with an M-matrix A and the vector f, we first present a preconditioner P by using the number of positive sign of the components in f,

Table 2 $\rho(B^{-1}C)$ and $\rho(\tilde{B}^{-1}\tilde{C})$ with $(\alpha_1,\ldots,\alpha_{n-1},\alpha_n)^T=(0.1,\ldots,0.1,\frac{2}{3})^T$ $w=0.9$ for Example 5.1

N	SSOR1	PSSOR1	SSOR11	PSSOR11
100	0.7193	0.6856	0.7193	0.6871
400	0.7218	0.6877	0.7218	0.6893
900	0.7233	0.6881	0.7223	0.6898
1600	0.7225	0.6883	0.7225	0.6899

and prove that the original LCP (1.1) is equivalent to the LCP (3.1). Then, on the basis of the preconditioner P , two preconditioned SSOR methods for linear complementarity problem are proposed and the convergence analysis is provided.Also we achieve comparison theorems on the preconditioned SSOR methods for the linear complementarity problem, which show that the PSSOR methods improve considerably the convergence rate of the original SSOR methods for solving the LCP (1.1). Numerical examples tested show the prominent efficiency of the proposed methods. How to extend this technique to other methods for solving the LCPs is the content of future research.

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Dan Zhang, College of Mathematics and Statistics, Northwest Normal University,Lanzhou,China,86-13893130913,(e-mail: zd943663677@163.com).