Discrete-time Geo^X/D/c/N Queueing System with Batch Geometric Arrivals, Multiple Servers, and Retention of Reneging Customers

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Abstract— **This paper considers the design and analysis of a discrete-time $Geo^X/D/c/N$ queueing system with batch geometric arrivals, multiple servers, and retention of reneging customers. This paper derives the limiting probability distribution of the number of customers in the system and also obtains the expression for the proportion of arriving customers that are blocked.**

Index Terms— **Batch geometric arrivals, multiple servers, proportion of blocked customers, retention of reneging customers.**

I. INTRODUCTION

Customer reneging is a phenomenon observed commonly in queueing systems, in which customers may wait for a certain time in the queue and leave the service system without getting service due to certain reasons such as a long queue. The queueing problems with customer reneging were first analyzed by Palm [1]. A bibliography can be found in Gross et al. [2]. A truncated heterogeneous two-server M/M/2/N queue with reneging and general balk function was considered by El-Sherbiny [3]. Kumar and Sharma [4] considered a queue, in which there are chances that a certain fraction of impatient customers can be retained for his future service owing to a certain customer retention strategy and analyzed an M/M/1/N queueing system with retention of reneging customers. Kumar and Sharma [5] studied an M/M/c/N queueing system with reneging and retention of reneging customers. Kumar [6] analyzed an M/M/c/N queueing model with balking, reneging, and retention of reneging customers. Kumar and Sharma [7] considered a finite capacity Markovian queueing system with two heterogeneous servers, discouraged arrivals, reneging, and retention of reneging customers. Bouchentouf and Yahiaoui [8] presented an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations, and Bernoulli schedule vacation interruption, where customers' impatience is due to an absence of server upon arrival.

Discrete-time queues with retention of reneging customers were also considered. Lee considered a discrete-time $Geo^X/D/1/N$ queueing system with retention of reneging customers [9] and its extension to state dependent retention [10]. Lee [11] also considered $Geo/Geo/c/N$ queue with retention of reneging customers.

This paper considers a discrete-time $Geo^X/D/c/N$ queue with retention of reneging customers. The discrete-time $Geo^X/D/c/N$ queueing system is the generalization of the

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previous discrete-time version $Geo^X/D/1/N$. This paper derives the limiting distribution of the number of customers in the system and also obtains the expression for the proportion of arriving customers that are blocked.

The paper is structured as follows. In Section II, we described the queueing model. In Section III, we formulate the system as a discrete-time Markov chain and derive the limiting probability distribution of the number of customers in the system and the expression for the proportion of arriving customers that are blocked. Conclusion is provided in Section IV.

II. SYSTEM MODEL

We formulate the queueing model, which is based on the following assumptions:

1. A discrete-time queue is considered in which the time axis is divided into fixed-length contiguous intervals, referred to as slots.
2. Customers arrive according to a batch geometric process.
3. The numbers of arrivals during the consecutive slots are assumed to be independent and identically distributed random variables with distribution $\{a_k, k = 0, 1, \ldots\}$.
4. There are c servers in the system.
5. The services of customers are synchronized to (i.e., can only start and end at) slot boundaries.
6. The service time of customers is one slot.
7. The buffer of the system has finite capacity N.
8. Customers are served in FIFO order.
9. The reneging times follow a geometric distribution with parameter r.
10. Reneging customers may leave the queue without getting service with probability s.
11. During each slot, reneging occurs after service initiation epoch and before customer arrival epochs.

III. STATIONARY DISTRIBUTION

Random variable N_k is defined as the total number of customers in the system at the end of slot k. Then, the stochastic process $\{N_k, k \geq 0\}$ becomes a discrete-time Markov chain. The state space of the Markov chain $\{N_k, k \geq 0\}$ is $\{0, 1, 2, \ldots, N\}$.

We define $r_{i,j}$ as the probability that j customers among i waiting customers in the buffer leave the system at a slot due to reneging. Then, $r_{i,j}$ can be obtained as: if $i \geq j$ and $i \geq 1$, then

$$r_{i,j} = \sum_{k=j}^{i} \binom{i}{k} r^k (1-r)^{i-k} \binom{k}{j} s^j (1-s)^{k-j}; \qquad (1)$$

Otherwise, the value $r_{i,j}$ equals 0. Note that

$$\binom{i}{k} r^k (1-r)^{i-k} \text{ and } \binom{k}{j} s^j (1-s)^{k-j} \tag{2}$$

are the probability that k customers among i waiting customers renege and the probability that j customers among k reneging customers leave the system at a slot, respectively.

The one-step state transition probability matrix $\boldsymbol{P}$ of the Markov chain $\{N_k, k \geq 0\}$ with state space $\{0, 1, 2, \dots, N\}$ is given by

$$\boldsymbol{P} =$$

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & r_{1,1} & r_{1,0} & 0 & \cdots & 0 \\ 0 & \cdots & r_{2,2} & r_{2,1} & r_{2,0} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & r_{N-c,N-c} & r_{N-c,N-c-1} & r_{N-c,N-c-2} & \cdots & r_{N-c,0} \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_N^+ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & a_2 & \cdots & a_N^+ \\ 0 & a_0 & a_1 & \cdots & a_{N-1}^+ \\ 0 & 0 & a_0 & \cdots & a_{N-2}^+ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_c^+ \end{pmatrix},$$

$$\tag{3}$$

where the probability a_k^+ is given by:

$$a_k^+ \equiv 1 - \sum_{i=0}^{k-1} a_i. \tag{4}$$

Note that the size of the one-step state transition probability matrix $\boldsymbol{P}$ is $(N+1) \times (N+1)$.

Let

$$\boldsymbol{x} \equiv (x_0, x_1, \dots, x_N) \tag{5}$$

be the limiting probability distribution associated with the discrete-time Markov chain $\{N_k, k \geq 0\}$, where

$$x_i \equiv \lim_{k \to \infty} P\{N_k = i\}. \tag{6}$$

Then, the limiting probability $\boldsymbol{x}$ is obtained by solving

$$\boldsymbol{x}\boldsymbol{P} = \boldsymbol{x}, \text{ and } \boldsymbol{x}\boldsymbol{e} = 1, \tag{7}$$

where $\boldsymbol{e}$ is the column vector with all elements 1.

The average number of customers in the system is given by

$$L = \sum_{i=1}^{N} i \, x_i. \tag{8}$$

The expression for the proportion of arriving customers that are blocked is given by:

$$P_b = \sum_{i=0}^{N} x_i \sum_{j=0}^{i-(i,c)^-} r_{i-(i,c)^-,j} \sum_{k=N-i+j}^{\infty} \frac{(k+i-j-N)a_k}{\sum_{l=1}^{\infty} la_l},$$

where

$(i,c)^- = \min(i,c),$
$r_{0,0} \equiv 1.$

IV. Conclusion

This paper has considered the design and analysis of a discrete-time Geo$^{\text{X}}$/D/c/N queueing system with impatient customers, batch geometric customer arrivals, and multiple servers, where customers may renege and reneging customers may be retained. This paper has derived the limiting probability distribution of the number of customers in the system and also obtained the expression for the proportion of arriving customers that are blocked.

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