Geo/Geo/c/N Queue with Multiple Servers and Retention of Reneging Customers

Yutae Lee

Abstract— **This paper considers a discrete-time Geo/Geo/c/N queueing system with geometric arrivals, multiple servers, geometric service times, and retention of reneging customers. The limiting probability distribution of the number of customers in the system is derived. The blocking probability of arriving customers is also obtained.**

Index Terms— **Blocking probability, geometric service times, multi-server, retention of reneging customers.**

I. INTRODUCTION

The phenomenon of customer reneging is commonly observed in many queueing systems, where customers may leave the service system before receiving their service due to the long waiting time. The problem of queueing systems with customer reneging was first analyzed by Palm [1]. A bibliography can be found in Gross et al. [2]. El-Sherbiny [3] considered a truncated heterogeneous two-server M/M/2/N queueing system with reneging and general balk function. Kumar and Sharma [4] considered a queueing system, where reneging customers may be retained for his future service owing to a certain customer retention strategy and analyzed an M/M/1/N queueing system with retention of reneging customers. Kumar and Sharma [5] studied an M/M/c/N queueing system with reneging and retention of reneging customers. Kumar [6] analyzed an M/M/c/N queueing model with balking, reneging, and retention of reneging customers. Kumar and Sharma [7] considered a finite capacity Markovian queueing system with two heterogeneous servers, discouraged arrivals, reneging, and retention of reneging customers. Bouchentouf and Yahiaoui [8] presented an analysis of a Markovian feedback queueing system with reneging and retention of reneged customers, multiple working vacations, and Bernoulli schedule vacation interruption, where customers' impatience is due to the server's vacation.

Discrete-time queueing systems with retention of reneging customers were considered. Lee [9] considered a Discrete-time $\mathrm{Geo}/\mathrm{Geo}/1/N$ queueing system with retention of reneging customers. Lee considered a discrete-time $\mathrm{Geo}^{\mathrm{X}}/\mathrm{D}/1/N$ queueing system with retention of reneging customers [10] and its extension to state dependent retention [11]. There are some errors in the analysis of the above discrete-time versions.

This paper considers a discrete-time $\mathrm{Geo}^{\mathrm{X}}/\mathrm{D}/\mathrm{c}/N$ queueing system with retention of reneging customers. The discrete-time $\mathrm{Geo}^{\mathrm{X}}/\mathrm{D}/\mathrm{c}/N$ queueing system is an extension of the previous discrete-time version $\mathrm{Geo}/\mathrm{Geo}/1/N$. The limiting distribution of the number of customers in the system is derived. The blocking probability of arriving customers is also obtained.

The paper is organized as follows. In Section II, we described the queueing model. In Section III, we formulate the system as a discrete-time Markov chain and analyze the limiting probability distribution of the number of customers in the system and the blocking probability of arriving customers. Conclusion is provided in Section IV.

Yutae Lee, Department of Information and Communications Engineering, Dongeui University, Busan, Republic of Korea

II. SYSTEM MODEL

In this section, we formulate the queueing model, which is based on the following assumptions:

1. We consider a discrete-time queueing system in which the time axis is divided into fixed-length contiguous intervals, referred to as slots.
2. Customers arrive according to a geometric process. Let p be the probability that a customer enters the system during a slot.
3. There are c servers in the system.
4. The service of a customer can start only at a slot boundary.
5. The service times of customers follow a geometric distribution with parameter q.
6. The system has a buffer of finite capacity N.
7. Customers are served in FCFS order.
8. The reneging times follow a geometric distribution with parameter r.
9. Reneging customers may leave the queue without getting service with probability s.
10. During each slot, reneging occurs before customer arrivals.

III. STATIONARY DISTRIBUTION

To model this system, we define the random variable N_k as the total number of customers in the system at the end of slot k. Then, the stochastic process $\{N_k, k \geq 0\}$ becomes a discrete-time Markov chain. The state space of this Markov chain $\{N_k, k \geq 0\}$ is $\{0, 1, 2, \dots, N\}$.

Define $r_{i,j}$ as the probability that j customers among i waiting customers in the buffer leave the system at a slot due to reneging. Then, the probability $r_{i,j}$ can be obtained as: if $i \geq j$ and $i \geq 1$, then

$$r_{i,j} = \sum_{k=j}^{i} \binom{i}{k} r^k (1-r)^{i-k} \binom{k}{j} s^j (1-s)^{k-j}; \qquad (1)$$

Otherwise, $r_{i,j} = 0$. Note that

$\binom{i}{k} r^k (1-r)^{i-k}$ and $\binom{k}{j} s^j (1-s)^{k-j}$

are the probability that k customers among i waiting customers renege and the probability that j customers among k reneging customers leave the system at a slot, respectively.

Define $u_{i,j}$ as the probability that j customers among i customers in service complete their service at a slot. Then, the probability $u_{i,j}$ can be obtained as

$$u_{i,j} = \binom{i}{j} q^j (1-q)^{i-j} \tag{5}$$

for $i \geq j$.

The one-step state transition probability matrix $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & r_{1,1} & r_{1,0} & 0 & \cdots & 0 \\ 0 & \cdots & r_{2,2} & r_{2,1} & r_{2,0} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & r_{N-c,N-c} & r_{N-c,N-c-1} & r_{N-c,N-c-2} & \cdots & r_{N-c,0} \end{pmatrix} \boldsymbol{Q}$$

where the matrix $\boldsymbol{Q}$ is an $(N+1) \times (N+1)$ matrix with its ij elements $\boldsymbol{Q}_{i,j}$:

$$\boldsymbol{Q}_{0,0} = 1 - p,$$

$$\boldsymbol{Q}_{0,1} = p$$

$$\boldsymbol{Q}_{i,0} = (1-p) u_{i,i}, 1 \leq i \leq c,$$

$$\boldsymbol{Q}_{i,j} = p u_{i,i-j+1} + (1-p) u_{i,i-j}, 1 \leq i \leq c, 1 \leq j \leq i,$$

$$\boldsymbol{Q}_{i,i+1} = p u_{i,0}, \quad 1 \leq i \leq c,$$

$$\boldsymbol{Q}_{c+i,i} = (1-p) u_{c,c}, 1 \leq i \leq N-1-c,$$

$$\boldsymbol{Q}_{c+i,i+j} = p u_{c,c+1-j} + (1-p) u_{c,c-j},$$
$$1 \leq i \leq N-1-c, 1 \leq j \leq c,$$

$$\boldsymbol{Q}_{c+i,c+i+1} = p u_{c,0}, \quad 1 \leq i \leq N-1-c,$$

$$\boldsymbol{Q}_{N,N-c+j} = u_{c,c-j}, \qquad 0 \leq j \leq c.$$

Note that the size of the one-step state transition probability matrix $\boldsymbol{P}$ is $(N+1) \times (N+1)$.

Let $\boldsymbol{x} \equiv (x_0, x_1, \ldots, x_N)$ be the limiting probability vector associated with the discrete-time Markov chain $\{N_k, k \geq 0\}$, where $x_i \equiv \lim_{k\to\infty} P\{N_k = i\}$. Then, the limiting probability $\boldsymbol{x}$ is obtained by solving $\boldsymbol{xP} = \boldsymbol{x}$ and $\boldsymbol{xe} = 1$, where $\boldsymbol{e}$ is the column vector with all elements 1.

The mean number of customers in the system is given by

$$L = \sum_{i=1}^{N} i\, x_i. \tag{12}$$

The blocking probability of an arriving customer is given by x_N.

IV. Conclusion

In this paper, we have considered a $\mathrm{Geo}/\mathrm{Geo}/c/N$ queueing system with impatient customers, geometric customer arrivals, and multiple servers, where customers may renege and reneging customers may be retained. The limiting probability distribution of the number of customers in the system has been derived. The blocking probability of arriving customers has also been obtained.

References

[1] C. Palm, "Methods of judging the annoyance caused by congestion," *Telecommunications*, vol. 4, 1953, pp. 153-163.

[2] D. Gross, J. F. Shortle, J. M. Thompson, and C. M. Harris, *Fundamentals of Queueing Theory*, John Wiley & Sons, 4th Ed., 2008.

[3] A. A. El-Sherbiny, "The truncated heterogeneous two-server queue: M/M/2/N with reneging and general balk function," *International Journal of Mathematical Archive*, vol. 3, 2012, pp. 2745-2754.

[4] R. Kumar and S. K. Sharma, "M/M/1/N queueing system with retention of reneged customers," *Pakistan Journal of Statistics and Operation Research*, vol. 8, no. 4, 2012, pp.859-866.

[5] R. Kumar and S. K. Sharma, "An M/M/c/N queueing system with reneging and retention of reneged customers," *International Journal of Operational Research*, vol. 17, no. 3, 2013, pp. 333-334.

[6] R. Kumar, "Economic analysis of an M/M/c/N queueing model with balking, reneging and retention of reneged customers," *Opsearch*, vol. 50, no. 3, 2013, pp. 383-403.

[7] R. Kumar and S. K. Sharma, "Two-heterogeneous server Markovian queueing system with discouraged arrivals, reneging and retention of reneged customers," *International Journal of Operational Research*, vol. 11, no. 2, 2014, pp. 64-68.

[8] A. A. Bouchentouf and L. Yahiaoui, "On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption," *Arabian Journal of Mathematics*, vol. 6, no. 1, 2017, pp. 1-11.

[9] Y. Lee, "Geo/Geo/1/N queue with retention of reneging customers," *World Journal of Research and Review*, vol. 4, no. 2, 2017, pp. 20-21.

[10] Y. Lee, "Discrete-time queue with batch geometric arrivals and retention of reneging customers," *World Journal of Research and Review*, vol. 4, no. 2, 2017, pp. 27-28.

[11] Y. Lee, "Discrete-time queue with batch geometric arrivals and state dependent retention of reneging customers," *International Journal of New Technology and Research*, vol. 3, no. 2, 2017, pp. 32-33.