A Class of One-Step Hybrid Third derivative Block Method for the Direct Solution of Initial Value Problems of Second-order Ordinary Differential Equations

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***Abstract*— In this paper, we consider the development of a class of one-step hybrid third derivative block method with three off-grid points for the direct solution of initial value problems of second order Ordinary Differential Equations. We adopted method of interpolation and collocation of power series approximate solution to generate the continuous hybrid linear multistep method, which was evaluated at grid points to give a contnuous block method. The discrete block method was recovered when the continuous block method was evaluated at selected grid points. The basic propertise of the method was investigated and was found to be zero-stable, consistent and convergent. The efficiency of the method was tested on some stiff equations and was found to give better approximation than the existing method, which we compared our result with.**

***Index Terms*— One-step, Hybrid Block Method, Third derivative, stiff ODEs, Collocation and Interpolation Method.**

I. INTRODUCTION

This paper solves second order initial value problems in the form

$$y''=f(x,y(x),y'(x)),\quad y(x_0)=y_0,\ y'(x_0)=y'_0 \qquad (1).$$

where f is continuous within the interval of integration. Solving higher order derivatives method by reducing them to a system of first-order approach involves more functions to evaluate which then leads to a computational burden as in [1-3].The setbacks of this approach had been reported by scholars, among them are Bun and Vasil'yer [4] and Awoyemi et al [5].

The method of collocation and interpolation of the power series approximation to generate continuous linear multistep method has been adopted by many scholars, among them are Fatunla [7], Awoyemi [5], Olabode [10], Vigor Aquilar and Ramos [6], Adeniran et al [15], Abdelrahim et al[14], Mohammad et al[13] to mention a few. Block method generates independent solution at selected grid points without overlapping. It is less expensive in terms of number of function evaluation compared to predictor corrector method, moreover it possess the properties of Runge Kutta method for being self-starting and does not require starting values. Some of the authors that proposed block method are: [8, 9, 11, 12, 17,18].

In this paper, we developed a on e-step hybrid third derivative method with three offgrid, which is implemented in block method. The method is self-starting and does not require starting values or predictors. The implementation of the method is cheaper than the predictor-corrector method. This method harnesses the properties of hybrid and third derivative, this makes it e¢cient for sti¤ problems.

The paper is organised as follows: In section 2, we discuss the methods and the materials for the development of the method. Section 3 considers analysis of the basis properties of the method which include convergence and stability region, numerical experiments where the e¢ciency of the derived method is tested on some numerical examples and discussion of results. Lastly, we concluded in section 4.

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II. DERIVATION OF THE METHOD

We consider a power series approximate solution of the form

$$y(x)=\sum_{j=0}^{2r+s-1} a_i\left(\frac{x-x_n}{h}\right)^j \qquad (2)$$

where $r=2$ and $s=5$ are the numbers of interpolation and collocation points respectively, is considered to be a solution to (1).

The second and third derivative of (2) gives

$$\begin{aligned} y''(x)&=\sum_{j=2}^{2r+s-1}\frac{a_j\, j!}{h^2(j-2)}\left(\frac{x-x_n}{h}\right)^{j-2} \\ &=f(x,y,y'), \\ y'''(x)&=\sum_{j=3}^{2r+s-1}\frac{a_j\, j!}{h^3(j-3)}\left(\frac{x-x_n}{h}\right)^{j-3} \\ &=g(x,y,y'), \end{aligned} \qquad (3)$$

Substituting (3) into (1) gives

$$f(x,y,y'')=\sum_{j=2}^{2r+s-1}\frac{a_j\, j!}{h^2(j-2)}\left(\frac{x-x_n}{h}\right)^{j-2}+\sum_{j=3}^{2r+s-1}\frac{a_j\, j!}{h^3(j-3)}\left(\frac{x-x_n}{h}\right)^{j-3} \qquad (4)$$

Collocating (4) at all points $x_{n+r}, r = 0\left(\frac{1}{4}\right)1$ and Interpolating Equation (2) at $x_{n+s}, s = 0, \frac{1}{4}$, gives a system of non linear equation of the form

$$AX = U \qquad (5)$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]^T,$$

and

$$U = \left[y_n, y_{n+\frac{1}{4}}, f_n, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}, g_n, g_{n+\frac{1}{4}}, g_{n+\frac{1}{2}}, g_{n+\frac{3}{4}}, g_{n+1}\right]^T,$$

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} \\
1 & x_{n+\frac{1}{4}} & x^2_{n+\frac{1}{4}} & x^3_{n+\frac{1}{4}} & x^4_{n+\frac{1}{4}} & x^5_{n+\frac{1}{4}} & x^6_{n+\frac{1}{4}} & x^7_{n+\frac{1}{4}} & x^8_{n+\frac{1}{4}} & x^9_{n+\frac{1}{4}} & x^{10}_{n+\frac{1}{4}} & x^{11}_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 \\
0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x^2_{n+\frac{1}{4}} & 20x^3_{n+\frac{1}{4}} & 30x^4_{n+\frac{1}{4}} & 42x^5_{n+\frac{1}{4}} & 56x^6_{n+\frac{1}{4}} & 72x^7_{n+\frac{1}{4}} & 90x^8_{n+\frac{1}{4}} & 110x^9_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} & 42x^5_{n+\frac{1}{2}} & 56x^6_{n+\frac{1}{2}} & 72x^7_{n+\frac{1}{2}} & 90x^8_{n+\frac{1}{2}} & 110x^9_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12x^2_{n+\frac{3}{4}} & 20x^3_{n+\frac{3}{4}} & 30x^4_{n+\frac{3}{4}} & 42x^5_{n+\frac{3}{4}} & 56x^6_{n+\frac{3}{4}} & 72x^7_{n+\frac{3}{4}} & 90x^8_{n+\frac{3}{4}} & 110x^9_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} & 72x^7_{n+1} & 90x^8_{n+1} & 110x^9_{n+1} \\
0 & 0 & 0 & 6 & 24x_n & 60_n^2 & 120_n^3 & 210_n^4 & 336_n^5 & 504_n^6 & 720_n^7 & 990_n^8 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{4}} & 60^2_{n+\frac{1}{4}} & 120^3_{n+\frac{1}{4}} & 210^4_{n+\frac{1}{4}} & 336^5_{n+\frac{1}{4}} & 504^6_{n+\frac{1}{4}} & 720^7_{n+\frac{1}{4}} & 990^8_{n+\frac{1}{4}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{2}} & 60^2_{n+\frac{1}{2}} & 120^3_{n+\frac{1}{2}} & 210^4_{n+\frac{1}{2}} & 336^5_{n+\frac{1}{2}} & 504^6_{n+\frac{1}{2}} & 720^7_{n+\frac{1}{2}} & 990^8_{n+\frac{1}{2}} \\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{4}} & 60^2_{n+\frac{3}{4}} & 120^3_{n+\frac{3}{4}} & 210^4_{n+\frac{3}{4}} & 336^5_{n+\frac{3}{4}} & 504^6_{n+\frac{3}{4}} & 720^7_{n+\frac{3}{4}} & 990^8_{n+\frac{3}{4}} \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60^2_{n+1} & 120^3_{n+1} & 210^4_{n+1} & 336^5_{n+1} & 504^6_{n+1} & 720^7_{n+1} & 990^8_{n+1}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{bmatrix}
=
\begin{bmatrix} y_n \\ y_{n+\frac{1}{4}} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \\ g_n \\ g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix}$$

Solving (5) for $a_i's$ using Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$y(x) = \sum_{j=0,\frac{1}{4}} \alpha_j y_{n+j} + h^2\left[\sum_{j=0}^{1} \beta_j f_{n+j} + \beta_k f_{n+k}\right] + h^3\left[\sum_{j=0}^{1} \gamma_j g_{n+j} + \gamma_k g_{n+k}\right],\ k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \qquad (6)$$

Differentiating (6) once yields

$$p'(x) = \frac{1}{h}\sum_{j=0,\frac{1}{4}} \alpha_j y_{n+j} + h\left[\sum_{j=\frac{1}{4},\frac{1}{2},\frac{3}{4}} \beta_j f_{n+j} + \sum_{j=0}^{1} \beta_j f_{n+j}\right] + h^2\left[\sum_{j=\frac{1}{4},\frac{1}{2},\frac{3}{4}} \gamma_j g_{n+j} + \sum_{j=0}^{1} \gamma_j g_{n+j}\right] \qquad (7)$$

where

. $$\alpha_0 = 1 - \frac{4(x - x_n)}{h}$$

$$\alpha_{\frac{1}{4}} = \frac{4\,(x - x_n)}{h}$$

$$\beta_0 = -\frac{2602339}{38320128}(x - x_n)\,h + \frac{1}{2}(x - x_n)^2 - \frac{485}{36}\frac{(x - x_n)^4}{h^2} + \frac{4031}{54}\frac{(x - x_n)^5}{h^3} - \frac{85862}{405}\frac{(x - x_n)^6}{h^4}$$
$$+ \frac{208100}{567}\frac{(x - x_n)^7}{h^5} - \frac{75920}{189}\frac{(x - x_n)^8}{h^6} + \frac{66080}{243}\frac{(x - x_n)^9}{h^7} - \frac{126464}{1215}\frac{(x - x_n)^{10}}{h^8} + \frac{5120}{297}\frac{(x - x_n)^{11}}{h^9}$$

$$\beta_{\frac{1}{4}} = -\frac{148231}{11975040}(x-x_n)h - \frac{128}{9}\frac{(x-x_n)^4}{h^2} + \frac{7168}{45}\frac{(x-x_n)^5}{h^3} - \frac{270592}{405}\frac{(x-x_n)^6}{h^4} + \frac{841216}{567}\frac{(x-x_n)^7}{h^5}$$
$$-\frac{364288}{189}\frac{(x-x_n)^8}{h^6} + \frac{358912}{243}\frac{(x-x_n)^9}{h^7} - \frac{753664}{1215}\frac{(x-x_n)^{10}}{h^8} + \frac{32768}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\beta_{\frac{1}{2}} = -\frac{1807}{80640}(x-x_n)h + \frac{12(x-x_n)^4}{h^2} - \frac{456}{5}\frac{(x-x_n)^5}{h^3} + \frac{4424}{15}\frac{(x-x_n)^6}{h^4} - \frac{10496}{21}\frac{(x-x_n)^7}{h^5}$$
$$+\frac{3264}{7}\frac{(x-x_n)^8}{h^6} - \frac{2048}{9}\frac{(x-x_n)^9}{h^7} + \frac{2048}{45}\frac{(x-x_n)^{10}}{h^8}$$

$$\beta_{\frac{3}{4}} = -\frac{243193}{11975040}(x-x_n)h + \frac{128}{9}\frac{(x-x_n)^4}{h^2} - \frac{17408}{135}\frac{(x-x_n)^5}{h^3} + \frac{213248}{405}\frac{(x-x_n)^6}{h^4}$$
$$-\frac{685568}{567}\frac{(x-x_n)^7}{h^5} + \frac{313088}{189}\frac{(x-x_n)^8}{h^6} - \frac{326144}{243}\frac{(x-x_n)^9}{h^7} + \frac{720896}{1215}\frac{(x-x_n)^{10}}{h^8} - \frac{32768}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\beta_1 = -\frac{382169}{191600640}(x-x_n)h + \frac{53}{36}\frac{(x-x_n)^4}{h^2} - \frac{1241}{90}\frac{(x-x_n)^5}{h^3} + \frac{23758}{405}\frac{(x-x_n)^6}{h^4}$$
$$-\frac{80356}{567}\frac{(x-x_n)^7}{h^5} + \frac{38992}{189}\frac{(x-x_n)^8}{h^6} - \frac{43552}{243}\frac{(x-x_n)^9}{h^7} + \frac{103936}{1215}\frac{(x-x_n)^{10}}{h^8} - \frac{5120}{297}\frac{(x-x_n)^{11}}{h^9}$$

$$\gamma_0 = -\frac{28343}{18247680}(x-x_n)h^2 + \frac{1}{6}(x-x_n)^3 - \frac{25}{18}\frac{(x-x_n)^4}{h} + \frac{209}{36}\frac{(x-x_n)^5}{h^2} - \frac{398}{27}\frac{(x-x_n)^6}{h^3}$$
$$+\frac{4546}{189}\frac{(x-x_n)^7}{h^4} - \frac{1600}{63}\frac{(x-x_n)^8}{h^5} + \frac{1360}{81}\frac{(x-x_n)^9}{h^6} - \frac{512}{81}\frac{(x-x_n)^{10}}{h^7} + \frac{512}{495}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_{\frac{1}{4}} = \frac{551}{49896}(x-x_n)h^2 - \frac{16}{3}\frac{(x-x_n)^4}{h} + \frac{608}{15}\frac{(x-x_n)^5}{h^2} - \frac{18848}{135}\frac{(x-x_n)^6}{h^3} + \frac{7424}{27}\frac{(x-x_n)^7}{h^4}$$
$$-\frac{20768}{63}\frac{(x-x_n)^8}{h^5} + \frac{19328}{81}\frac{(x-x_n)^9}{h^6} - \frac{38912}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{8192}{495}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_{\frac{1}{2}} = \frac{32027}{3548160}(x-x_n)h^2 - \frac{6(x-x_n)^4}{h} + \frac{264}{5}\frac{(x-x_n)^5}{h^2} - \frac{3124}{15}\frac{(x-x_n)^6}{h^3} + \frac{3224}{7}\frac{(x-x_n)^7}{h^4}$$
$$-\frac{608(x-x_n)^8}{h^5} + \frac{4288}{9}\frac{(x-x_n)^9}{h^6} - \frac{1024}{5}\frac{(x-x_n)^{10}}{h^7} + \frac{2048}{55}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_{\frac{3}{4}} = \frac{3959}{1596672}(x-x_n)h^2 - \frac{16}{9}\frac{(x-x_n)^4}{h} + \frac{736}{45}\frac{(x-x_n)^5}{h^2} - \frac{9184}{135}\frac{(x-x_n)^6}{h^3} + \frac{30208}{189}\frac{(x-x_n)^7}{h^4}$$
$$-\frac{14176}{63}\frac{(x-x_n)^8}{h^5} + \frac{15232}{81}\frac{(x-x_n)^9}{h^6} - \frac{34816}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{8192}{495}\frac{(x-x_n)^{11}}{h^8}$$

$$\gamma_1 = \frac{14339}{127733760}(x-x_n)h^2 - \frac{1}{12}\frac{(x-x_n)^4}{h} + \frac{47}{60}\frac{(x-x_n)^5}{h^2} - \frac{452}{135}\frac{(x-x_n)^6}{h^3} + \frac{1538}{189}\frac{(x-x_n)^7}{h^4}$$
$$-\frac{752}{63}\frac{(x-x_n)^8}{h^5} + \frac{848}{81}\frac{(x-x_n)^9}{h^6} - \frac{2048}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{512}{495}\frac{(x-x_n)^{11}}{h^8}$$

Equation (6) is evaluated at the non-interpolating points $\left\{x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\right\}$ and (7) at all points $\left\{x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}\right\}$, produces the following general equations in block form

$$AY_L = BR_1 + CR_2 + DR_3 + ER_4 + GR_5 \qquad (8)$$

Where

$$A=\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{4}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Y=\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{3}{4}} \\ y'_{n+1} \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ \frac{4}{h} & 1 \\ \frac{4}{h} & 0 \\ \frac{4}{h} & 0 \\ \frac{4}{h} & 0 \\ \frac{4}{h} & 0 \end{bmatrix}, \quad R_1=\begin{bmatrix} y_n \\ y'_n \end{bmatrix}, \quad C=\begin{bmatrix} \frac{37111h^2}{6967296} \\ \frac{12679h^2}{1161216} \\ \frac{19709h^2}{1161216} \\ -\frac{260233h}{38320128} \\ \frac{1961683h}{95800320} \\ \frac{1403419h}{63866880} \\ \frac{2182739h}{95800320} \\ \frac{5021969h}{191600640} \end{bmatrix}, \quad R_2=[f_n]$$

$$D=\begin{bmatrix} \frac{16463h^2}{435456} & \frac{445h^2}{32256} & \frac{2225h^2}{435456} & \frac{3217h^2}{6967296} \\ \frac{6149h^2}{72576} & \frac{767h^2}{10752} & \frac{1403h^2}{72576} & \frac{1381h^2}{1161216} \\ \frac{9925h^2}{72576} & \frac{767h^2}{5376} & \frac{5179h^2}{72576} & \frac{8411h^2}{1161216} \\ \frac{-148231h}{11975040} & \frac{-1807h}{80640} & \frac{-243193h}{11975040} & \frac{-382169h}{191600640} \\ \frac{83485h}{1197504} & \frac{1489h}{80640} & \frac{89279h}{5987520} & \frac{137161h}{95800320} \\ \frac{26921h}{147840} & \frac{767h}{5376} & \frac{103853h}{3991680} & \frac{5303h}{2365440} \\ \frac{1156801h}{5987520} & \frac{21521h}{80640} & \frac{165731h}{1197504} & \frac{358217h}{95800320} \\ \frac{2735353h}{11975040} & \frac{24817h}{80640} & \frac{2640391h}{11975040} & \frac{3530299h}{38320128} \end{bmatrix}, \quad R_3=\begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \quad E=\begin{bmatrix} \frac{4253h^3}{23224320} \\ \frac{23h^3}{60480} \\ \frac{467h^3}{774144} \\ \frac{-28343h^2}{18247680} \\ \frac{551h^2}{798336} \\ \frac{3629h^2}{4730880} \\ \frac{3239h^2}{3991680} \\ \frac{25651h^2}{25546752} \end{bmatrix}, \quad R_4=[g_n]$$

$$G=\begin{bmatrix} \frac{-17h^3}{9072} & \frac{-83h^3}{30720} & \frac{-871h^3}{1451520} & \frac{-599h^3}{23224320} \\ \frac{-3053h^3}{967680} & \frac{-83h^3}{15360} & \frac{-347h^3}{193536} & \frac{-127h^3}{1935360} \\ \frac{-29h^3}{7560} & \frac{-83h^3}{15360} & \frac{-269h^3}{241920} & \frac{-1117h^3}{3870720} \\ \frac{551h^2}{49896} & \frac{32027h^2}{3548160} & \frac{3959h^2}{1596672} & \frac{14339h^2}{127733760} \\ \frac{-23851h^2}{2280960} & \frac{-12053h^2}{1774080} & \frac{-5755h^2}{3193344} & \frac{-2567h^2}{31933440} \\ \frac{-95h^2}{16632} & \frac{-56677h^2}{3548160} & \frac{-7951h^2}{2661120} & \frac{-5311h^2}{42577920} \\ \frac{-72269h^2}{15966720} & \frac{-12053h^2}{1774080} & \frac{-123463h^2}{15966720} & \frac{-6439h^2}{31933440} \\ \frac{-61h^2}{249480} & \frac{32027h^2}{3548160} & \frac{15701h^2}{1140480} & \frac{-312317h^2}{127733760} \end{bmatrix}, \quad R_5=\begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix}$$

$$IY_L=\bar{B}R_1+\bar{C}R_2+\bar{D}R_3+\bar{E}R_4+\bar{G}R_5$$

$$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{4}}\\y_{n+\frac{1}{2}}\\y_{n+\frac{3}{4}}\\y_{n+1}\\y'_{n+\frac{1}{4}}\\y'_{n+\frac{1}{2}}\\y'_{n+\frac{3}{4}}\\y'_{n+1}\end{bmatrix}=\begin{bmatrix}1&\frac{h}{4}\\1&\frac{h}{2}\\1&\frac{3h}{4}\\1&h\\0&1\\0&1\\0&1\\0&1\end{bmatrix}\begin{bmatrix}y_n\\y'_n\end{bmatrix}+\begin{bmatrix}\frac{2602339h^2}{153280512}\\\frac{35h^2}{891}\\\frac{39015h^2}{630784}\\\frac{6353h^2}{74844}\\\frac{1539551h}{17418240}\\\frac{24463h}{272160}\\\frac{6501h}{71680}\\\frac{1601h}{17010}\end{bmatrix}\left[f_n\right]$$

$$+\begin{bmatrix}\frac{148231h^2}{47900160}&\frac{1807h^2}{322560}&\frac{243193h^2}{47900160}&\frac{382169h^2}{766402560}\\\frac{196h^2}{4455}&\frac{h^2}{40}&\frac{68h^2}{4455}&\frac{13h^2}{8910}\\\frac{18531h^2}{197120}&\frac{3159h^2}{35840}&\frac{6813h^2}{197120}&\frac{8469h^2}{3153920}\\\frac{13952h^2}{93555}&\frac{52h^2}{315}&\frac{8578h^2}{93555}&\frac{3457h^2}{374220}\\\frac{89371h}{1088640}&\frac{103h}{2520}&\frac{38341h}{1088640}&\frac{59681h}{17418240}\\\frac{1654h}{8505}&\frac{52h}{315}&\frac{394h}{8505}&\frac{1153h}{272160}\\\frac{921h}{4480}&\frac{81h}{280}&\frac{711h}{4480}&\frac{411h}{71680}\\\frac{2048h}{8505}&\frac{104h}{315}&\frac{2048h}{8505}&\frac{1601h}{17010}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{4}}\\f_{n+\frac{1}{2}}\\f_{n+\frac{3}{4}}\\f_{n+1}\end{bmatrix}+\begin{bmatrix}\frac{28343h^3}{72990720}\\\frac{1277h^3}{1330560}\\\frac{9747h^3}{6307840}\\\frac{269h^2}{124740}\\\frac{26051h^2}{11612160}\\\frac{421h^2}{181440}\\\frac{339h^2}{143360}\\\frac{29h^2}{11340}\end{bmatrix}\left[g_n\right]+\begin{bmatrix}\frac{-551h^3}{199584}&\frac{-32027h^3}{14192640}&\frac{-3959h^3}{6386688}&\frac{-14339h^2}{510935040}\\\frac{-41h^3}{5544}&\frac{-5h^3}{693}&\frac{-17h^3}{9240}&\frac{-109h^2}{1330560}\\\frac{-4509h^3}{394240}&\frac{-19197h^3}{1576960}&\frac{-9h^3}{2464}&\frac{-27h^2}{180224}\\\frac{-464h^3}{31185}&\frac{-10h^3}{693}&\frac{-16h^3}{4455}&\frac{-5h^2}{12474}\\\frac{-31207h^2}{1451520}&\frac{-81h^2}{1520}&\frac{-1243h^2}{290304}&\frac{-2237h^2}{11612160}\\\frac{-19h^2}{1134}&\frac{-h^2}{40}&\frac{-31h^2}{5670}&\frac{-43h^2}{181440}\\\frac{-279h^2}{17920}&\frac{-81h^2}{1520}&\frac{-183h^2}{17920}&\frac{-9h^2}{28672}\\\frac{-32h^2}{2835}&0&\frac{32h^2}{2835}&\frac{-29h^2}{11340}\end{bmatrix}\begin{bmatrix}g_{n+\frac{1}{4}}\\g_{n+\frac{1}{2}}\\g_{n+\frac{3}{4}}\\g_{n+1}\end{bmatrix}$$

III. Analysis of Basic Properties of the Method

3.1 Order of the Block

According to [7] the order of the new method in Equation (8) is obtained by using the Taylor series and it is found that the developed method has a uniformly order eleven, with an error constants vector of:

$$C_{11}=\begin{bmatrix}6.1829\times10^{-14},1.752\times10^{-13},3.0454\times10^{-13}, & 4.8542\times10^{-13},\\ 4.1791\times10^{-13},4.8542\times10^{-13},5.5292\times10^{-13},9.7083\times10^{-13}\end{bmatrix}^T$$

3.2 Consistency

The hybrid block method [4] is said to be consistent if it has an order more than or equal to one.
Therefore, our method is consistent.

3.3 Zero Stability of Our Method

Definition: A block method is said to be zero-stable if as $h\to0$, the root $z_i, i=1(1)k$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k}A^{(i)}z^{k-i}\right]=0$

Satisfies $|z_i|\le1$ and for those roots with $|z_i|=1$, multiplicity must not exceed two. The block method for k=1, with three off grid collocation point expressed in the form

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&1&0&0&0&\frac{h}{4}\\0&0&0&1&0&0&0&\frac{h}{2}\\0&0&0&1&0&0&0&\frac{3h}{4}\\0&0&0&1&0&0&0&h\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{bmatrix}\right|=z^6(z-1)^2$$

$$\rho(z)=z^6(z-1)^2=0,$$

Hence, our method is zero-stable.

3.3 Regions of Absolute Stability (RAS)

Using Mat Lab package, we were able to plot the stability region of the block methods (see figs below).
The stability polynomial for K=1 with three offstep point using Scientific Workplace soft ware package we have

$$= h^{12}\left(\left(\frac{7219}{15107327262720000}\right)w^4 - \left(\frac{79}{156959244288000}\right)w^3\right) - h^{11}\left(\left(\frac{3423139}{139591703907532800}\right)w^4 + \left(\frac{1171}{30519853056000}\right)w^3\right)$$

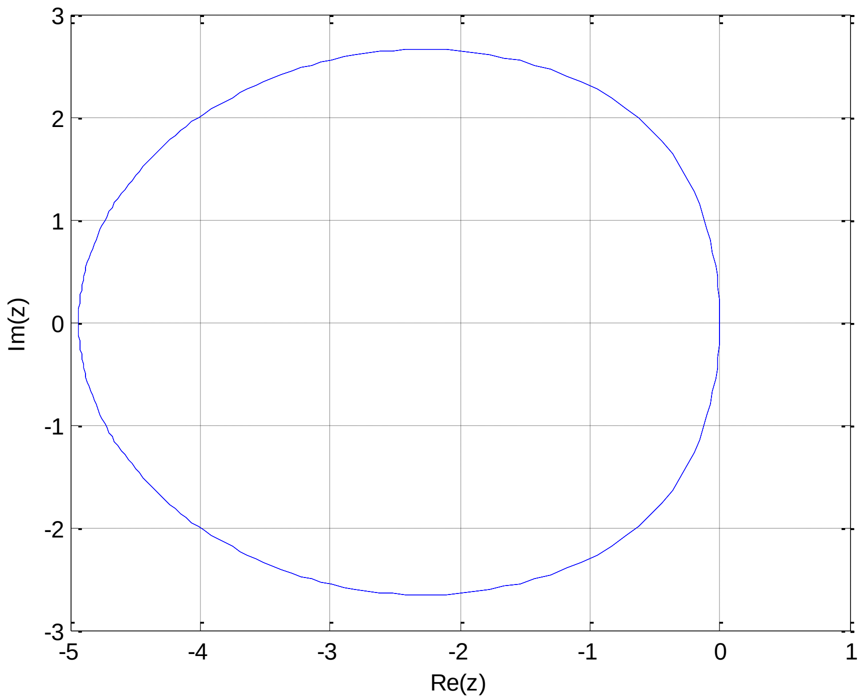
$$- h^{10}\left(\left(\frac{4666483}{3625758543052800}\right)w^3 + \left(\frac{585349}{23930006384148480}\right)w^4\right) + h^9\left(\left(\frac{73634299}{2913555972096000}\right)w^3 - \left(\frac{2899283}{1495625399009280}\right)w^3\right)$$

$$- h^8\left(\left(\frac{12604072069}{149562539900928000}\right)w^4 + \left(\frac{6897035147}{8157956721868800}\right)w^3\right) - h^7\left(\left(\frac{775601}{9442079539200}\right)w^4 + \left(\frac{707100649}{109258348953600}\right)w^3\right)$$

$$+ h^6\left(\left(\frac{1953881}{256927334400}\right)w^4 - \left(\frac{26693444639}{145677798604800}\right)w^3\right) - h^5\left(\left(\frac{2850443}{36883123200}\right)w^4 + \left(\frac{357859}{1077753600}\right)w^3\right)$$

$$- h^4\left(\left(\frac{2587}{4257792}\right)w^4 + \left(\frac{12221035}{1034643456}\right)w^3\right) + h^3\left(\left(\frac{25}{1782}\right)w^4 + \left(\frac{269}{114740}\right)w^3\right) - h^2\left(\left(\frac{16399}{228096}\right)w^4 + \left(\frac{1009733}{4790016}\right)w^3\right) + w^4 - w^3$$

Using Mat Lab software, the absolute stability region of the new method is plotted and shown in figure 2.



3.3 Numerical Example

Problem I. We consider a highly stiff problem

$y''+1001y'+1000y, \quad y(0)=1, y'(0)=-1$

Exact Solution: $y(x)=\exp(-x)\ h=\frac{1}{10}$

Table 2. Comparison of the proposed method with

x-values	Exact Solution	Computed Solution	Error in our method	Error in [13]
0.100	0.904837418035957316	0.904837418035957245	7.100E(-19)	1.054712E(-14)
0.200	0.818730753077981858 67	0.818730753077981857 95	7.200E(-19)	1.776357E(-14)
0.300	0.740818220681717866 07	0.740818220681717865 34	7.300E(-19)	2.342571E(-14)
0.400	0.670320046035639300 74	0.670320046035639300 01	7.300E(-19)	2.797762E(-14)
0.500	0.606530659712633423 60	0.606530659712633422 85	7.500E(-19)	3.130829E(-14)
0.600	0.548811636094026432 63	0.548811636094026431 85	7.800E(-19)	3.397282E(-14)
0.700	0.496585303791409514 70	0.496585303791409513 90	8.000E(-19)	3.563816E(-14)
0.800	0.449328964117221591 43	0.449328964117221590 58	8.500E(-19)	3.674838E(-14)
0.900	0.406569659740599111 88	0.406569659740599110 99	8.900E(-19)	3.730349E(-14)
1.00	0.367879441171442321 60	0.367879441171442320 65	9.500E(-19)	3.741452E(-14)

Problem II. $f(x,y,y')=y'$, $y(0)=1$, $y'(0)=\frac{1}{2}$, $0\le x\le 1$.

Exact Solution: $y(x)=1-e^{x}$ *with* $h=\frac{1}{100}$

x-values	Exact Solution	Computed Solution	Error in our method	Error in [16]
.0.100	-0.10517091807564762480	-0.10517091807564762481	1.000E(-20)	8.326679(-17)
0.200	-0.22140275816016983390	-0.22140275816016983391	1.000E(-20)	2.775557(-16)
0.300	-0.34985880757600310400	-0.34985880757600310397	-3.000E(-20)	5.551115(-16)
0.400	-0.49182469764127031780	-0.49182469764127031779	-1.000E(-20)	9.436896(-16)
0.500	-0.64872127070012814680	-0.64872127070012814679	-1.000E(-20)	2.109424(-15)
0.600	-0.82211880039050897490	-0.82211880039050897478	-1.000E(-19)	3.219647(-15)
0.700	-1.01375270747047652160	-1.01375270747047652150	-1.000E(-19)	4.440892(-15)
0.800	-1.22554092849246760460	-1.22554092849246760440	-2.000E(-19)	5.995204(-15)
0.900	-1.45960311115694966380	-1.45960311115694966350	-3.000E(-19)	7.771561(-15)
1.00	-1.71828182845904523540	-1.71828182845904523500	-4.000E(-19)	1.065814(-14)

IV. CONCLUSIONS

It is evident from the above tables that our proposed methods are indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is $A-stable$. Comparing the new method with the existing method [13,16],the result presented in the tables 1 and 2 shows that the new method performs better than the existing method [13,16]and even the order of new method is higher than the order of the existing method [13,16]. In this article, a one-step block method with three off-step points is derived via the interpolationand collocation approach. The developed method is consistent, $A-stable$, convergent, with a region of absolute stability and order Ten.

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