Directional q-Derivative

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***Abstract*— In this paper partial q-derivative of a two variable function f and directional q-derivative of function f at the point $P=(p_1,p_2)$ in the direction of a unit vector are introduced and some properties of q- directional derivative are investigated.**

***Index Terms*—Partial q-derivative, directional q- derivative.**

I. INTRODUCTION

A quantum calculus is a version of calculus in which we do not take limits. Derivatives are differences and anti derivatives are sums. It is a theory, where smoothness is no more required[1].

The general idea in this paper is to generalize the concept d-derivative of a real function f to a two variable function and to construct q- directional derivative of a function.

II. PRELIMINARIES

Consider an arbitrary function $f(x)$. The q-derivative $D_q f$ of the function $f(x)$ is given by

$$\left(D_q f\right)(x)=\frac{f(qx)-f(x)}{qx-x}, \qquad (1)$$

if $x\neq 0$ and $\left(D_q f\right)(0)=f'(0)$ provided $f'(0)$ exists. Note that

$$\lim_{q\to 1} D_q f(x)=\frac{d}{dx}f(x)$$

if $f(x)$ differentiable[2]. The Leibniz notation $\frac{d}{dx}f(x)$, a ratio of two "infinitesimals" is rather confusing, since the notion of the differential $df(x)$ requires an elaborate explanation. In contrast, the notion of q-differential is obvious and plain ratio[3].

It is clear that as with ordinary derivative, the action of taking the q-derivative of a function is a linear operator. In other words, for any constants a and b, we have

$$D_q\{af(x)+bg(x)\}=aD_q f(x)+bD_q g(x)$$

The formulas for the q-derivative of a product and a quotient of functions are [3]

$$D_q\{f(x)g(x)\}=D_q f(x)g(x)+f(qx)D_q g(x)$$

and

$$D_q\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(x)D_q f(x)-f(x)D_q g(x)}{g(qx)g(x)}.$$

If f is q-differentiable at x, then

$$f(qx)=f(x)+(q-1)xD_q f(x).$$

The q-analogue of the chain rule is more complicated since it involves q-derivatives for different values of q depending on the composed functions. The chain rule for general functions $f(x)$ and $g(x)$ is[3]

$$D_q(f\circ g)(x)=D_{\frac{g(qx)}{g(x)}}f\big(g(x)\big)D_q\,g(x).$$

III. PARTIAL Q-DERIVATIVE

In this section we will define to partial q-derivative of a two variable functions by using the definition one variable case and give a version of a chain rule for two variable functions.

For i=1,2, I_i is a nonempty closed subset of the real numbers $\mathbb{R}$. Let us set

$$I^2=I_1\times I_2=\{t=(t_1,t_2): t_i\in I_i, i=1,2\}.$$

Definition 3.1. Let $f: I^2\to\mathbb{R}$ be a two variable function. The partial q-derivative of f with respect to t_1 and t_2 is defined by

$$\frac{\partial f(t)}{\partial_{q_1}t_1}=\frac{f(q_1t_1,t_2)-f(t_1,t_2)}{q_1t_1-t_1}$$

and

$$\frac{\partial f(t)}{\partial_{q_2}t_2}=\frac{f(t_1,q_2t_2)-f(t_1,t_2)}{q_2t_2-t_2}$$

respectively.

Note that

$$\lim_{q_i\to 1}\frac{\partial f(t)}{\partial_{q_i}t_i}=\frac{\partial f(t)}{\partial t_i}, \qquad i=1,2$$

if $f(t)$ differentiable.

Lemma 3.2 Let $f,g: I^2\to\mathbb{R}$ are two variable functions. Then, for $a,b\in\mathbb{R}$, i=1,2,

$$\frac{\partial}{\partial_{q_i}t_i}\{af(t)\pm bg(t)\}=a\frac{\partial f(t)}{\partial_{q_i}t_i}\pm b\frac{\partial g(t)}{\partial_{q_i}t_i}$$

and

$$\frac{\partial}{\partial_{q_1}t_1}\{f(t)g(t)\}=f(q_1t_1,t_2)\frac{\partial g(t)}{\partial_{q_1}t_1}+g(t)\frac{\partial f(t)}{\partial_{q_1}t_1},$$
$$\frac{\partial}{\partial_{q_2}t_2}\{f(t)g(t)\}=f(t_1,q_2t_2)\frac{\partial g(t)}{\partial_{q_2}t_2}+g(t)\frac{\partial f(t)}{\partial_{q_2}t_2}.$$

Proof: By the Definition 3.1. we get easily linearity. And the partial q-derivative of product f and g is

$$\frac{\partial(fg)(t_1,t_2)}{\partial_{q_1}t_1}=\frac{(fg)(q_1t_1,t_2)-(fg)(t_1,t_2)}{q_1t_1-t_1}$$
$$=\frac{f(q_1t_1,t_2)g(q_1t_1,t_2)-f(t_1,t_2)g(t_1,t_2)}{q_1t_1-t_1}$$
$$=\frac{f(q_1t_1,t_2)g(q_1t_1,t_2)-f(t_1,t_2)g(t_1,t_2)\pm f(q_1t_1,t_2)g(t_1,t_2)}{q_1t_1-t_1}$$

$$=\frac{f(q_1t_1,t_2)g(q_1t_1,t_2)-f(q_1t_1,t_2)g(t_1,t_2)}{q_1t_1-t_1}$$
$$+\frac{f(q_1t_1,t_2)g(t_1,t_2)-f(t_1,t_2)g(t_1,t_2)}{q_1t_1-t_1}$$
$$=f(q_1t_1,t_2)\frac{\{g(q_1t_1,t_2)-g(t_1,t_2)\}}{q_1t_1-t_1}$$
$$+\frac{\{f(q_1t_1,t_2)-f(t_1,t_2)\}}{q_1t_1-t_1}g(t_1,t_2)$$
$$=f(q_1t_1,t_2)\frac{\partial g(t_1,t_2)}{\partial_{q_1}t_1}+g(t_1,t_2)\frac{\partial f(t_1,t_2)}{\partial_{q_1}t_1}.$$

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Lemma 3.3. Let $u_1(t)$ and $u_2(t)$ are real functions and $f: I^2 \to \mathbb{R}$ be a two variable function. Then $f(u_1(t), u_2(t))$ is a real function of variable t and
$D_q f(u_1(t), u_2(t)) = \frac{\partial f(u_1(t), u_2(qt))}{\partial_{q_1^*} u_1} D_q u_1(t) + \frac{\partial f(u_1(t), u_2(t))}{\partial_{q_2^*} u_1} D_q u_2(t).$

Proof: Let $g(t) = f(u_1(t), u_2(t))$. Then the q-derivative of $g(t)$, we have

$$D_q g(t) = \frac{g(qt) - g(t)}{qt - t}$$
$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t))}{qt - t}$$
$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t)) \pm f(u_1(t), u_2(qt))}{qt - t}$$
$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(qt))}{qt - t}$$
$$+ \frac{f(u_1(t), u_2(qt)) - f(u_1(t), u_2(t))}{qt - t} = R_1 + R_2.$$

By the chain rule, we obtain

$$R_1 = \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(qt))}{qt - t} \frac{(u_1(qt) - u_1(t))}{(u_1(qt) - u_1(t))}$$
$$= \frac{f\left(u_1(qt)\frac{u_1(t)}{u_1(t)}, u_2(qt)\right) - f(u_1(t), u_2(qt))}{\left(\frac{u_1(qt)}{u_1(t)} - 1\right) u_1(t)} \frac{(u_1(qt) - u_1(t))}{qt - t}$$
$$= \frac{f(q_1^* u_1(t), u_2(qt)) - f(u_1(t), u_2(qt))}{(q_1^* - 1) u_1(t)} \frac{(u_1(qt) - u_1(t))}{qt - t}$$
$$= \frac{\partial f(u_1(t), u_2(qt))}{\partial_{q_1^*} u_1} D_q u_1(t)$$

and

$$R_2 = \frac{\partial f(u_1(t), u_2(t))}{\partial_{q_2^*} u_1} D_q u_2(t)$$

where $q_1^* = \frac{u_1(qt)}{u_1(t)}$ and $q_2^* = \frac{u_2(qt)}{u_2(t)}$. Hence
$D_q f(u_1(t), u_2(t)) = \frac{\partial f(u_1(t), u_2(qt))}{\partial_{q_1^*} u_1} D_q u_1(t) + \frac{\partial f(u_1(t), u_2(t))}{\partial_{q_2^*} u_1} D_q u_2(t).$

IV. Directional q-Derivative

Definition 4.1. Let $f: I^2 \to \mathbb{R}$ be a two variable function. The directional q-derivative of f function at the point $P = (p_1, p_2)$ in the direction of the unit vector $\vec{v} = (v_1, v_2)$ is defined as the number

$$\frac{\partial f(P)}{\partial_q \vec{v}} = D_q f(P + \lambda v)\big|_{\lambda=0}.$$

Theorem 4.2. Let $f: I^2 \to \mathbb{R}$ be a two variable function. The directional q-derivative of f function at the point $P = (p_1, p_2)$ in the direction of the unit vector $\vec{v} = (v_1, v_2)$ is

$$\frac{\partial f(P)}{\partial_q \vec{v}} = \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$

Proof: $D_q u_i(\lambda)\big|_{\lambda=0} = v_i$, since $u_i(\lambda) = p_i + \lambda v_i$,i=1,2. Then by the Lemma 3.3 the theorem is proved.

Theorem 4.3. Let $a, b \in \mathbb{R}$, $f, g: I^2 \to \mathbb{R}$ are two variable function, $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ are unit vectors. Then

i) $\frac{\partial f(P)}{\partial_q \overrightarrow{v+w}} = \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial f(P)}{\partial_q \vec{w}}$

ii) $\frac{\partial (f+g)(P)}{\partial_q \vec{v}} = \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial g(P)}{\partial_q \vec{v}}$

Proof: i) By the Definition 4.1 and Theorem 4.2, we have

$$\frac{\partial f(P)}{\partial_q \overrightarrow{v+w}} = \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1}(v_1 + w_1) + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2}(v_2 + w_2)$$
$$= \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1}(v_1) + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2}(v_2)$$
$$+ \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1}(w_1) + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2}(w_2)$$
$$= \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial f(P)}{\partial_q \vec{w}}.$$

ii)

$$\frac{\partial (f+g)(P)}{\partial_q \vec{v}} = \frac{\partial (f+g)(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial (f+g)(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$
$$= \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$
$$+ \frac{\partial (g)(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial (g)(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$
$$= \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial g(P)}{\partial_q \vec{v}}$$

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