Abstract—In this paper partial q-derivative of a two variable function \( f \) and directional q-derivative of function \( f \) at the point \( P = (p_1, p_2) \) in the direction of a unit vector are introduced and some properties of q-directional derivative are investigated.

Index Terms—Partial q-derivative, directional q-derivative.

I. INTRODUCTION

A quantum calculus is a version of calculus in which we do not take limits. Derivatives are differences and anti-derivatives are sums. It is a theory, where smoothness is no more required[1].

The general idea in this paper is to generalize the concept d-derivative of a real function \( f \) to a two variable function and to construct q-directional derivative of a function.

II. PRELIMINARIES

Consider an arbitrary function \( f(x) \). The q-derivative \( D_q f \) of the function \( f(x) \) is given by

\[
(D_q f)(x) = \frac{f(qx) - f(x)}{qx - x},
\]

if \( x \neq 0 \) and \( (D_q f)(0) = f'(0) \) provided \( f'(0) \) exists. Note that

\[
\lim_{q \to 1} D_q f(x) = \frac{df}{dx}(x)
\]

if \( f(x) \) differentiable[2]. The Leibniz notation \( \frac{d}{dx} \), a ratio of two "infinitesimals" is rather confusing, since the notion of the differential \( df(x) \) requires an elaborate explanation. In contrast, the notion of q-differential is obvious and plain ratio[3].

It is clear that as with ordinary derivative, the action of taking the q-derivative of a function is a linear operator. In other words, for any constants \( a \) and \( b \), we have

\[
D_q(af(x) + bg(x)) = aD_q f(x) + bD_q g(x)
\]

The formulas for the q-derivative of a product and a quotient of functions are [3]

\[
D_q \left( \frac{f(x)g(x)}{g(x)} \right) = \frac{D_q f(x)g(x) + f(x)D_q g(x)}{g(x)g(x)}
\]

and

\[
D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(x)}
\]

If \( f \) is q-differentiable at \( x \), then

\[
f(qx) = f(x) + (q - 1)x D_q f(x).
\]

The q-analogue of the chain rule is more complicated since it involves q-derivatives for different values of \( q \) depending on the composed functions. The chain rule for general functions \( f(x) \) and \( g(x) \) is[3]

\[
D_q(f \circ g)(x) = D_q g \left( \frac{f(qx) - f(x)}{qx - x} \right) D_q g(x).
\]

III. PARTIAL Q-DERIVATIVE

In this section we will define to partial q-derivative of a two variable functions by using the definition one variable case and give a version of a chain rule for two variable functions.

For \( i=1,2 \), \( l_i \) is a nonempty closed subset of the real numbers \( \mathbb{R} \). Let us set

\[
l^2 = l_1 \times l_2 = \{ t = (t_1, t_2) : t_i \in l_i, i = 1,2 \}.
\]

Definition 3.1. Let \( f : l^2 \to \mathbb{R} \) be a two variable function. The partial q-derivative of \( f \) with respect to \( t_1 \) and \( t_2 \) is defined by

\[
\frac{\partial f(t)}{\partial t_1} = \frac{f(q_1 t_1, t_2) - f(t_1, t_2)}{q_1 t_1 - t_1}
\]

and

\[
\frac{\partial f(t)}{\partial t_2} = \frac{f(t_1, q_2 t_2) - f(t_1, t_2)}{q_2 t_2 - t_2}
\]

respectively.

Note that

\[
\lim_{q_i \to 1} \frac{\partial f(t)}{\partial t_i} = \frac{\partial f(t)}{\partial t_i}, \quad i = 1,2
\]

if \( f(t) \) differentiable.

Lemma 3.2 Let \( f, g : l^2 \to \mathbb{R} \) are two variable functions. Then, for \( a, b \in \mathbb{R} \), \( i=1,2 \),

\[
\frac{\partial}{\partial t_i} \left( af(t) \pm bg(t) \right) = a \frac{\partial f(t)}{\partial t_i} \pm b \frac{\partial g(t)}{\partial t_i}
\]

and

\[
\frac{\partial}{\partial t_i} \left( f(t)g(t) \right) = f(q_1 t_1, t_2) \frac{\partial g(t)}{\partial t_i} + g(t) \frac{\partial f(t)}{\partial t_i} + f(t) \frac{\partial g(t)}{\partial t_i} + g(t) \frac{\partial f(t)}{\partial t_i}
\]

Proof: By the Definition 3.1. we get easily linearity. And the partial q-derivative of product \( f \) and \( g \) is

\[
\frac{\partial (fg)(t_1, t_2)}{\partial t_1} = \frac{f(q_1 t_1, t_2)g(t_1, t_2) - f(t_1, t_2)g(q_1 t_1, t_2)}{q_1 t_1 - t_1}
\]

and

\[
\frac{\partial (fg)(t_1, t_2)}{\partial t_2} = \frac{f(q_1 t_1, t_2)g(t_1, t_2) - f(t_1, t_2)g(q_1 t_1, t_2)}{q_1 t_1 - t_1}
\]

on the composed functions. The chain rule for general functions \( f(x) \) and \( g(x) \) is[3]

\[
D_q(f \circ g)(x) = D_q g \left( \frac{f(qx) - f(x)}{qx - x} \right) D_q g(x).
\]
Lemma 3.3. Let $u_1(t)$ and $u_2(t)$ are real functions and $f: t^2 \to \mathbb{R}$ be a two variable function. Then $f(u_1(t), u_2(t))$ is a real function of variable $t$ and
$$D_qf(u_1(t), u_2(t)) = \frac{\partial f(u_1(t), u_2(t))}{\partial q_1 u_1} D_q u_1(t) + \frac{\partial f(u_1(t), u_2(t))}{\partial q_2 u_2} D_q u_2(t).$$

Proof: Let $g(t) = f(u_1(t), u_2(t))$. Then the q-derivative of $g(t)$, we have
$$D_q g(t) = \frac{g(qt) - g(t)}{qt - t} = \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t))}{qt - t} + f(u_1(t), u_2(t))$$
$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t))}{qt - t} + \frac{f(u_1(t), u_2(t))}{qt - t} = R_1 + R_2.$$

By the chain rule, we obtain
$$R_1 = f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t)) \frac{u_1(qt) - u_1(t)}{q-1}$$
and
$$R_2 = \frac{\partial f(u_1(t), u_2(t))}{\partial q_1 u_1} D_q u_1(t) = \frac{\partial f(u_1(qt), u_2(qt))}{\partial q_1 u_1} D_q u_1(t).$$

Definition 4.1. Let $f: t^2 \to \mathbb{R}$ be a two variable function. The directional q-derivative of $f$ function at the point $P = (p_1, p_2)$ in the direction of the unit vector $\vec{v} = (v_1, v_2)$ is defined as the number
$$\frac{\partial f(P)}{\partial q \vec{v}} = D_q f(P + \lambda \vec{v}) \bigg|_{\lambda=0}.$$

Theorem 4.2. Let $f: t^2 \to \mathbb{R}$ be a two variable function. The directional q-derivative of $f$ function at the point $P = (p_1, p_2)$ in the direction of the unit vector $\vec{v} = (v_1, v_2)$ is
$$\frac{\partial f(P)}{\partial q \vec{v}} = \frac{\partial f(q_1 p_1, p_2)}{\partial q_1 u_1} v_1 + \frac{\partial f(p_1, p_2)}{\partial q_2 u_2} v_2.$$

Proof: By the Definition 4.1 and Theorem 4.2, we have
$$\frac{\partial f(P)}{\partial q \vec{v}} + w = \frac{\partial f(q_1 p_1, p_2)}{\partial q_1 u_1} (v_1 + w_1) + \frac{\partial f(p_1, p_2)}{\partial q_2 u_2} (v_2 + w_2)$$
$$= \frac{\partial f(q_1 p_1, p_2)}{\partial q_1 u_1} v_1 + \frac{\partial f(p_1, p_2)}{\partial q_2 u_2} v_2$$
$$= \frac{\partial f(P)}{\partial q \vec{v}} + \frac{\partial f(P)}{\partial q \vec{w}}.$$

REFERENCES