On the Approximate Solution of Fractional Logistic Equation by Shannon Wavelets

Mesut Karabacak, Ebubekir Karabacak

***Abstract*— In this paper, the numerical solution of fractional order logistic equation (FOLE) is considered by Shannon wavelet functions. We derive the wavelet operational matrix of the fractional order integration and by using it to solve the fractional order logistic equation. An illustrative example is presented to demonstrate the applicability and validity of this wavelet base technique. The results obtained are in good-agreement with the exact solutions. It is shown that the technique used here is effective and easy to apply.**

***Index Terms*— fractional order logistic equation (FOLE), Shannon wavelets, operational matrix**

I. INTRODUCTION

In recent years, fractional modeling and fractional differential equations has been used widely to deal with some engineering problems such as aerodynamics, signal and image processing, economics, electrodynamics, biophysics, polymer rheology, economics, control theory and many others. Since the wide range applications of fractional calculus and dynamics, this area caught the interest of many researchers. The main advantage of fractional derivatives lies in that it is more suitable for describing memory and hereditary properties of various materials and process in comparison with classical integer-order derivative. Most fractional differential equations do not have exact solutions, so approximations and numerical techniques should be used for them [1-7]

The fractional Logistic model can be obtained by applying the fractional derivative operator to the Logistic equation. As known, Pierre F. Verhulst introduced the nonlinear term into the rate equation in 1838 and obtained what today is known as the logistic equation [8]

$$u'(t) = k\, u(t)\big(1 - u(t)\big) \qquad t \geq 0 \qquad (1)$$

This differential equation is one of the few nonlinear differential equation that has a known exact closed form solution

$$u(t) = \frac{u_0}{u_0 + (1-u_0)\, exp\,(-kt)} \qquad t \geq 0 \qquad (2)$$

where u_0 is the initial state $\left(u_0 = u(0) = \frac{N(0)}{N_{max}} \right.$, where $N(0)$ is the total population at the initial time and N_{max} is the carrying capacity of the ecosystem) [9]. Here we consider the fractional order version of the standard logistic equation as

$$D_t^{\alpha}\,[u(t)] = k\, u(t)\, \big(1 - u(t)\big) \quad t > 0, \ \ k > 0 \qquad (3)$$

with an initial condition $u_0 = u(0)$.

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Fractional logistic equation has no known exact solution yet. Therefore, we study on the numerical solution of the equation. Most known application of the logistic equation is the modeling of population growth. In this model $k > 0$ defines the growth rate and $u(t)$ and t represent population size and time respectively.

Another application of the logistic model is in medicine, where it has been used to model the growth of tumors [10]. Fisher and Fry [11] and the adaptability of society to innovation use the logistic equation to model the social dynamics of replacement technologies.

In addition, wavelet analysis is a relatively new area in mathematical researches. Wavelets are localized functions and a useful tool in many different applications: signal analysis, vibration analysis, solving PDEs, data compression, solid mechanics and operator analysis. Usually, wavelets have been used only as any other kind of orthogonal functions, without taking into consideration their fundamental properties. [12,13]

In this paper, we show to use Shannon wavelet bases to solve the fractional order logistic equation. Firstly, we derive Shannon wavelet operational matrix of the fractional order integration and then we use the Shannon wavelet operational matrix to transform the fractional order equation into algebraic systems of equations completely.

Finally, we solve this transformed complicated algebraic equations system by Matlab software.

The paper is organized as follows. In Section 2, we introduce some preliminaries of the fractional calculus theory. In Section 3, some relevant properties of the Shannon wavelet bases and function approximation by these bases are presented. In addition, operational matrix of integration for Shannon wavelet is obtained. In Section 4, we demonstrate a numerical example and we end with some conclusions and remarks in Section 5.

II. BASIC CONCEPTS

Definition 2.1. A real function $f(x), x > 0.$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$. Clearly, $C_\mu \subset C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x), x < 0.$ is said to be in the space $C_\mu^{\,m}, m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function,

$f \in C_\mu, \ \mu \geq -1$, is defined as

i. $I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \,, \alpha > 0, x > 0$

ii. $I^0 f(x) = f(x)$

The properties of the operator I^α can be found in [14-16]. We make use of the followings.

For $f \in C_\mu, \ \ \mu \geq -1, \ \ \alpha, \beta \geq 0$ and $\gamma > -1$

1. $I^{\alpha}I^{\beta}f(x) = I^{\alpha+\beta}f(x)$
2. $I^{\alpha}I^{\beta}f(x) = I^{\beta}I^{\alpha}f(x)$
3. $I^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$

Definition 2.4. The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ of a function is defined as

$$D^{\alpha}f(t) = \begin{cases} \frac{d^m f(t)}{dt^m} & \alpha = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\alpha)}\frac{d}{dt^m}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau & t > 0, 0 < m-1 < \alpha < m \end{cases}$$

III. Shannon Wavelets and Operational Matrix of the Fractional Integration

This section is devoted to introduction of Shannon wavelet bases, function approximation with these bases and establish the operational matrix of fractional integration.

i. Shannon wavelets

Wavelets are a family of functions constructed from dilation and translation of a single function called the mother wavelet. The scaling function for the Shannon multiresolution analysis is sinc function that defined on $\mathbb{R}$, and is given below

$$\varphi(t) = sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Theorem 3.1 ([17]). The function $\varphi(t)$ is a scaling function of a multiresolution analysis and the corresponding mother wavelet is defined by

$$\psi\left(t+\frac{1}{2}\right) = 2\varphi(2t) - \varphi(t)$$

Theorem 3.2 ([17]). Let j, k be non-negative integers. Then the family

$$\{\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)\}_{j,k=0}^{\infty}$$

is an orthonormal bases of $L_2(\mathbb{R})$. j, k are dilatation and translation parameters in above theorems respectively.

ii. Reconstruction of a Function by Shannon Wavelets

In this section we express the convergence of orthogonal wavelet series when the mother wavelet is of Shannon-type. Also we show how to approximate a reasonable function with these wavelet bases.

Theorem 3.3 ([17]). Let $u(t) \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, if $u(t)$ is of bounded variation on every bounded interval, then the wavelet series

$$u_j(t) = \sum_k \langle u, \psi_{j,k}\rangle \psi_{j,k}(t)$$

converges to $u(t)$ as $j \to \infty$ at every point of continuity of $u(t)$

Therefore, any function $u(t) \in L_2([0,1])$ have an Shannon expansion as

$$u(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{j,k}\psi_{j,k}(t)$$

where $c_{j,k} = < u(t), \psi_{j,k}(t) >$ in which $< \cdot , \cdot >$ denotes the inner product. The serie (3.1) is truncated after m-terms as

$$u(t) \cong u_m(t) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1} c_{j,k}\psi_{j,k}(t) = C^T\Psi(t)$$

where m denotes a positive integer, C and $\Psi(t)$ are vectors given by

$$C = [c_{0,0}, c_{0,1}, \dots, c_{0,m-1}, c_{1,0}, c_{1,1} \dots, c_{1,m-1}, \dots, c_{m-1,0}, c_{m-1,1} \dots, c_{m-1,m-1}]^T$$
$$\Psi(t) = [\psi_{0,0}(t), \psi_{0,1}(t), \dots, \psi_{0,m-1}(t), \psi_{1,0}(t), \dots, \psi_{1,m-1}(t), \dots, \psi_{m-1,0}(t), \dots, \psi_{m-1,m-1}(t)]^T$$

iii. Operational matrix of the fractional integration

In this part, we may simply introduce the operational matrix of fractional integration of Shannon wavelets.

Firstly, taking the collocation points as

$$t_i = \frac{i}{m^2-1}, \quad i = mj + k, \quad j, k = 0,1, \dots m-1$$

we define Shannon matrix $\Psi_{m^2 \times m^2}$ as

$$\Psi_{m^2 \times m^2} = \left[\Psi(0), \Psi\left(\frac{1}{m^2-1}\right), \Psi\left(\frac{2}{m^2-1}\right) \dots, \Psi\left(\frac{m^2-2}{m^2-1}\right), \Psi(1)\right]$$

Finally, for $u_m = [u_m(t_0), u_m(t_1) \cdots, u_m(t_{m^2-1})]^T$, the Shannon coefficients $c_{j,k}, (j, k = 0,1, \dots m-1)$ can be obtained by

$$C^T = u_m \Psi_{m^2 \times m^2}^{-1}$$

For example, when $m = 2$ Shannon matrix is written as

$$\Psi_{4\times 4} = \begin{bmatrix} 1 & 0{,}826993 & 0{,}413497 & 3.89817\times 10^{-17} \\ -0{,}63662 & 0{,}699057 & 0{,}699057 & -0{,}63662 \\ 0{,}212207 & 0{,}372702 & -0{,}521783 & -0{,}63662 \\ -0{,}900316 & 0{,}988616 & -0{,}737913 & 0{,}300105 \end{bmatrix}$$

Now, we need to integrate the $\Psi(t)$. It can be approximated by Shannon expansion with Shannon coefficient matrix P,

$$\int_0^t \Psi(\tau)\,d\tau \approx P_{m^2 \times m^2}\Psi(t)$$

where the m^2-square matrix P is called the Shannon wavelet operational matrix of integration. Our goal is to derive the Shannon wavelet operational matrix of the fractional order integration namely $P^{\alpha}_{m^2 \times m^2}$. For this, we use m^2 −term of Block Pulse Functions on $[0, 1)$ (BPFs) as follows

$$b_i(t) = \begin{cases} 1, & i\frac{1}{m^2} \leq t < (i+1)\frac{1}{m^2} \\ 0, & \text{otherwise} \end{cases}$$

where $i = 0,1,2, \cdots, (m^2-1)$

$b_i(t)$ functions have some useful properties like disjointness and orthogonality. Respectively that is,

$$b_i(t)\,b_l(t) = \begin{cases} 0, & i \neq l \\ b_i(t), & i = l \end{cases}$$

$$\int_0^1 b_i(\tau)\, b_l(\tau)\, d\tau = \begin{cases} 0, & i \neq l \\ 1/m^2, & i = l \end{cases}$$

Then, the Shannon wavelets can be transformed into an m^2-term block pulse functions (BPF) as

$$\mathbf{\Psi}(t) = \mathbf{\Psi}_{m^2 \times m^2} B(t) \tag{4}$$

where

$$B(t) \triangleq [b_0(t), b_1(t), \cdots, b_{m^2-1}(t)]^T$$

Kilicman and Al Zhour [18], have introduced the Block Pulse operational matrix of the fractional order integration F^α as follows

$$(I^\alpha B)(t) \approx F^\alpha_{m^2 \times m^2} B(t) \tag{5}$$

where

$$F^\alpha = \left(\frac{1}{m^2}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m^2-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m^2-2} \\ 0 & 0 & 1 & \cdots & \xi_{m^2-3} \\ & & & & \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$$

for $k = 1, 2, \ldots m^2 - 1$

Next, we derive the Shannon wavelet operational matrix for the fractional order integration.

Let

$$(I^\alpha \mathbf{\Psi})(t) \approx P^\alpha_{m^2 \times m^2} \mathbf{\Psi}(t) \tag{6}$$

where the $m^2 -$ square matrix $P^\alpha_{m^2 \times m^2}$ is called the Shannon wavelet operational matrix of the fractional order integration.

Using Eqs. (4), (5), we get

$$(I^\alpha \mathbf{\Psi})(t) \approx (I^\alpha \mathbf{\Psi}_{m^2 \times m^2} B)(t) = \mathbf{\Psi}_{m^2 \times m^2} (I^\alpha B)(t) \approx \mathbf{\Psi}_{m^2 \times m^2} F^\alpha B(t) \tag{7}$$

from Eqs. (6) and (7), we get

$$P^\alpha_{m^2 \times m^2} \mathbf{\Psi}(t) = P^\alpha_{m^2 \times m^2} \mathbf{\Psi}_{m^2 \times m^2} B(t) = \mathbf{\Psi}_{m^2 \times m^2} F^\alpha B(t) \tag{8}$$

Then, the Shannon wavelet operational matrix $P^\alpha_{m^2 \times m^2}$ is written by

$$P^\alpha_{m^2 \times m^2} = \mathbf{\Psi}_{m^2 \times m^2} F^\alpha \mathbf{\Psi}^{-1}_{m^2 \times m^2} \tag{9}$$

For example, $\alpha = 1.5$ and $m = 2$, the operational matrix $P^\alpha_{m^2 \times m^2}$ is computed below

$$P^{1.5}_{4 \times 4} = \begin{bmatrix} 0{,}344784 & 0{,}029417 & -0{,}648696 & 0{,}167481 \\ -0{,}017471 & -0{,}045238 & -0{,}152993 & 0{,}003118 \\ 0{,}049062 & 0{,}062126 & -0{,}048808 & -0{,}009804 \\ -0{,}125848 & 0{,}032364 & 0{,}105076 & -0{,}100288 \end{bmatrix}$$

IV. NUMERICAL EXAMPLE

Showing the efficiency of the method, we take the following fractional logistic equation. All the numerical results were obtained by using the Matlab software.

Example 4.1. Consider the following fractional order logistic equation.

$$D^\alpha u(t) = \frac{1}{2} u(t)\big(1 - u(t)\big) \quad t > 0$$
$$u(0) = 0{,}25 \qquad 0 < \alpha \leq 1 \tag{10}$$

Now, we transform all terms of the equation into Shannon series form below. Firstly, let

$$Du(t) = S^T \mathbf{\Psi}(t) \tag{11}$$

with the initial states, we get

$$D^\alpha u(t) = S^T P^{1-\alpha}_{m^2 \times m^2} \mathbf{\Psi}(t) \tag{12}$$

$$u(t) = S^T P^1_{m^2 \times m^2} \mathbf{\Psi}(t) + \underbrace{0{,}25}_{u(0)} \tag{13}$$

Substituting Eqs. (11-13) into (10), we get

$$S^T P^{1-\alpha}_{m^2 \times m^2} \mathbf{\Psi}(t) = k(S^T P^1_{m^2 \times m^2} \mathbf{\Psi}(t)) \left(1 - S^T P^1_{m^2 \times m^2} \mathbf{\Psi}(t)\right) \tag{14}$$

Hereby, Eq. (10) has been transformed into a system of algebraic equations. Substituting values and solving the algebraic equations system, we can find the coefficients S^T. Then using Eq. (13), we can get $u(t)$.
The numerical results for $m = 5$ are shown Fig 1,2,3,4 under the different values of α. The numerical solution is in good agreement with the exact solution.

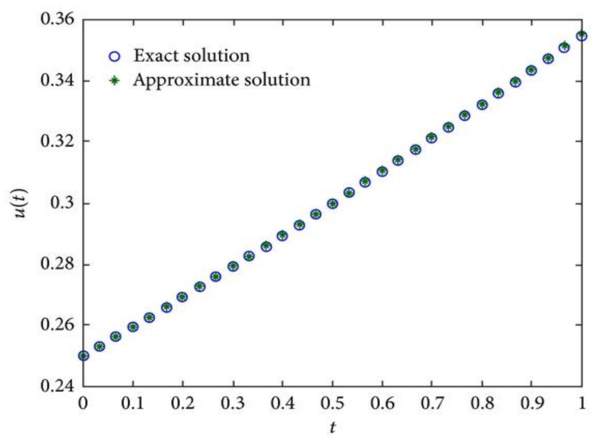

Fig 1. The comparison between the approximate solution and the exact solution when $\alpha = 1$

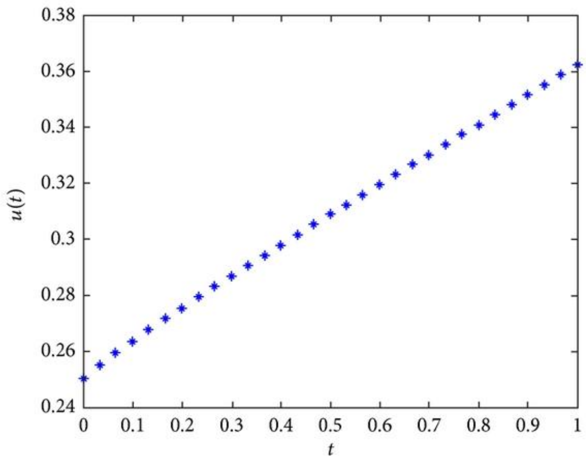

Fig 2. The graph of $u(t)$ for $\alpha = 0{,}85$

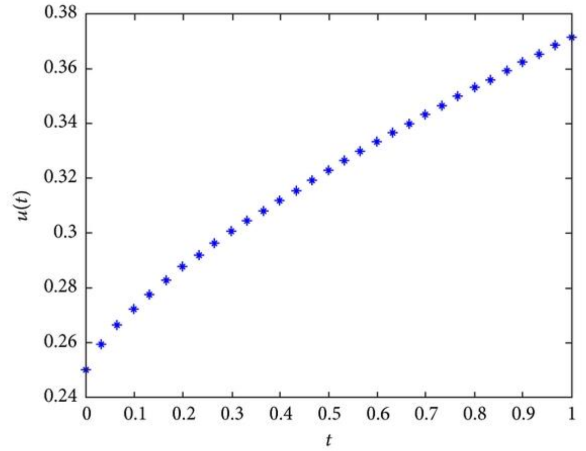

Fig 3. The graph of $u(t)$ for $\alpha = 0{,}65$

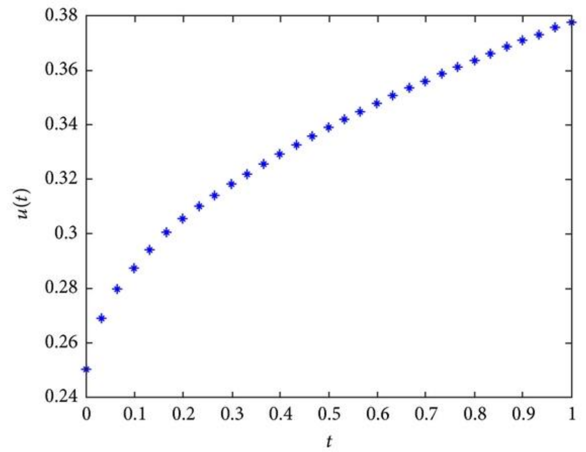

Fig 4. The graph of $u(t)$ for $\alpha = 0{,}45$

V. Conclusion

In this paper, the Shannon wavelet functions has been employed to solve fractional logistic equation. The results obtained by the method are in good agreement with the given exact solutions. The study show that the method is effective techniques to solve fractional logistic equations, and the method presents real advantages in terms of comprehensible applicability and precision.

References

[1] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997, pp.291-348.

[2] R. Hilfer, Applications of Fractional Calculus in Physics. World Scientific, Singapore, 2000.

[3] A.B. Malinowska, D.F.M. Torres, Towards a combined fractional mechanics and quantization. Fractional Calculus and Applied Analysis 15, No 3 (2012), 407–417

[4] A. Carpinteri, F Mainardi. Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, NewYork, 1997

[5] K. Nouri, NB. Siavashani, Application of Shannon wavelet for solving boundary value problems of fractional differential equations, Wavelets and Linear Algebra 1 (2014) 33-42

[6] B. İbiş. M. Bayram . Numerical comparison of methods for solving fractional differential-algebraic equations (FDAEs), Computers and Mathematics with Applications 62 (2011) 3270-3278

[7] M. Karabacak, E. Çelik. The numerical solution of fractional differential-algebraic equations (FDAEs), New Trends in Mathematical Sciences, Vol.1 No.1 (2013), pp 106.

[8] P.F.Verhulst, Notice sur la loi que la population sint dons son accroissement, Math. Phys.10(1838)113.

[9] A.M.A. El-Sayed, A.E.M. El-Mesiry, H.A.A. El-Saka, On the fractional order Logistic equation, Appl. Math. Letters, 20(7) (2007), 817-823

[10] U. Forys, A. Marciniak-Czochra, Logistic equations in tumor growth modelling, Int. J.Appl. Math. Comput. Sci. 13 (2003) 317.

[11] T.C. Fisher, R.H. Fry, A simple substitution model of technological change, Technol. Forecast. Soc. Change 3 (1971) 75.

[12] C. Cattani, "Shannon wavelets theory," Mathematical Problems in Engineering, vol. 2008, Article ID 164808, 24 pages, 2008.

[13] C. Cattani, "Connection coefficients of Shannon wavelets," Mathematical Modelling and Analysis, vol. 11, no. 2, pp. 117–132, 2006.

[14] I. Podlubny. Fractional Differential Equations. An Introduction to Fractional Derivatives Fractional Differential Equations Some Methods of their Solution and Some of their Applications, Academic Press, San Diego, 1999.

[15] KS Miller, B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons Inc., New York, 1993.

[16] K. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

[17] JJ. Benedetto, PJSG. Ferreira, Modern Sampling Theory, Springer Science and Business Media, New York, 2001.

[18] A. Kilicman, ZAA. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput. 187 (2007) 250–265