Study and Evaluation of a European Option by Heston Model

Mohammed Lakhdar Hadji

***Abstract*— In this work we propose an approximate numerical method for an option pricing by the Heston model. First we prove the existence and uniqueness of the solution in a weighted Sobolev space, and then we propose the finite element and finite difference methods to solve the considered problem. Therefore, we compare the obtained results for the two approaches, with those by the Monte Carlo method in Broadie-Kaya. To show the efficiency of the numerical approaches, we use different values of the interest rate and show improvements in the results for the convergence and cputime.**

Subject Classification (2000): 35R60, 58J65, 60H15, 60J69, 65C05.

***Index Terms*—European Option, Stochastic Volatility, Finite Elements, Finite Differences.**

I. INTRODUCTION

The mathematical modeling in finance is a subject that has attracted many researchers for the last years. Specially the European options, showed that its price is dependent on its volatility. The Black Scholes formula[5] which considered the volatility as a constant is clearly an assumption that does not reflect the reality of the market. Empirical studies show that the volatility is random and depends on the time variable. The option price is done by solving a system of two stochastic differential equations (SDE), one for the underlying and the other for its volatility, taking into account the correlation coefficient between the two noise sources. Hull-White [14], Stein & Stein[16] and Heston[12] proposed models with stochastic volatility that can be solved analytically. Cox and Ross[7] introduced these models to the dynamics of the underlying to explain the empirical bias exhibited by the Black-Scholes option pricing model. One very simple model that is based on a stochastic volatility is the Heston model which is an extended version of the Black Scholes stochastic one. The problem that occurs in Heston model is the non existence of analytical solutions and to overcome this problem, approximations have been derived. Many authors have used approaches by the use of Monte-Carlo method, but it has to be pointed out that this method is quite expensive. Here we use approximation techniques based on finite element and finite difference schemes and compare the obtained results with the Monte-Carlo method in order to point out the advantages and inconveniences.

This paper is organized as follows: section 2 introduces the Heston model, its dynamics and the associated pricing problem. In section 3 we introduce the weighted Sobolev space to be used in the variational formulation of our problem and prove the existence and uniqueness of the weak solution. Section 4 is dedicated to the use of the finite element and finite difference methods for the pricing of the European option, we conclude in section 5. Broadie et Al [6] have performed exact and approximate solutions of the SDE for the evaluation of the European call under the stock index S&P500 by the Monte Carlo method. Here we propose to use the finite element and the finite difference methods in order to improve the convergence of the root mean square (RMS) error. In section 6 we present a comparison of the obtained results (the Cputime and RMS error) by the two methods with the ones by Broadie et Al [6] for different values of the interest rate. Finally, we highlight the performances of our approaches by comments and concluding remarks.

Mohammed Lakhdar Hadji, Department of Mathematics, Badji Mokhtar University/ Laboratory Laps, Annaba, Algeria, Phone/No.(213)662170889

.

II. PRESENTATION OF THE MODEL

A. Dynamic system of a European option

We consider the following stochastic volatility model for the stock price under a neutral risk probability P chosen by the market:

$$\begin{cases} dS_t = rS_t dt + \sqrt{\sigma_t} S_t dW_{t,S} \\ d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu\sqrt{\sigma_t} dW_{t,\sigma} \\ dW_{t,S} dW_{t,\sigma} = \rho dt \end{cases} \tag{1}$$

S_t represents the price of the underlying;
σ_t : the volatility supposed to be stochastic;
r : the positive constant instantaneous interest rate;
θ : the long-term variance, when t tends to infinity;
κ : the return speed of σ_t at θ;
ρ : the correlation coefficient between the two Brownian motions; where κ, θ and ν are positive constants satisfying the condition $\frac{2\kappa\theta}{\nu^2} \geq 1$. The variables $W_{t,S}$ and $W_{t,\sigma}$ are independent standard Brownian defined on a complete probability space (Ω; F; P).

B. European option price model

In this section we write down the partial differential equation (PDE) modeling a European option price. For, we consider an European derivative on S_t, denoted by V (t; S_t; σ_t) with expiration date T, the strike K and a payoff -function $h(S_T) = \max(S_T - K, 0)$

The price at the time t will depend on t, on the price of the underlying asset S_t and on the volatility σ_t is a solution of the following Garman PDE [9].

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma S^2 \frac{\partial^2 V}{\partial S^2} + \rho\nu\sigma S^2 \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}\nu^2 \sigma \frac{\partial^2 V}{\partial \sigma^2} +$$
$$rS\frac{\partial V}{\partial S} + \kappa(\theta - \sigma)S\frac{\partial V}{\partial \sigma} - rV = 0$$

With boundary conditions $h(S_T) = \max(S_T - K, 0)$.

This equation can be written as:

$$\begin{cases} \dfrac{\partial V(t,S_t,\sigma_t)}{\partial t} + L(V(t,S_t,\sigma_t)) = 0 \quad \forall t \in [0,T] \\ V(T,S_t,\sigma_t) = h(s,\sigma) = h(s) \end{cases} \quad (2)$$

$$L(V) = \frac{1}{2}\sigma S^2 \frac{\partial^2 V}{\partial S^2} + \rho\nu\sigma S^2 \frac{\partial^2 V}{\partial S \partial \sigma} +$$

$$\frac{1}{2}\nu^2\sigma \frac{\partial^2 V}{\partial \sigma^2} + rS\frac{\partial V}{\partial S} + \kappa(\theta-\sigma)S\frac{\partial V}{\partial \sigma} - rV = 0.$$

III. Analytical study

A. Variational analysis

Let $U = [0,+\infty[\times]a,b[$ with $0 < a < b < +\infty$. We consider the weighted space Sobolev Ξ defined by

$$\Xi = \left\{ (s\frac{\partial u}{\partial s}, \frac{\partial u}{\partial \sigma}, u) \in (L^2(U))^3 \right\}$$

And equipped with the norm

$$\|u\| = \left(\int_U s^2 \left|\frac{\partial u}{\partial s}\right|^2 dsd\sigma + \int_U \left|\frac{\partial u}{\partial \sigma}\right|^2 dsd\sigma + \int_U |u|^2 dsd\sigma + \right)^{\frac{1}{2}}$$

$$|u|_1^{\ 2} = \int_U s^2 \left|\frac{\partial u}{\partial s}\right|^2 dsd\sigma, |u|_2^{\ 2} = \int_U \left|\frac{\partial u}{\partial \sigma}\right|^2 dsd\sigma, |u|_3^{\ 2} \int_U |u|^2 dsd\sigma$$

This space is a Hilbert separable reflexive space, dense in $L^2(U)$, see (Theorem 4.2 in [1],[8]).

By multiplying equation (2) by a test function v, we obtain the associated variational formulation

$$\begin{cases} (\dfrac{\partial u}{\partial t}, v) + (Lu, v) = 0 \forall t \in [0,T] \ and \ s,\sigma \in U \\ (u(0,..,0), v) = (h(s,\sigma), v) \end{cases} \quad (3)$$

With $u(t,s,\sigma)=V(T-t,s,\sigma)$.

By using the Green's formula and the Direchlet boundary conditions, we obtain

$$\begin{cases} (\dfrac{\partial u}{\partial t}, v) + a(u, v) = 0 \forall t \in [0,T] \ and \ s,\sigma \in U \\ (u(0,..,0), v) = (h(s,\sigma), v) \end{cases} \quad (4)$$

And

$$a(u,v) = \frac{1}{2}\int_U \sigma s^2 \frac{\partial u \partial v}{\partial s \partial s} dsd\sigma + \int_U \sigma s \frac{\partial u}{\partial s} v dsd\sigma -$$

$$\int_U rs \frac{\partial u}{\partial s} v dsd\sigma \frac{1}{2}\int_U \sigma \nu^2 \frac{\partial u \partial v}{\partial \sigma \partial \sigma} dsd\sigma + \frac{1}{2}\int_U \nu^2 \frac{\partial u}{\partial s} v dsd\sigma -$$

$$\int_U \kappa(\theta-\sigma)\frac{\partial u}{\partial \sigma} v dsd\sigma + \int_U \rho\nu\sigma s \frac{\partial u \partial v}{\partial \sigma \partial s} dsd\sigma +$$

$$\int_U \rho\nu\sigma \frac{\partial u}{\partial s} v dsd\sigma + \int_U ruv dsd\sigma$$

with

$$(Lu, v) = a(u, v)$$

B. Main Result

By using theorem due to Lax Miligram (see J. L. Lions[15], we establish the existence and uniqueness of the solution of problem (4).

Theorem 1 *the Variational problem (4) admits an unique solution in* Ξ.

Proof. To show this result, we need to prove the continuity and the coercivity or the ellipticity of the bilinear a(.,.).

- For the continuity, we use the Cauchy-Schwartz's inequality for $\rho < 1$.
- For the coercivity, we use the Cauchy-Schwarz's inequality for $\rho > -1$, the Poincare's inequality and Young's inequality.

IV. Numerical Resolution

In this section, we discuss two of the main techniques used for pricing options and present numerical simulations. We implement a European Call using the associated PDE (2) to the Heston model (1). Numerical results by the use of a finite element scheme using the open source software FreeFem++ (see www.freefem++.org) [13] and a finite difference method using Matlab.

A. Finite Element Discretization

In this subsection we solve the variational problem for the pricing of a European option using finit elements method for the *space variable,* and the finit differences explicit Euler method for the time variable. The space Ξ is generally of infinite dimension, we construct by Galerkin method a subspace Ξ h such that dim Ξ h $< \infty$. We consider a bounded domain, with a Lipchitz boundary $\partial\Omega$, we introduce a regular family $(T_h)_h$ of triangulations of Ω, in the usual sense that :

* For each h, the closure of Ω is the union of all elements of T_h.

* The intersection of two different elements of T_h is the empty, a vertex or a whole edge.

Let us first recall some standard notations: For each element E in a discretization $T_{h\in E}$ denotes the set of all elements of E that are not contained in $\partial\Omega$:

* The ratio of the diameter h_E of any element E of T_h to the diameter of its inscribed circle is smaller than a constant independent of the discretization parameter h.

As standard, h stands for the maximum of the diameters $h \in E \in T_h$. For u_h, $v_h \in \Omega$, the discrete fomulation is writen as

$$\begin{cases} (\dfrac{\partial u_h}{\partial t}, v_h) + a(u_h, v_h) = 0 \forall t \in [0,T] \ and \ s,\sigma \in \Omega \\ (u_h(0,..,0), v_h) = (h(s,\sigma), v) \end{cases} \quad (5)$$

Where $a(u_h,v_h)$ is the analogous formulation to a(u,v)

$$a(u_h,v_h) = \frac{1}{2}\int_U \sigma s^2 \frac{\partial u_h \partial v_h}{\partial s \partial s} dsd\sigma + \int_U \sigma s \frac{\partial u_h}{\partial s} v_h dsd\sigma -$$

$$\int_U rs \frac{\partial u_h}{\partial s} v_h dsd\sigma \frac{1}{2}\int_U \sigma \nu^2 \frac{\partial u_h \partial v_h}{\partial \sigma \partial \sigma} dsd\sigma + \frac{1}{2}\int_U \nu^2 \frac{\partial u_h}{\partial s} v_h dsd\sigma -$$

$$\int_U \kappa(\theta-\sigma)\frac{\partial u_h}{\partial \sigma} v_h dsd\sigma + \int_U \rho\nu\sigma s \frac{\partial u_h \partial v_h}{\partial \sigma \partial s} dsd\sigma +$$

$$\int_U \rho\nu\sigma \frac{\partial u_h}{\partial s} v_h dsd\sigma + \int_U ru_h v_h dsd\sigma \quad (6)$$

The resolution of this problem using FreeFem++ with P1finite element, provides the following numerical results.

For more details on finite element methods the reader is refered to [17].

Numerical Results by the finite elements method

We compare the results given by Heston model using the finite elements method described above, with those obtained by Broadie- Kaya[6] using the Monte Carlo method.

Table 1 gives results for a European call option. The set of parameters for the SV model are taken from Duffie et al. (2000). These were found by minimizing the mean squared errors for market option prices for S&P 500 on November 2, 1993. The bias column is estimated using 40 million simulation trials. The number of time steps for the Euler discretization is set to be equal to the square root of the number of simulation trials.

Table 1: Simulation results for European call using Monte Carlo in [6]

(a)Simulations with the exact method

No of Simul	RMC error	Comp Time
10000	0.0750	3.8
40000	0.0373	15.2
160000	0.0186	60.0
640000	0.0093	239.4
256 0000	0.0046	955.7

(b)Simulations with the Euler Discretization

No of Simul	No of Time	RMS error	Comp Time
10000	100	0.1725	0.2
40000	200	0.1073	1.9
160000	400	0.0689	15.2
640000	800	0.0406	121.3
256 0000	1600	0.0272	970.0

Note that the parameters used for the experimentations are: S = 100, K = 100, V_0 = 0.010201, κ= 6.21, θ= 0.019, σ_v = 0.61, ρ= -0.70, r =3:19%, T = 1.0 year, true option price=6.8061.

Table 2: Simulation results for European call option using Finite Element Method

No of time steps	Price European Call	RMC error	Executing Times(sec)	Interest Rate
100	7.07989	0.387192	3.86	3.19%
200	7.00494	0.281204	3.891	4%
400	6.91287	0.150994	3.844	5%
800	6.82131	0.021516	3.813	6%
1600	6.80307	0.00429	3.907	6.2%

For our numerical simulations, we choose the parameters to be: S =100; K = 100, κ = 2. , θ = 2.3, ν= -0.5; ρ=0.5;
T = 1.0 year.

We can conclude that this method is more efficient than the Monte Carlo method used by Broadie and Kaya [6] in terms of execution time by varying the interest rate.

B. Finite differenceapproximation

In this subsection we present an approximation of problem (4) using a finite difference scheme. For we introduce a partition of the bounded domain

$$\Omega_T = [0.T]\times([S\min, S\max]\times[\sigma\min,\sigma\max])$$

into subintervals $[t_k; t_{k+1}]\times[S_i\times S_{i+1}]\times[\sigma_j;\sigma_{j+1}]$ such that

$t_k= k\Delta t$, $0\le k\le N$, $S_i= i\Delta S$, $0\le i\le I$ and $\sigma_j = j\Delta\sigma$, $0\le j\le J$ with $\Delta t=\frac{T}{N}$, $\Delta S=\frac{S\max - S\min}{I+1}$, $\frac{\sigma\max-\sigma\min}{J+1}$

For simplicity, we introduce the notation $(t_k, S_i, \sigma_j) = (k,i, j)$, $V(t_k,S_i, \sigma_j) = V_{i,j}^k$

Implicite scheme

Let ΔX; ΔY be the space steps and Δt are the discretization time step size. The European option value is in the cronological anti-sense, from the maturity date T. The domain price of the underlying and its volatility is infinite in theory, but numerically we take a rectangle, its center is the price of the underlying and its volatility at t, for which we seak to evaluate the option. If the price of the underlying and its volatility are negative, the option value is zero. Hence the value of the option V(S,σ,t) is approximated by

$$V_{i,j}^k = V(i\Delta S, j\Delta\sigma,(T-k\Delta t))$$

The derivatives can be written as follows

$$\frac{\partial V}{\partial S}=\frac{V_{i+1,j}^{k+1}-V_{i-1,j}^{k+1}}{2\Delta s} \qquad \frac{\partial^2 V}{\partial S^2}=\frac{V_{i+1,j}^{k+1}-2V_{i,j}^{k+1}+V_{i-1,j}^{k+1}}{\Delta s^2}$$

$$\frac{\partial V}{\partial\sigma}=\frac{V_{i,j+1}^{k+1}-V_{i,j-}^{k+1}}{2\Delta\sigma} \quad \frac{\partial^2 V}{\partial\sigma^2}=\frac{V_{i,j+1}^{k+1}-2V_{i,j}^{k+1}+V_{i,j-1}^{k+1}}{\Delta\sigma^2}$$

$$\frac{\partial^2 V}{\partial S\partial\sigma}=\frac{V_{i+1,j+1}^{k+1}-V_{i+1,j-1}^{k+1}-V_{i-1,j+1}^{k+1}+V_{i-1,j-1}^{k+1}}{4\Delta S\Delta\sigma}$$

$$\frac{\partial V}{\partial t}=\frac{V_{i,j}^{k}-V_{i,j}^{k+1}}{\Delta t}$$

$$V_{i,j}^k = \begin{array}{l} aV_{i,j}^{k+1}+b(V_{i+1,j+1}^{k+1}+V_{i+1,j-1}^{k+1}+V_{i-1,j+1}^{k+1}+V_{i-1,j-1}^{k+1})+ \\ cV_{i+1,j}^{k+1}+dV_{i,j+1}^{k+1}+eV_{i,j-1}^{k+1}+fV_{i-1,j}^{k+1}\end{array}$$

With

$$a=1+\left[r+ji^2\Delta\sigma+\frac{j\nu^2}{\Delta\sigma}\right]\Delta t,\ b=-\frac{\rho\nu ij}{4}\Delta t$$

$$c=-\frac{i\Delta t}{2}\left[ji\Delta\sigma+\nu\right],\ d=-\frac{\Delta t}{2\Delta\sigma}\left[\nu^2 j+\kappa(\theta-j\Delta\sigma)\right]$$

$$e=\frac{\Delta t}{2\Delta\sigma}\left[\kappa(\theta-j\Delta\sigma)-\nu^2 j\right],\ f=\frac{\nu i}{4}\Delta t$$

With the initial condition $V_{i,j}^k=(S_i-K)^+$ and the boundary conditions $V_{0,j}^k=V_{I+1,j}^k=0$ for $0\le j\le J+1$

$V_{i,0}^k=V_{i,J+1}^k=0$ for $0\le i\le I+1$

C. Numerical results by the finite difference method

In this subsection, we compare the obtained numerical results using the finite difference method described in the previous section implemented in Matlab, with those obtained by Broadie-Kaya[6] using the Monte Carlo method.

Table 3: Simulation results for European call option using Finite Difference Method

No of time steps	Price European Call	RMC error	Executing Times(sec)	Interest Rate
100	6.7272	0.1115	0.5460	3.19%
200	6.7752	0.03437	0.5772	4%
400	6.8413	0.0498	0.6396	5%
800	6.9097	0.1465	0.8892	6%
1600	6.9226	0.1647	1.2012	6.2%

Note that the option parameters taken for simulations are:
$S = 100$; $K = 100$, $\kappa = 0.021$, $\theta = 17.94$, $\nu = 0.5$; $T = 1.0$ year.
We remark that this method is more efficient than the Monte Carlo method used by Broadie and Kaya[6] in terms of Cputime. These tests confirm that the finite difference method is faster for different values of the interest rate when it is less than 5%.

D. Comments the numerical results

In this subsection we compare the results obtained by the two methods: the Cputime and the RMS error for different values of the interest rate.

Execution Time

Comparison of the Execution Time between F.E.M & F.D.M.

Table 4: Cputime

Interest Rate	F.E	F.D
3.19	3.86	0.6460
4	3.891	0.5772
5	3.844	0.6396
6	3.813	0.8892
6.2	3.907	1.2012

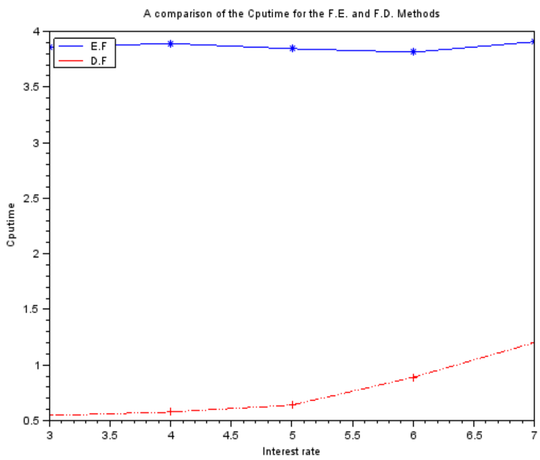

Figure 1: Cputime

By comparing the Cputime for the two methods, we note that the finite difference method is faster than the finite element method (see Figure 1)

RMS error

Comparison of RMS error between F.E.M & F.D.M

Table 5: RMS error

Interest Rate	F.E	F.D
3.19	0.387192	0.1115
4	0.281204	0.0437
5	0.150994	0.0498
6	0.0215161	0.1465
6.2	0.00429025	0.1647

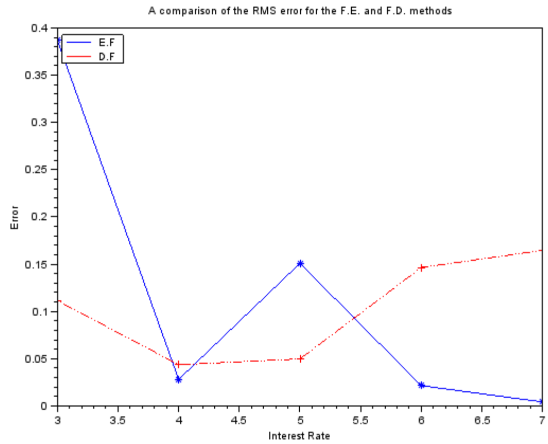

Figure 2 : RMS error

Figure 2 shows that the finite element method is more accurate than the finite difference one following the different values of the interest rate, when it is less than 5.

V. Conclusion

The aim of this paper is the pricing of the European option, by a diffusion with a stochastic volatility, where the volatility follows a Heston model. We proved the existence and uniqueness of the weak solution in a weighted Sobolev space. We present results for a comparison between two methods, the finite element and the finite difference methods for some values of the interest rate.
The obratined results show that the finite element method is more accurate compare to the finite differences in terms of RMS error when the interest rate is less than 5 (Table 5 and Figure 2), however in terms of Cputime the latter method is less expensive than the first one (see Table 4 and Figure 1).
Furtheremore a comparison with the Monte Carlo Method used by Broadie et Al [6] is presented and the obtained results lead us to conclude that the two proposed approaches are faster and more accurate for different values of the interest rate (See Tables 1,2,3,4 & 5 and Figures 1 & 2).
Finally we have to point out that our results are closer to the true market value of the European option exercised under the stock index S&P500 on November 2, 1993.

Acknowledgment

The author are grateful to Pierre Pierre Spiteri from INP-ENSEEIHT Toulouse for several enlightening discussions on this work . The author olso are thankful and grateful to the referees for their fruitful remarks, comments and suggestions.

REFERENCES

[1] Y. Achdou and N. Tchou, "Variational anaysis for the Black and Scholes Equation with Stochastic Volatility," Mathematical Modelling and Numerical Analysis, 3, 2002, pp 373 - 395.

[2] L. Andersen and V. Piterbarg, "Moment exposions in stochastic volatility models", Finance Stochatic, 11, 2007, pp29-50.

[3] R. Aboulaich, F. Baghery and A. Jraifi, "Option pricing for a stochastic volatility jump-diffusion model", International Journal of Mathematics and Statistics, 13, 2013, pp1 - 19.

[4] R. Aboulaich, M. L. Hadji and A. Jraifi, "Option Pricing with Constant Elasticity of Variance (CEV) Model," Applied Mathematical Sciences, Vol. 7, 2013, no. 109, 2013, pp 5443 - 5456.

[5] F. Black and Scholes, "The princing of options and corporate liabilities, Journal of Political Economy, "81, 1973, pp 635 - 654.

[6] M. Broadie and O. Kaya,"Exact simulation of stochastic volatility and other affine jump diffusion processes", Operations Research, 54 (2), 2006, pp 217- 231.

[7] J.C. Cox and S.A. Ross, "The Valuation of Options for Alternative Stochastic Processes," Journal of Financial Economics, 4, 1976, pp145 -166.

[8] K.O. Friedrichs, "The identity of Weak and Strong Extensions of Differential Operators," Trans, Amer, Math, Soc, 55, 1944, pp132 - 151.

[9] M. Garman. "A general theory of asset valuation under diffusion state processes," Technical Report50, Center for Research in Management Science, Berkeley,July, 1976.

[10] J. Gatheral and A. Jacquier, "Convergence of Heston to SVI", ArXiv: 1002.3633, 2010.

[11] P. S. Hagan, D. Kumar, A. Lesniewski and D. E. Woodward, "Managing smile risk," Willmot magazine 15, 2002, pp 84-108.

[12] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," Review of Financial Studies 6(2), 2013, pp 327-343.

[13] Hecht, F., Pironneau, O.: FreeFem++. http://www.freefem.org

[14] J. C. Hull and A. White, "The pricing of options on assets with stochastic volatilities."Journal of Finance, 42, 1987, pp 281-300.

[15] J.L. Lions and E. Magenes, "Methode Numerique pour les Equations aux Dérivées Partielles en Finance," Addison-Wesley Reading, Massachusetts, 1994.

[16] E. M. Stein and J. C. Stein, "Stock price distributions with stochastic volatility: An analytic approach." The Review of Financial Studies, 4, 1991, pp727-752.

[17] O.C. Zienkiewicz and R.L. Taylor, "The finite element method." Butter-worth Heinemann, fifth Edition, 2000.

Name: MOHAMMED LAKHDAR
First Name: HADJI
Date and place of birth: 24 May 1958, SOUK AHRAS **(Algeria)**
Algerian nationality
Grade: Class A Lecturer

Qualifications

* Baccalaureate (1979) Mathematical Series, Rebahi Nouar High School, Souk-Ahras (Algeria)
* Graduate degree (1983) Functional Analysis. University of Annaba (Algeria)
* Diploma of advanced studies, Doctoral School of Mathematics (2008), Badji Mokhtar University, Annaba
* Magistere (2009), University May 08, 1945, Guelma (Algeria)
* PhD (2013), Mathematics and Probability, Badji Moktar University, Annaba.
* Habilitation (2015), Badji Moktar University, Annaba.

Articles

M.L. Hadji, M. Maouni, F. Z. Nouri, "A Wavelet Inpainting by a Tixotrop Model," 14th International conference on Information Visualisation, IV 2010, 26-29 July 2010, London, UK. IEEE Computer Society 2010 ISBN 978-0-7695-4165.

M.L. Hadji, M. R. Remita, "A Numerical Resolution of a European Option Value with Stochastic Volatility," applied mathematical Sciences, vol. 7, 2013, no. 18, pp.883-891.

M.L. Hadji, "Etude de l'Equation de Navier-Stockes Stochastique non Homogène, Revue Synthèse, 2013, no. 27, pp.15-21.

M.L. Hadji, R. Aboulaich, A. Jraifi, "Option Pricing with Constant Elasticity of Variance (CEV) Model, "applied mathematical Sciences, vol. 7, 2013, no. 109, pp.5443-5456.

M.L. Hadji, A. Assala and F.Z. Nouri, "A posteriori Error Analysis for Navier Stokes Equations coupled with Darcy Problem, " Calcolo (2014) DOI 10.1007/S10092-014-0130-Z.

M.L. Hadji, "A Weak Solution of a Stochastic Nonlinear Problem, " Abstract and Applied Analysis, Publication 2015.

M.L. Hadji, M. E. Khalfallah, "Risk Measure from Investment in Finance by Value at Risk, " Colloque 18 th International Conference on Actuarial Mathematics, Istanbul, Turkey, 19-20 Decembre 2016.

.