Study and Evaluation of a European Option by Heston Model

Mohammed Lakhdar Hadji

Abstract— In this work we propose an approximate numerical method for an option pricing by the Heston model. First we prove the existence and uniqueness of the solution in a weighted Sobolev space, and then we propose the finite element and finite difference methods to solve the considered problem. Therefore, we compare the obtained results for the two approaches, with those by the Monte Carlo method in Broadie-Kaya. To show the efficiency of the numerical approaches, we use different values of the interest rate and show improvements in the results for the convergence and cpu time.

Subject Classification (2000): 35R60, 58J65, 60H15, 60J69, 65C05.

Index Terms—European Option, Stochastic Volatility, Finite Elements, Finite Differences.

I. INTRODUCTION

The mathematical modeling in finance is a subject that has attracted many researchers for the last years. Specially the European options, showed that its price is dependent on its volatility. The Black Scholes formula[5] which considered the volatility as a constant is clearly an assumption that does not reflect the reality of the market. Empirical studies show that the volatility is random and depends on the time variable. The option price is done by solving a system of two stochastic differential equations (SDE), one for the underlying and the other for its volatility, taking into account the correlation coefficient between the two noise sources. Hull-White [14], Stein & Stein[16] and Heston[12] proposed models with stochastic volatility that can be solved analytically. Cox and Ross[7] introduced these models to the dynamics of the underlying to explain the empirical bias exhibited by the Black-Scholes option pricing model. One very simple model that is based on a stochastic volatility is the Heston model which is an extended version of the Black Scholes stochastic one. The problem that occurs in Heston model is the non existence of analytical solutions and to overcome this problem, approximations have been derived. Many authors have used approaches by the use of Monte-Carlo method, but it has to be pointed out that this method is quite expensive. Here we use approximation techniques based on finite element and finite difference schemes and compare the obtained results with the Monte-Carlo method in order to point out the advantages and inconveniences.

This paper is organized as follows: section 2 introduces the Heston model, its dynamics and the associated pricing problem. In section 3 we introduce the weighted Sobolev space to be used in the variational formulation of our problem and prove the existence and uniqueness of the weak solution.

Section 4 is dedicated to the use of the finite element and finite difference methods for the pricing of the European option, we conclude in section 5. Broadie et Al [6] have performed exact and approximate solutions of the SDE for the evaluation of the European call under the stock index S&P500 by the Monte Carlo method. Here we propose to use the finite element and the finite difference methods in order to improve the convergence of the root mean square (RMS) error. In section 6 we present a comparison of the obtained results (the Cputime and RMS error) by the two methods with the ones by Broadie et Al [6] for different values of the interest rate. Finally, we highlight the performances of our approaches by comments and concluding remarks.

II. PRESENTATION OF THE MODEL

A. Dynamic system of a European option

We consider the following stochastic volatility model for the stock price under a neutral risk probability $P$ chosen by the market:

$$\begin{aligned}
\frac{dS_t}{S_t} &= r dt + \sqrt{\sigma_t} dW_{t,S} \\
\frac{d\sigma_t}{\sigma_t} &= \kappa(\theta - \sigma_t) dt + \nu \sqrt{\sigma_t} dW_{t,\sigma} \\
W_{t,S}dW_{t,\sigma} &= \rho dt
\end{aligned}$$

(1)

$S_t$ represents the price of the underlying; $\sigma_t$: the volatility supposed to be stochastic; $r$: the positive constant instantaneous interest rate; $\theta$: the long term variance, when $t$ tends to infinity; $\kappa$: the return speed of $\sigma_t$ at $0$; $\rho$: the correlation coefficient between the two Brownian motions; where $\kappa, \theta$ and $\nu$ are positive constants satisfying the condition $\frac{2\kappa \theta}{\nu^2} \geq 1$. The variables $W_{t,S}$ and $W_{t,\sigma}$ are independent standard Brownian defined on a complete probability space ($\Omega, F; P$).

B. European option price model

In this section we write down the partial differential equation (PDE) modeling a European option price. For, we consider an European derivative on $S_t$, denoted by $V(t; S_t; \sigma_t)$ with expiration date $T$, the strike $K$ and a payoff -function $h(S_T) = \max(S_T - K, 0)$.

The price at the time $t$ will depend on $t$, on the price of the underlying asset $S_t$ and on the volatility $\sigma_t$ is a solution of the following Garman PDE [9].

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} + \rho \nu \sigma \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} \nu^2 \sigma^2 \frac{\partial^3 V}{\partial \sigma^2} + \frac{r S}{\sigma} \frac{\partial V}{\partial S} + \kappa(\theta - \sigma) \frac{\partial V}{\partial \sigma} - r V = 0$$

With boundary conditions $h(S_T) = \max(S_T - K, 0)$.
This equation can be written as:
\[
\begin{aligned}
&\left\{ \frac{\partial V(t, S, \sigma)}{\partial t} + L(V(t, S, \sigma)) \right\} = 0 \quad \forall t \in [0, T] \\
&V(T, S, \sigma) = h(s, \sigma) = h(s) \\
&L(V) = \frac{1}{2} \sigma S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma \nu S^2 \frac{\partial^2 V}{\partial S \partial \sigma} + \\
&\frac{1}{2} \nu^2 \sigma \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial \sigma} + \kappa(\theta - \sigma) S \frac{\partial V}{\partial \sigma} - r V = 0.
\end{aligned}
\]

**Proof.** To show this result, we need to prove the continuity and coercivity or the ellipticity of the bilinear \(a(\cdot, \cdot)\).

- For the continuity, we use the Cauchy-Schwartz's inequality for \( \rho < 1 \).
- For the coercivity, we use the Cauchy-Schwarz's inequality for \( \rho > -1 \), the Poincare’s inequality and Young’s inequality.

IV. NUMERICAL RESOLUTION

In this section, we discuss two of the main techniques used for pricing options and present numerical simulations. We implement a European Call using the associated PDE (2) to the Heston model (1). Numerical results by the use of a finite element scheme using the open source software FreeFem++ (see www.freefem++.org) [13] and a finite difference method using Matlab.

A. Finite Element Discretization

In this subsection we solve the variational problem for the pricing of a European option using finit element method for the space variable, and the finit differences explicit Euler method for the time variable. The space \( \Xi \) is generally of infinite dimension, we construct by Galerkin method a subspace \( \Xi h \) such that \( \dim \Xi h \leq c < \infty \). We consider a bounded domain, with a Lipchitz boundary \( \partial \Omega \), we introduce a regular family \( (T_h)_h \) of triangulations of \( \Omega \), in the usual sense that :

- For each \( h \), the closure of \( \Omega_h \) is the union of all elements of \( T_h \).
- The intersection of two different elements of \( T_h \) is the empty, a vertex or a whole edge.

Let us first recall some standard notations: For each element \( E \) in a discretization \( T_{\text{h,e}} \) denotes the set of all elements of \( E \) that are not contained in \( \partial \Omega_h \).

* The ratio of the diameter \( h_0 \) of any element \( E \) of \( T_h \) to the diameter of its inscribed circle is smaller than a constant independent of the discretization parameter \( h \).

As standard, \( h \) stands for the maximum of the diameters \( h \in E \in T_h \). For \( u_h, v_h \in \Omega \), the discrete formulation is written as

\[
\begin{aligned}
&\left\{ \frac{\partial u_h}{\partial t} + a_h(u_h, v_h) \right\} = 0 \quad \forall t \in [0, T] \quad \text{and} \quad s, \sigma \in \Omega \\
&(u_h(0,\ldots,0), v_h) = (h(s, \sigma), v)
\end{aligned}
\]

With \( u \in L^2(\mathbb{T},\mathbb{R}) \).

By using the Green’s formula and the Direchlet boundary conditions, we obtain

\[
\begin{aligned}
&\left\{ \frac{\partial u_h}{\partial t} + a_h(u_h, v_h) \right\} = 0 \quad \forall t \in [0, T] \quad \text{and} \quad s, \sigma \in \Omega \\
&(u_h(0,\ldots,0), v_h) = (h(s, \sigma), v)
\end{aligned}
\]

And

\[
\begin{aligned}
&\frac{\partial u_h}{\partial t} = \frac{1}{2} \int_{\Omega_h} \sigma^2 \frac{\partial^2 u_h}{\partial S^2} dSd\sigma + \int_{\Omega_h} \sigma \frac{\partial u_h}{\partial S} dvdsd\sigma - \\
&\int_{\Omega_h} \frac{\partial u_h}{\partial S} dSd\sigma \int_{\Omega_h} \frac{\partial^2 u_h}{\partial S^2} dSd\sigma + \frac{1}{2} \nu^2 \sigma \frac{\partial u_h}{\partial S} dvdsd\sigma - \\
&\int_{\Omega_h} \kappa(\theta - \sigma) \frac{\partial u_h}{\partial \sigma} dvdsd\sigma + \int_{\Omega_h} \rho \sigma \nu \frac{\partial u_h}{\partial S} dvdsd\sigma + \\
&\int_{\Omega_h} \rho \sigma \nu \frac{\partial u_h}{\partial S} dvdsd\sigma + \int_{\Omega_h} ruvdvdsd\sigma
\end{aligned}
\]

with

\[
(Lu, v) = (u, v)
\]

B. Main Result

By using theorem due to Lax Miligram (see J. L. Lions[15]), we establish the existence and uniqueness of the solution of problem (4).

**Theorem 1** The variational problem (4) admits an unique solution in \( \Xi \).
Numerical Results by the finite elements method

We compare the results given by Heston model using the finite elements method described above, with those obtained by Broadie- Kaya[6] using the Monte Carlo method.

Table 1 gives results for a European call option. The set of parameters for the SV model are taken from Duffie et al. (2000). These were found by minimizing the mean squared errors for market option prices for S&P 500 on November 2, 1993. The bias column is estimated using 40 million simulation trials. The number of time steps for the Euler discretization is set to be equal to the square root of the number of simulation trials.

Table 1: Simulation results for European call using Monte Carlo in [6]

(a) Simulations with the exact method

<table>
<thead>
<tr>
<th>No of Simul</th>
<th>RMC error</th>
<th>Comp Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000</td>
<td>0.0750</td>
<td>3.8</td>
</tr>
<tr>
<td>40000</td>
<td>0.0373</td>
<td>15.2</td>
</tr>
<tr>
<td>160000</td>
<td>0.0186</td>
<td>60.0</td>
</tr>
<tr>
<td>640000</td>
<td>0.0093</td>
<td>239.4</td>
</tr>
<tr>
<td>256 0000</td>
<td>0.0046</td>
<td>955.7</td>
</tr>
</tbody>
</table>

(b) Simulations with the Euler Discretization

<table>
<thead>
<tr>
<th>No of Simul</th>
<th>No of Time</th>
<th>RMS error</th>
<th>Comp Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000</td>
<td>100</td>
<td>0.1725</td>
<td>0.2</td>
</tr>
<tr>
<td>40000</td>
<td>200</td>
<td>0.1073</td>
<td>1.9</td>
</tr>
<tr>
<td>160000</td>
<td>400</td>
<td>0.0689</td>
<td>15.2</td>
</tr>
<tr>
<td>640000</td>
<td>800</td>
<td>0.0406</td>
<td>121.3</td>
</tr>
<tr>
<td>256 0000</td>
<td>1600</td>
<td>0.0272</td>
<td>970.0</td>
</tr>
</tbody>
</table>

Note that the parameters used for the experimentations are: S = 100, K = 100, V0 = 0.010201, k = 6.21, 0 = 0.019, σ = 0.61, ρ = -0.70, r = 3.19%, T = 1.0 year, true option price=6.8061.

Table 2: Simulation results for European call option using Finite Element Method

<table>
<thead>
<tr>
<th>No of time steps</th>
<th>Price European Call</th>
<th>RMC error</th>
<th>Executing Times(sec)</th>
<th>Interest Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>7.07989</td>
<td>0.387192</td>
<td>3.86</td>
<td>3.19%</td>
</tr>
<tr>
<td>200</td>
<td>7.00494</td>
<td>0.281204</td>
<td>3.891</td>
<td>4%</td>
</tr>
<tr>
<td>400</td>
<td>6.91287</td>
<td>0.150994</td>
<td>3.844</td>
<td>5%</td>
</tr>
<tr>
<td>800</td>
<td>6.82131</td>
<td>0.021516</td>
<td>3.813</td>
<td>6%</td>
</tr>
<tr>
<td>1600</td>
<td>6.80307</td>
<td>0.00429</td>
<td>3.907</td>
<td>6.2%</td>
</tr>
</tbody>
</table>

For our numerical simulations, we choose the parameters to be: S =100; K = 100, k = 2. , ω = 2.3, v= -0.5; ρ=0.5; T = 1.0 year.
We can conclude that this method is more efficient than the Monte Carlo method used by Broadie and Kaya [6] in terms of execution time by varying the interest rate.

B. Finite difference approximation

In this subsection we present an approximation of problem (4) using a finite difference scheme. For we introduce a partition of the bounded domain

\[ \Omega = [0.7] \times ([S \min, S \max] \times [\sigma \min, \sigma \max]) \]

into subintervals \([t_i, t_{i+1}] \times [S_i, S_{i+1}] \times [\sigma_i, \sigma_{i+1}]\) such that \(t_0 = 0\), \(0 < k \leq S \Delta S, 0 \leq i \leq 1 \text{and } \Delta \sigma, 0 \leq j \leq J\)

with \(\Delta t = \frac{T}{N}, \Delta S = \frac{S \max - S \min}{J + 1}, \Delta \sigma = \frac{\sigma \max - \sigma \min}{J + 1}\)

For simplicity, we introduce the notation \((t_i, S_i, \sigma_j) = (k, i, j), V(t_i, S_i, \sigma_j) = V_{i,j}\)

Implicit scheme

Let \(\Delta X; \Delta Y\) be the space steps and \(\Delta t\) are the discretization time step size. The European option value is in the cronological anti-sense, from the maturity date \(T\). The domain price of the underlying and its volatility is infinite in theory, but numerically we take a rectangle, its center is the price of the underlying and its volatility \(t\), for which we seek to evaluate the option. If the price of the underlying and its volatility are negative, the option value is zero. Hence the value of the option \(V(S,\sigma,t)\) is approximated by

\[ V_{i,j}^k = V(i\Delta S, j\Delta \sigma, (T - k\Delta t)) \]

The derivatives can be written as follows

\[ \frac{\partial V}{\partial S} = V_{i+1,j}^{k+1} - V_{i-1,j}^{k+1} \]
\[ \frac{\partial^2 V}{\partial S^2} = \frac{2}{\Delta S^2} \]
\[ \frac{\partial V}{\partial \sigma} = V_{i,j+1}^{k+1} - V_{i,j-1}^{k+1} \]
\[ \frac{\partial^2 V}{\partial \sigma^2} = \frac{2}{\Delta \sigma^2} \]
\[ \frac{\partial^2 V}{\partial S \partial \sigma} = \frac{4 \Delta S \Delta \sigma}{\Delta t} \]
\[ \frac{\partial V}{\partial t} = \frac{V_{i,j+1}^{k+1} - V_{i,j-1}^{k+1}}{\Delta t} \]

With

\[ a = 1 + \left[ r + jv^2 \Delta \sigma + \frac{jv^2}{\Delta \sigma} \right] \Delta t, b = -\frac{\rho \sigma v j}{4} \Delta t \]
\[ c = -\frac{i \Delta t}{2} [j\Delta \sigma + v], d = -\frac{\Delta t}{2\Delta \sigma} [v^2 j + \kappa (\theta - j\Delta \sigma)] \]
\[ e = \frac{\Delta t}{2\Delta \sigma} [\kappa (\theta - j\Delta \sigma) - v^2 j] \]
\[ f = \frac{v^4}{4} \Delta t \]

With the initial condition \(V_{i,j}^{0} = (S_i - K)^+\) and the boundary conditions \(V_{i,0}^{k} = V_{i,1}^{k} = 0\) for \(0 \leq j \leq J + 1\)
\[ V_{i,0}^{k} = V_{i,J+1}^{k} = 0 \text{ for } 0 \leq i \leq I + 1 \]

C. Numerical results by the finite difference method

In this subsection, we compare the obtained numerical results using the finite difference method described in the previous
section implemented in Matlab, with those obtained by Broadie-Kaya[6] using the Monte Carlo method.

**Table 3**: Simulation results for European call option using Finite Difference Method

<table>
<thead>
<tr>
<th>No of time steps</th>
<th>Price European Call</th>
<th>RMC error</th>
<th>Executing Times (sec)</th>
<th>Interest Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>6.7272</td>
<td>0.1115</td>
<td>0.5460</td>
<td>3.19%</td>
</tr>
<tr>
<td>200</td>
<td>6.7752</td>
<td>0.03437</td>
<td>0.5772</td>
<td>4%</td>
</tr>
<tr>
<td>400</td>
<td>6.8413</td>
<td>0.0498</td>
<td>0.6396</td>
<td>5%</td>
</tr>
<tr>
<td>800</td>
<td>6.9097</td>
<td>0.1465</td>
<td>0.8892</td>
<td>6%</td>
</tr>
<tr>
<td>1600</td>
<td>6.9226</td>
<td>0.1647</td>
<td>1.2012</td>
<td>6.2%</td>
</tr>
</tbody>
</table>

Note that the option parameters taken for simulations are: S = 100; K = 100, κ = 0.021, θ = 17.94, ν = 0.5; T = 1.0 year. We remark that this method is more efficient than the Monte Carlo method used by Broadie and Kaya[6] in terms of Cputime. These tests confirm that the finite difference method is faster for different values of the interest rate when it is less than 5%.

**D. Comments the numerical results**

In this subsection we compare the results obtained by the two methods: the Cputime and the RMS error for different values of the interest rate.

**Execution Time**

Comparison of the Execution Time between F.E.M & F.D.M.

**Table 4**: Cputime

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>F.E</th>
<th>F.D</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.19</td>
<td>3.86</td>
<td>0.6460</td>
</tr>
<tr>
<td>4</td>
<td>3.891</td>
<td>0.5772</td>
</tr>
<tr>
<td>5</td>
<td>3.844</td>
<td>0.6396</td>
</tr>
<tr>
<td>6</td>
<td>3.813</td>
<td>0.8892</td>
</tr>
<tr>
<td>6.2</td>
<td>3.907</td>
<td>1.2012</td>
</tr>
</tbody>
</table>

By comparing the Cputime for the two methods, we note that the finite difference method is faster than the finite element method (see Figure 1).

**RMS error**

Comparison of RMS error between F.E.M & F.D.M

**Table 5**: RMS error

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>F.E</th>
<th>F.D</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.19</td>
<td>0.387192</td>
<td>0.1115</td>
</tr>
<tr>
<td>4</td>
<td>0.281204</td>
<td>0.0437</td>
</tr>
<tr>
<td>5</td>
<td>0.150994</td>
<td>0.0498</td>
</tr>
<tr>
<td>6</td>
<td>0.0215161</td>
<td>0.1465</td>
</tr>
<tr>
<td>6.2</td>
<td>0.00429025</td>
<td>0.1647</td>
</tr>
</tbody>
</table>

Figure 2 shows that the finite element method is more accurate than the finite difference one following the different values of the interest rate, when it is less than 5.

V. **CONCLUSION**

The aim of this paper is the pricing of the European option, by a diffusion with a stochastic volatility, where the volatility follows a Heston model. We proved the existence and uniqueness of the weak solution in a weighted Sobolev space. We present results for a comparison between two methods, the finite element and the finite difference methods for some values of the interest rate. The obtained results show that the finite element method is more accurate compare to the finite differences in terms of RMS error when the interest rate is less than 5 (Table 5 and Figure 2), however in terms of Cputime the latter method is less expensive than the first one (see Table 4 and Figure 1). Furthermore a comparison with the Monte Carlo Method used by Broadie et Al [6] is presented and the obtained results lead us to conclude that the two proposed approaches are faster and more accurate for different values of the interest rate (See Tables 1,2,3,4 & 5 and Figures 1 & 2).

Finally we have to point out that our results are closer to the true market value of the European option exercised under the stock index S&P500 on November 2, 1993.

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