Order Ten Implicit One-Step Hybrid Block Method for The Solution of Stiff Second-order Ordinary Differential Equations

Y. Skwame, J. Sabo, P. Tumba, T. Y. kyagya

Abstract— A one-step hybrid block method for initial value problems of general second order Ordinary Differential Equations has been studied in this paper. The method is developed using interpolation and collocation techniques. The use of the power series approximate solution as an interpolation polynomial and its second derivative as a collocation equation is considered in deriving the method. Numerical analysis shows that the developed new method is consistent, convergent, and order ten. The new method is then applied to solve the system of second-order ordinary differential equations and the accuracy is better when compared with the existing methods in terms of error.

Index Terms— Collocation and Interpolation Method, Hybrid Block Method, Stiff ODEs, System of Second order ODEs.

I. INTRODUCTION

Numerous problems such as chemical kinetics, orbital dynamics, circuit and control theory and Newton's second law applications involve second-order ODEs [1]. Ordinary differential equations (ODEs) are commonly used for mathematical modeling in many diverse fields such as engineering, operation research, industrial mathematics, behavioral sciences, artificial intelligence, management and sociology. This mathematical modeling is the art of translating

problem from an application area into tractable mathematical formulations whose theoretical and numerical analysis provide insight, answers and guidance useful for the originating application [2]. This type of problem can be formulated either in terms of first-order or higher order ODEs. In this article, the system of second-order ODEs of the following form is considered.

$${}^{1} y'' f(x^{1}, y^{1}, {}^{2}y'), {}^{1} y(x_{0}) = a_{0}, {}^{1}y'(x_{0}) = b_{0}$$

$${}^{2} y'' f(x^{2}, y^{2}, {}^{2}y'), {}^{2}y(x_{0}) = a_{1}, {}^{2}y'(x_{0}) = b_{1}$$

$$\vdots$$

$$(1)$$

^{*m*} $y^{m} f(x^{m}, y^{m}, {}^{m} y), {}^{m} y(x_{0}) = a_{0}, {}^{m} y(x_{0}) = b_{m}$ The method of solving higher-order ODEs by reducing them to a system of first-order approach involves more functions to

evaluate them and then leads to a computational burden as

mentioned in [3]-[5]. The multistep methods for solving higher-order ODEs directly have been developed by many scholars such as [6]-[9]. The aim of this paper is to develop a new numerical method for solving systems of second-order stiff ODEs.

II. DERIVATION OF THE METHOD

In this section, a one-step hybrid block method with three

off-step points,
$$x_{n+\frac{1}{4}}$$
, $x_{n+\frac{1}{2}}$ and $x_{n+\frac{3}{4}}$

for solving Equation (1) is derived. Let the power series of the form

$${}^{j}y(x) = \sum_{i=0}^{\nu+m-1} a_i \left(\frac{x-x_n}{h}\right)^i, \quad j = 1, \dots, m.$$
 (2)

be the approximate solution to Equation (1) for $x \in [x_n, x_{n+1}]$ where $n = 0, 1, 2, \dots N - 1$, *a*'s are the real coefficients to be determined, *v* is the number of collocation points, *m* is the number of interpolation points and $h = x_n - x_{n-1}$ is a constant step size of the partition of interval [a, b], which is given by $a = x_0 < x_1 < \dots < x_N = b$.

Differentiating Equation (2) once and twice yields:

$${}^{j}y'(x) = {}^{j}f(x^{j}, y^{j}, {}^{j}y) = \sum_{i=1}^{\nu+m-1} \frac{ia_{i}}{h} \left(\frac{x-x_{n}}{h}\right)^{i-1}, \quad j = 1, \cdots, m.$$
(3)

$${}^{j}y''(x) = {}^{j}f(x^{j}, y^{j}, y^{j}) = \sum_{i=2}^{\nu+m-1} \frac{i(i-1)a_{i}}{h^{2}} \left(\frac{x-x_{n}}{h}\right)^{i-2}, \quad j = 1, \cdots, m.$$
(4)

Interpolating Equation (2) at the selected intervals, i.e., X_n , and collocating Equation (3) and (4) at all points in the selected interval, i.e., $x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}$ and x_{n+1} , gives the following equations which can be written in matrix

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($\left(a_{0}\right)$	
1	0	0	0	0	0	0	0	0	0	0		
0	$\frac{1}{h}$	0	0	0	0	0	0	0	0	0	$\ a_1\ $	$\int_{-1}^{j} \mathcal{Y}_{n}$
0	$\frac{1}{h}$	$\frac{1}{2h}$	$\frac{3}{16h}$	$\frac{1}{16h}$	$\frac{5}{256h}$	$\frac{3}{512h}$	$\frac{7}{4096h}$	$\frac{1}{2048h}$	$\frac{9}{65536h}$	$\frac{5}{131072h}$		$\int_{n+\frac{1}{2}}^{j} \mathcal{Y}_{n+\frac{1}{2}}$
0	$\frac{1}{h}$	$\frac{1}{h}$	$\frac{3}{4h}$	$\frac{1}{2h}$	$\frac{5}{16h}$	$\frac{3}{16h}$	$\frac{2}{64h}$	$\frac{1}{16h}$	$\frac{9}{256h}$	$\frac{5}{256h}$	$ a_3 $	$\int_{n+\frac{1}{2}} y_{n+\frac{1}{2}}$
0	$\frac{1}{h}$	$\frac{3}{2h}$	$\frac{27}{16k}$	$\frac{2n}{16h}$	$\frac{405}{256h}$	$\frac{1458}{3125k}$	$\frac{5103}{4096^{h}}$	$\frac{2187}{2048^{k}}$	<u>59049</u>	$\frac{98415}{131072h}$	$ a_4 $	$\int_{n+\frac{1}{2}}^{j} \mathcal{Y}_{n+\frac{1}{2}}$
0	$\frac{1}{h}$	$\frac{2n}{h}$	$\frac{3}{h}$	$\frac{4}{h}$	$\frac{5}{h}$	$\frac{6}{h}$	$\frac{7}{h}$	$\frac{8}{h}$	$\frac{9}{h}$	$\frac{10}{h}$	a_5	$= \begin{vmatrix} {}^{j} \mathcal{Y}_{n+1} \\ {}^{j} f_{n} \end{vmatrix}$
0	0	$\frac{2}{h^2}$	0	0	0	0	0	0	0	0	$ a_6 $	$ \begin{bmatrix} ^{j} \mathbf{y}_{n+} \\ ^{j} f \end{bmatrix} $
0	0	$\frac{2}{h^2}$	$\frac{6}{4h^2}$	$\frac{1}{h^2}$	$\frac{5}{16h^2}$	$\frac{15}{128h^2}$	$\frac{21}{512h}$	$\frac{7}{512h^2}$	$\frac{9}{2048h^2}$	$\frac{45}{32768h^2}$	$ _{a_7}$	$\int_{j}^{n+1} f_{n+1}$
0	0	$\frac{2}{h^2}$	$\frac{3}{h^2}$	$\frac{3}{h^2}$	$\frac{5}{2h^2}$	$\frac{15}{8h^2}$	$\frac{21}{16h}$	$\frac{1}{2}$ $\frac{7}{8h^2}$	$\frac{9}{16h^2}$	$\frac{45}{128h^2}$		$\int_{n+1}^{n+1} f_{n+1}$
0	0	$\frac{2}{h^2}$	$\frac{9}{2h^2}$	$\frac{27}{4h^2}$	$\frac{135}{16h^2}$	$\frac{1215}{128h^2}$	$\frac{510}{2}$ $\frac{510}{512}$	$\frac{0}{h^2} = \frac{510}{512h}$	$\frac{13}{h^2} = \frac{1968}{2048}$	$\frac{33}{h^2} = \frac{590490}{65536h^2}$		$\int f_{n+1}$
0	0	$\frac{2}{h^2}$	$\frac{6}{h^2}$	$\frac{12}{h^2}$	$\frac{20}{h^2}$	$\frac{30}{h^2}$	$\frac{42}{h^2}$	$\frac{512}{12}$	$\frac{5}{2}$ $\frac{72}{h^2}$	$\frac{90}{h^2}$	$ a_9 $	
		п	п	п	n	п	п	n	n	n ,	$\left(a_{10}\right)$	

$$j=1,\cdots,m$$
.

(5)

Applying the Gaussian elimination method on Equation (5) gives the coefficient a_i 's, for i = 0 (1)10. These values are then substituted into Equation (2) to give the implicit continuous hybrid method of the form:

$${}^{j}y(x) = \sum_{i=\frac{1}{4},\frac{1}{2},\frac{3}{4}} {}^{j}\beta_{i}(x){}^{j}f_{n+i} + \sum_{i=0}^{1} {}^{j}\beta_{i}(x){}^{j}f_{n+i}, \quad j = 1, \cdots, m_{\perp}$$
(6)

Differentiating Equation (6) once yields:

$${}^{j}y'(x) = \sum_{i=\frac{1}{4},\frac{1}{2},\frac{3}{4}} \frac{d}{dx}{}^{j}\beta_{i}(x){}^{j}f_{n+i} + \sum_{i=0}^{1}\frac{d}{dx}{}^{j}\beta_{i}(x){}^{j}f_{n+i}, \quad j = 1, \cdots, m.$$
(7)

Where

$${}^{j}\beta_{0} = x - x_{n} - \frac{485}{9} \frac{(x - x_{n})^{3}}{h^{2}} + \frac{20155}{54} \frac{(x - x_{n})^{4}}{h^{3}} - \frac{171724}{135} \frac{(x - x_{n})^{5}}{h^{4}} + \frac{208100}{81} \frac{(x - x_{n})^{6}}{h^{5}} - \frac{607360}{189} \frac{(x - x_{n})^{7}}{h^{6}}$$

$$+ \frac{66080}{27} \frac{(x-x_n)^8}{h^7} - \frac{252928}{243} \frac{(x-x_n)^9}{h^8} + \frac{5120}{27} \frac{(x-x_n)^{10}}{h^9}$$

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$$\begin{split} & (\beta_{\frac{1}{4}} = -\frac{512}{9} \frac{(x-x_{a})^{3}}{h^{2}} + \frac{7168}{9} \frac{(x-x_{a})^{4}}{h^{3}} - \frac{541184}{135} \frac{(x-x_{a})^{5}}{h^{4}} + \frac{841216}{81} \frac{(x-x_{a})^{5}}{h^{5}} \\ & -\frac{2914304}{189} \frac{(x-x_{a})^{7}}{h^{6}} + \frac{358912}{27} \frac{(x-x_{a})^{8}}{h^{7}} - \frac{1507328}{233} \frac{(x-x_{a})^{9}}{h^{8}} + \frac{32768}{27} \frac{(x-x_{a})^{6}}{h^{9}} \\ & -\frac{48(x-x_{a})^{7}}{h^{2}} - \frac{456(x-x_{a})^{4}}{h^{3}} + \frac{8848}{5} \frac{(x-x_{a})^{5}}{h^{4}} - \frac{10036}{103} \frac{(x-x_{a})^{6}}{h^{5}} + \frac{26112}{7} \frac{(x-x_{a})^{7}}{h^{6}} \\ & -\frac{2048(x-x_{a})^{8}}{h^{7}} + \frac{4006}{9} \frac{(x-x_{a})^{2}}{h^{8}} \\ & -\frac{2048(x-x_{a})^{7}}{h^{7}} - \frac{326144}{h^{2}} \frac{(x-x_{a})^{8}}{127} + \frac{1441792}{243} \frac{(x-x_{a})^{9}}{h^{8}} - \frac{32768}{135} \frac{(x-x_{a})^{6}}{h^{4}} - \frac{685568}{81} \frac{(x-x_{a})^{6}}{h^{5}} \\ & + \frac{2504704}{189} \frac{(x-x_{a})^{7}}{h^{6}} - \frac{326144}{277} \frac{(x-x_{a})^{8}}{h^{7}} + \frac{1441792}{243} \frac{(x-x_{a})^{9}}{h^{8}} - \frac{32768}{81} \frac{(x-x_{a})^{10}}{h^{9}} \\ & + \frac{2504704}{h^{9}} \frac{(x-x_{a})^{7}}{h^{6}} - \frac{326144}{118} \frac{(x-x_{a})^{8}}{h^{7}} + \frac{1441792}{243} \frac{(x-x_{a})^{9}}{h^{8}} - \frac{32768}{81} \frac{(x-x_{a})^{10}}{h^{9}} \\ & + \frac{2504704}{h^{9}} \frac{(x-x_{a})^{2}}{h^{7}} - \frac{326144}{227} \frac{(x-x_{a})^{8}}{h^{7}} + \frac{1411792}{243} \frac{(x-x_{a})^{9}}{h^{8}} - \frac{32768}{81} \frac{(x-x_{a})^{10}}{h^{5}} \\ & -\frac{43552}{27} \frac{(x-x_{a})^{8}}{h^{9}} - \frac{32768}{212} \frac{(x-x_{a})^{10}}{h^{9}} \\ & -\frac{43552}{27} \frac{(x-x_{a})^{8}}{h^{7}} + \frac{207822}{243} \frac{(x-x_{a})^{9}}{h^{8}} - \frac{5120}{27} \frac{(x-x_{a})^{10}}{h^{9}} \\ & -\frac{12}{2}(x-x_{a})^{2} - \frac{50}{9} \frac{(x-x_{a})^{3}}{h^{8}} - \frac{5120}{217} \frac{(x-x_{a})^{4}}{h^{2}} \\ & -\frac{12}{45} \frac{(x-x_{a})^{3}}{h} + \frac{264}{3} \frac{(x-x_{a})^{3}}{h^{2}} - \frac{5120}{217} \frac{(x-x_{a})^{4}}{h^{3}} \\ & -\frac{166144}{63} \frac{(x-x_{a})^{7}}{h^{5}} \\ & +\frac{19228}{4} \frac{(x-x_{a})^{8}}{h^{6}} - \frac{786}{81} \frac{(x-x_{a})^{8}}{h^{7}} \\ & -\frac{166144}{63} \frac{(x-x_{a})^{7}}{h^{5}} \\ & +\frac{19228}{4} \frac{(x-x_{a})^{8}}{h^{6}} - \frac{2048(x-x_{a})^{9}}{h^{7}} + \frac{512}{5} \frac{(x-x_{a})^{5}}{h^{3}} \\ & +\frac{1528}{27} \frac{(x-x_{a})^{6}}{h^{4}} - \frac{113408}{63} \frac{(x-x_{a$$

III. CONVERGENCE ANALYSIS

3.1 Order and error Constants of the Methods According to [9] the order of the new method in Equation (5) is obtained by using the Taylor series and it is found that the

developed method has an uniformly order Ten, with an error constants vector of: $C_{10} = \begin{bmatrix} 4.1791 \times 10^{-13}, & 4.8542 \times 10^{-13}, & 5.5292 \times 10^{-13}, & 9.7083 \times 10^{-13} \end{bmatrix}^T$

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3.2 Consistency

The hybrid block method (5) is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent.

3.3 Regions of Absolute Stability (RAS)

Using the MATLAB package, we were able to plot the stability regions of the block method (see fig. below). This is done by reformulating the block method as general linear method to obtain the values of the matrices according to [10], [11]. The matrices are substituted into the stability matrix and using MATLAB software, the absolute stability regions of the new methods are plotted as shown in fig. below.



Figure: Region of Absolute Stability.

3.3 Numerical Implementation

To study the efficiency of the block hybrid method for k = 1, we present some numerical examples widely used by [14]. In this section, the performance of the developed one-step hybrid block method is examined using the following two systems of second-order initial value problems. Tables 1 and 2 show the comparison of the numerical results of the new method with the existing method [14] for solving problems 1 and 2 respectively.

Example 1

$$y_1^1 = -8y_1 + 7y_2$$
; $y_1(0) = 1$,
 $y_2^1 = 42y_1 - 43y_2$; $y_2(0) = 8$, $h = \frac{1}{10}$. With Exact

Solution

$$y_1(x) = 2e^{-x} - e^{-50x}$$
$$y_2(x) = 2e^{-x} - 6e^{-50x}$$

X-value	Error in [14] P=6 BHSM Three off-grid poin	ts	Error in New method P=10		
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$	
0.1	1.38×10^{0}	3.20×10^{0}	1.32×10^{-6}	8.10×10^{-2}	
0.2	9.02×10^{-1}	7.36×10^{-1}	1.90×10^{-8}	5.50×10^{-4}	
0.3	1.09×10^{0}	$2.58 \times 10^{\circ}$	4.00×10^{-9}	3.70×10^{-6}	
0.4	9.09×10^{-1}	$5.32 \times 10^{\circ}$	4.00×10^{-9}	2.10×10^{-8}	
0.5	8.84×10^{-1}	2.10×10^{0}	2.00×10^{-9}	3.00×10^{-9}	
0.6	7.22×10^{-1}	$3.75 \times 10^{\circ}$	3.00×10^{-9}	2.00×10^{-9}	
0.7	7.15×10^{-1}	1.71×10^{0}	4.50×10^{-9}	2.90×10^{-9}	
0.8	6.42×10^{-1}	$2.57 \times 10^{\circ}$	4.10×10^{-9}	3.70×10^{-9}	
0.9	5.78×10^{-1}	1.39×10^{0}	4.60×10^{-9}	4.00×10^{-9}	
1.0	5.68×10^{-1}	1.67×10^{-1}	4.80×10^{-9}	4.60×10^{-9}	

Table 1. Absolute Error for example 1.

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Example 2

$$y_1^1 = -y_1 + 95y_2; y_1(0) = 1, \qquad y_1(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x}$$
$$y_2^1 = -y_1 - 97y_2; y_1(0) = 1, \quad h = \frac{1}{10}. \quad \text{With Exact} \qquad y_2(x) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}$$

Solution

Table 2. Absolute Error for example 2.								
X-value	Error in	[14] P=6	Error in New method P=10					
	BHSM Three	off-grid points						
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$				
0.1	1.92×10^{0}	1.75×10^{0}	1.74×10^{-4}	1.74×10^{-4}				
0.2	$1.73 \times 10^{\circ}$	1.46×10^{0}	5.40×10^{-8}	5.30×10^{-8}				
0.3	1.85×10^{0}	1.47×10^{0}	1.00×10^{-9}	4.00×10^{-11}				
0.4	1.76×10^{0}	$1.32 \times 10^{\circ}$	2.30×10 ⁻⁹	3.50×10^{-11}				
0.5	1.68×10^{0}	1.21×10^{0}	2.20×10^{-9}	3.10×10^{-11}				
0.6	$1.59 \times 10^{\circ}$	1.10×10^{0}	1.80×10^{-9}	2.70×10^{-11}				
0.7	1.49×10^{0}	9.88×10^{-1}	1.60×10^{-9}	2.20×10^{-11}				
0.8	$1.39 \times 10^{\circ}$	8.84×10^{-1}	1.40×10^{-9}	2.00×10^{-11}				
0.9	$1.29 \times 10^{\circ}$	8.10×10^{-1}	1.20×10^{-9}	1.60×10^{-11}				
1.0	1.99×10^{0}	7.33×10^{-1}	9.00×10^{-10}	1.40×10^{-11}				

Table 2. Absolute Error for example 2

It is obvious from the result presented in the tables 1 and 2 that new method performs better than the existing method [14].

IV. CONCLUSIONS

It is evident from the above tables that our proposed methods are indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is A-stable

Comparing the new method with the existing method [14], the result presented in the tables 1 and 2 shows that the new method performs better than the existing method [14] and even the order of new method is higher than the order of the existing method [14]. In this article, a one-step block method with three off-step points is derived via the interpolation and collocation approach. The developed method is consistent, A-stable, convergent, with a region of absolute stability and order Ten.

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