

Order Ten Implicit One-Step Hybrid Block Method for The Solution of Stiff Second-order Ordinary Differential Equations

$$\begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & \frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & \frac{1}{h} & \frac{1}{2h} & \frac{3}{16h} & \frac{1}{16h} & \frac{5}{256h} & \frac{3}{512h} & \frac{7}{4096h} & \frac{1}{2048h} & \frac{9}{65536h} & \frac{5}{131072h} \\
 0 & \frac{1}{h} & \frac{1}{h} & \frac{3}{4h} & \frac{1}{2h} & \frac{5}{16h} & \frac{3}{16h} & \frac{2}{64h} & \frac{1}{16h} & \frac{9}{256h} & \frac{5}{256h} \\
 0 & \frac{1}{h} & \frac{3}{2h} & \frac{27}{16h} & \frac{27}{16h} & \frac{405}{256h} & \frac{1458}{3125h} & \frac{5103}{4096h} & \frac{2187}{2048h} & \frac{59049}{65536h} & \frac{98415}{131072h} \\
 0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} & \frac{6}{h} & \frac{7}{h} & \frac{8}{h} & \frac{9}{h} & \frac{10}{h} \\
 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \frac{2}{h^2} & \frac{6}{4h^2} & \frac{1}{h^2} & \frac{5}{16h^2} & \frac{15}{128h^2} & \frac{21}{512h^2} & \frac{7}{512h^2} & \frac{9}{2048h^2} & \frac{45}{32768h^2} \\
 0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{3}{h^2} & \frac{5}{2h^2} & \frac{15}{8h^2} & \frac{21}{16h^2} & \frac{7}{8h^2} & \frac{9}{16h^2} & \frac{45}{128h^2} \\
 0 & 0 & \frac{2}{h^2} & \frac{9}{2h^2} & \frac{27}{4h^2} & \frac{135}{16h^2} & \frac{1215}{128h^2} & \frac{510}{512h^2} & \frac{5103}{512h^2} & \frac{19683}{2048h^2} & \frac{590490}{65536h^2} \\
 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2}
 \end{pmatrix}
 \begin{pmatrix}
 a_0 \\
 a_1 \\
 a_2 \\
 a_3 \\
 a_4 \\
 a_5 \\
 a_6 \\
 a_7 \\
 a_8 \\
 a_9 \\
 a_{10}
 \end{pmatrix}
 =
 \begin{pmatrix}
 {}^j y_n \\
 {}^j y_{n+\frac{1}{4}} \\
 {}^j y_{n+\frac{1}{2}} \\
 {}^j y_{n+\frac{3}{4}} \\
 {}^j y_{n+1} \\
 {}^j f_n \\
 {}^j y_{n+1} \\
 {}^j f_{n+\frac{1}{4}} \\
 {}^j f_{n+\frac{1}{2}} \\
 {}^j f_{n+\frac{3}{4}} \\
 {}^j f_{n+1}
 \end{pmatrix}$$

$$j = 1, \dots, m. \tag{5}$$

Applying the Gaussian elimination method on Equation (5) gives the coefficient a_i 's, for $i = 0(1)10$. These values are then substituted into Equation (2) to give the implicit continuous hybrid method of the form:

$${}^j y(x) = \sum_{i=\frac{1}{4}, \frac{1}{2}, \frac{3}{4}} {}^j \beta_i(x) {}^j f_{n+i} + \sum_{i=0}^1 {}^j \beta_i(x) {}^j f_{n+i}, \quad j = 1, \dots, m. \tag{6}$$

Differentiating Equation (6) once yields:

$${}^j y'(x) = \sum_{i=\frac{1}{4}, \frac{1}{2}, \frac{3}{4}} \frac{d}{dx} {}^j \beta_i(x) {}^j f_{n+i} + \sum_{i=0}^1 \frac{d}{dx} {}^j \beta_i(x) {}^j f_{n+i}, \quad j = 1, \dots, m. \tag{7}$$

Where

$$\begin{aligned}
 {}^j \beta_0 = & x - x_n - \frac{485}{9} \frac{(x - x_n)^3}{h^2} + \frac{20155}{54} \frac{(x - x_n)^4}{h^3} - \frac{171724}{135} \frac{(x - x_n)^5}{h^4} + \frac{208100}{81} \frac{(x - x_n)^6}{h^5} \\
 & - \frac{607360}{189} \frac{(x - x_n)^7}{h^6} \\
 & + \frac{66080}{27} \frac{(x - x_n)^8}{h^7} - \frac{252928}{243} \frac{(x - x_n)^9}{h^8} + \frac{5120}{27} \frac{(x - x_n)^{10}}{h^9}
 \end{aligned}$$

$$\begin{aligned}
 {}^j\beta_{\frac{1}{4}} &= -\frac{512}{9} \frac{(x-x_n)^3}{h^2} + \frac{7168}{9} \frac{(x-x_n)^4}{h^3} - \frac{541184}{135} \frac{(x-x_n)^5}{h^4} + \frac{841216}{81} \frac{(x-x_n)^6}{h^5} \\
 &- \frac{2914304}{189} \frac{(x-x_n)^7}{h^6} + \frac{358912}{27} \frac{(x-x_n)^8}{h^7} - \frac{1507328}{243} \frac{(x-x_n)^9}{h^8} + \frac{32768}{27} \frac{(x-x_n)^{10}}{h^9} \\
 {}^j\beta_{\frac{1}{2}} &= \frac{48}{h^2} \frac{(x-x_n)^3}{h^2} - \frac{456}{h^3} \frac{(x-x_n)^4}{h^3} + \frac{8848}{5} \frac{(x-x_n)^5}{h^4} - \frac{10496}{3} \frac{(x-x_n)^6}{h^5} + \frac{26112}{7} \frac{(x-x_n)^7}{h^6} \\
 &- \frac{2048}{h^7} \frac{(x-x_n)^8}{h^7} + \frac{4096}{9} \frac{(x-x_n)^9}{h^8} \\
 {}^j\beta_{\frac{3}{4}} &= \frac{512}{9} \frac{(x-x_n)^3}{h^2} - \frac{17408}{27} \frac{(x-x_n)^4}{h^3} + \frac{426496}{135} \frac{(x-x_n)^5}{h^4} - \frac{685568}{81} \frac{(x-x_n)^6}{h^5} \\
 &+ \frac{2504704}{189} \frac{(x-x_n)^7}{h^6} - \frac{326144}{27} \frac{(x-x_n)^8}{h^7} + \frac{1441792}{243} \frac{(x-x_n)^9}{h^8} - \frac{32768}{27} \frac{(x-x_n)^{10}}{h^9} \\
 {}^j\beta_1 &= \frac{53}{9} \frac{(x-x_n)^3}{h^2} - \frac{1241}{18} \frac{(x-x_n)^4}{h^3} + \frac{47516}{135} \frac{(x-x_n)^5}{h^4} - \frac{80356}{81} \frac{(x-x_n)^6}{h^5} + \frac{311936}{189} \frac{(x-x_n)^7}{h^6} \\
 &- \frac{43552}{27} \frac{(x-x_n)^8}{h^7} + \frac{207872}{243} \frac{(x-x_n)^9}{h^8} - \frac{5120}{27} \frac{(x-x_n)^{10}}{h^9} \\
 {}^j\gamma_0 &= \frac{1}{2} (x-x_n)^2 - \frac{50}{9} \frac{(x-x_n)^3}{h} + \frac{1045}{36} \frac{(x-x_n)^4}{h^2} - \frac{796}{9} \frac{(x-x_n)^5}{h^3} + \frac{4546}{27} \frac{(x-x_n)^6}{h^4} \\
 &- \frac{12800}{63} \frac{(x-x_n)^7}{h^5} + \frac{1360}{9} \frac{(x-x_n)^8}{h^6} - \frac{5120}{81} \frac{(x-x_n)^9}{h^7} + \frac{512}{45} \frac{(x-x_n)^{10}}{h^8} \\
 {}^j\gamma_{\frac{1}{4}} &= -\frac{64}{3} \frac{(x-x_n)^3}{h} + \frac{608}{3} \frac{(x-x_n)^4}{h^2} - \frac{37696}{45} \frac{(x-x_n)^5}{h^3} + \frac{51968}{27} \frac{(x-x_n)^6}{h^4} - \frac{166144}{63} \frac{(x-x_n)^7}{h^5} \\
 &+ \frac{19328}{9} \frac{(x-x_n)^8}{h^6} - \frac{77824}{81} \frac{(x-x_n)^9}{h^7} + \frac{8192}{45} \frac{(x-x_n)^{10}}{h^8} \\
 {}^j\gamma_{\frac{1}{2}} &= -\frac{24}{h} \frac{(x-x_n)^3}{h} + \frac{264}{h^2} \frac{(x-x_n)^4}{h^2} - \frac{6248}{5} \frac{(x-x_n)^5}{h^3} + \frac{3224}{h^4} \frac{(x-x_n)^6}{h^4} - \frac{4864}{h^5} \frac{(x-x_n)^7}{h^5} \\
 &+ \frac{4288}{h^6} \frac{(x-x_n)^8}{h^6} - \frac{2048}{h^7} \frac{(x-x_n)^9}{h^7} + \frac{2048}{5} \frac{(x-x_n)^{10}}{h^8} \\
 {}^j\gamma_{\frac{3}{4}} &= -\frac{64}{9} \frac{(x-x_n)^3}{h} + \frac{736}{9} \frac{(x-x_n)^4}{h^2} - \frac{18368}{45} \frac{(x-x_n)^5}{h^3} + \frac{30208}{27} \frac{(x-x_n)^6}{h^4} - \frac{113408}{63} \frac{(x-x_n)^7}{h^5} \\
 &+ \frac{15232}{9} \frac{(x-x_n)^8}{h^6} - \frac{69632}{81} \frac{(x-x_n)^9}{h^7} + \frac{8192}{45} \frac{(x-x_n)^{10}}{h^8} \\
 {}^j\gamma_1 &= -\frac{1}{3} \frac{(x-x_n)^3}{h} + \frac{47}{12} \frac{(x-x_n)^4}{h^2} - \frac{904}{45} \frac{(x-x_n)^5}{h^3} + \frac{1538}{27} \frac{(x-x_n)^6}{h^4} - \frac{6016}{63} \frac{(x-x_n)^7}{h^5} \\
 &+ \frac{848}{9} \frac{(x-x_n)^8}{h^6} - \frac{4096}{81} \frac{(x-x_n)^9}{h^7} + \frac{512}{45} \frac{(x-x_n)^{10}}{h^8}
 \end{aligned}$$

III. CONVERGENCE ANALYSIS

3.1 Order and error Constants of the Methods

According to [9] the order of the new method in Equation (5) is obtained by using the Taylor series and it is found that the

developed method has a uniformly order Ten, with an error constants vector of:

$$C_{10} = [4.1791 \times 10^{-13}, 4.8542 \times 10^{-13}, 5.5292 \times 10^{-13}, 9.7083 \times 10^{-13}]^T$$

3.2 Consistency

The hybrid block method (5) is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent.

3.3 Regions of Absolute Stability (RAS)

Using the MATLAB package, we were able to plot the stability regions of the block method (see fig. below). This is done by reformulating the block method as general linear method to obtain the values of the matrices according to [10], [11]. The matrices are substituted into the stability matrix and using MATLAB software, the absolute stability regions of the new methods are plotted as shown in fig. below.

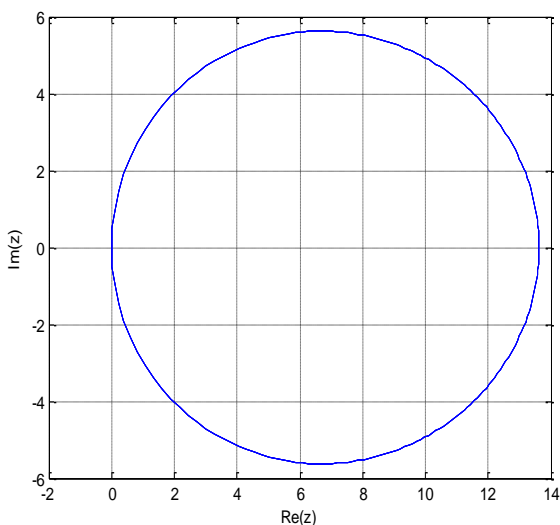


Figure: Region of Absolute Stability.

3.3 Numerical Implementation

To study the efficiency of the block hybrid method for $k = 1$, we present some numerical examples widely used by [14]. In this section, the performance of the developed one-step hybrid block method is examined using the following two systems of second-order initial value problems. Tables 1 and 2 show the comparison of the numerical results of the new method with the existing method [14] for solving problems 1 and 2 respectively.

Example 1

$$y_1^1 = -8y_1 + 7y_2 ; y_1(0) = 1,$$

$$y_2^1 = 42y_1 - 43y_2 ; y_2(0) = 8, \quad h = \frac{1}{10} . \text{ With Exact}$$

Solution

$$y_1(x) = 2e^{-x} - e^{-50x}$$

$$y_2(x) = 2e^{-x} - 6e^{-50x}$$

Table 1. Absolute Error for example 1.

<i>X - value</i>	Error in [14] P=6 BHSM Three off-grid points		Error in New method P=10	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	1.38×10^0	3.20×10^0	1.32×10^{-6}	8.10×10^{-2}
0.2	9.02×10^{-1}	7.36×10^{-1}	1.90×10^{-8}	5.50×10^{-4}
0.3	1.09×10^0	2.58×10^0	4.00×10^{-9}	3.70×10^{-6}
0.4	9.09×10^{-1}	5.32×10^0	4.00×10^{-9}	2.10×10^{-8}
0.5	8.84×10^{-1}	2.10×10^0	2.00×10^{-9}	3.00×10^{-9}
0.6	7.22×10^{-1}	3.75×10^0	3.00×10^{-9}	2.00×10^{-9}
0.7	7.15×10^{-1}	1.71×10^0	4.50×10^{-9}	2.90×10^{-9}
0.8	6.42×10^{-1}	2.57×10^0	4.10×10^{-9}	3.70×10^{-9}
0.9	5.78×10^{-1}	1.39×10^0	4.60×10^{-9}	4.00×10^{-9}
1.0	5.68×10^{-1}	1.67×10^{-1}	4.80×10^{-9}	4.60×10^{-9}

Example 2

$$y_1^1 = -y_1 + 95y_2; y_1(0) = 1,$$

$$y_2^1 = -y_1 - 97y_2; y_1(0) = 1, \quad h = \frac{1}{10} \cdot \text{With Exact}$$

$$y_1(x) = \frac{95}{47} e^{-2x} - \frac{48}{47} e^{-96x}$$

$$y_2(x) = \frac{48}{47} e^{-96x} - \frac{1}{47} e^{-2x}$$

Solution

Table 2. Absolute Error for example 2.

X – value	Error in [14] P=6 BHSM Three off-grid points		Error in New method P=10	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	1.92×10^0	1.75×10^0	1.74×10^{-4}	1.74×10^{-4}
0.2	1.73×10^0	1.46×10^0	5.40×10^{-8}	5.30×10^{-8}
0.3	1.85×10^0	1.47×10^0	1.00×10^{-9}	4.00×10^{-11}
0.4	1.76×10^0	1.32×10^0	2.30×10^{-9}	3.50×10^{-11}
0.5	1.68×10^0	1.21×10^0	2.20×10^{-9}	3.10×10^{-11}
0.6	1.59×10^0	1.10×10^0	1.80×10^{-9}	2.70×10^{-11}
0.7	1.49×10^0	9.88×10^{-1}	1.60×10^{-9}	2.20×10^{-11}
0.8	1.39×10^0	8.84×10^{-1}	1.40×10^{-9}	2.00×10^{-11}
0.9	1.29×10^0	8.10×10^{-1}	1.20×10^{-9}	1.60×10^{-11}
1.0	1.99×10^0	7.33×10^{-1}	9.00×10^{-10}	1.40×10^{-11}

It is obvious from the result presented in the tables 1 and 2 that new method performs better than the existing method [14].

IV. CONCLUSIONS

It is evident from the above tables that our proposed methods are indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is *A – stable*. Comparing the new method with the existing method [14], the result presented in the tables 1 and 2 shows that the new method performs better than the existing method [14] and even the order of new method is higher than the order of the existing method [14]. In this article, a one-step block method with three off-step points is derived via the interpolation and collocation approach. The developed method is consistent, *A – stable*, convergent, with a region of absolute stability and order Ten.

REFERENCES

- [1] M. Alkasasbeh; Zurni O. Implicit one-step block hybrid third-derivative method for the direct solution of initial value problems of second –order ordinary differential equations. J. apply.math. 2017, p. 8
- [2] Omar, Z.; Sulaiman, M. Parallel r-point implicit block method for solving higher order ordinary differential equations directly. J. ICT 2004, 3, 53–66.
- [3] Kayode, S.J.; Adeyeye, O. A 3-step hybrid method for direct solution of second order initial value problems. Aust. J. Basic Appl. Sci. 2011, 5, 2121–2126
- [4] James, A.; Adesanya, A.; Joshua, S. Continuous block method for the solution of second order initial value problems of ordinary differential equation. Int. J. Pure Appl. Math. 2013, 83, 405–416.
- [5] Omar, Z.; Suleiman, M.B. Parallel two-point explicit block method for solving high-order ordinary differential equations. Int. J. Simul. Process Model. 2006, 2, 227–231.
- [6] Vigo-Aguiar, J.; Ramos, H. Variable stepsize implementation of multistep methods for $y = f(x, y, y^1)$. J. Comput. Appl. Math. 2006, 192, 114–131.
- [7] Adesanya, A.O.; Anake, T.A.; Udo, O. Improved continuous method for direct solution of general second order ordinary differential equations. J. Niger. Assoc. Math. Phys. 2008, 13, 59–62.
- [8] Awoyemi, D.O. A P-stable linear multistep method for solving general third order of ordinary differential equations. Int. J. Comput. Math. 2003, 80, 985–991.
- [9] Henrici, P. Some Applications of the Quotient-Difference Algorithm. Proc. Symp. Appl. Math. 1963a, 15, 159–183.
- [10] Lambert, J.D. Computational Methods in ODEs; John Wiley and Sons: New York, NY, USA, 1973.
- [11] Y. Skwame.; J. Sabo.; T. Y. Kyagya. The construction of implicit one-step block hybrid methods with multiple off-grid points for the solution of stiff ODEs. JSRR 2017, 2320-0227
- [12] Kuboye, J.O.; Omar, Z. Derivation of a Six-Step Block method for direct solution of second order ordinary differential Equations. Math. Comput. Appl. 2015, 20, 151–159.
- [13] L.W. Jackson, S.K. Kenue, A fourth order exponentially fitted method, SIAM J. Numer. Anal., 11 (1974), 965-978.
- [14] Althemail, J. M.; Skwame, Y., Donald, J. Z. Multiple Off-Grid Hybrid Block Simpson’s Methods for Solution of Stiff Ordinary Differential Equations. IJSR 2014, 2319-7064