p-Convex Functions in Discrete Sets

Aboubakr Bayoumi, Ahmed fathy Ahmed

Abstract — We study the new concept of a p-convex function and A-p-convex sets for some set A of a vector space E. These concepts may have applications in convex and non linear analysis and other topics of mathematical sciences.

Index Terms — Non convex analysis, convex and p-convex sets, A-p-convex sets, p-convex functions, discrete sets

I. INTRODUCTION

In this paper, we extend some concepts and theorems to non convex analysis. In fact, we have proved that the family of p-convex sets form a vector space, (Th.2.3).

The epigraph of f is defined to be the set of all points lying on or above its graph. We proved that if S is a nonempty p-convex set in \( \mathbb{R}^n \) and \( f: S \rightarrow \mathbb{R}^+ \), then \( f \) is p-convex if and only if the epigraph of \( f \), p-epi \( f \), is a p-convex set (Th.3.3). As an example of a p-convex function is \( f(x) = \|x\|_p \) defined by a p-norm on a vector space \( E \).

We also proved equivalent properties to A-p-convex sets for some fixed set \( A \) of \( E \), (Th.3.5). Here the set \( B \) is said to be an operator between two power sets, \( \gamma = \gamma_{\mathcal{P}(E)}^{\mathcal{P}(A)} \), is defined by \( \gamma(B) = p - \text{cux}(B \cap Q) \land A \) for a real vector space \( E \) and for any subset \( A \) of \( E \) we proved that \( B \) is an A-p-convex if and only if \( B = \gamma(B) \) for all \( Q \supseteq A \), (Th. 3.5)

II. SOME PROPERTIES OF P-CONVEX SETS :

Throughout this paper we let \( 0 < p \leq 1 \). Let us first go to the following definitions and examples.

A “p-norm” on a vector space \( E \) over a field \( K \) is a mapping \( f(x) = \|x\|_p \) from \( E \) to \( R^+ \cup \{0\} \) satisfying the following axioms,

1. \( \|x\|_p = 0 \) if and only if \( x = 0 \), \( x \in E \)
2. \( \|tx\|_p = |t| \|x\|_p \), \( t \in K, x \in E \)
3. \( \|x + y\|_p \leq \|x\|_p + \|y\|_p \), \( x, y \in E \)

Let \( U \) be a set in a vector space \( E \) and \( x, y \in U, s, t \geq 0 \). The set

\[ A_x^y = \{sx+ty, s \leq 1 \} \]

is said to be the “closed arc segment” joining \( x, y \). \( A_x^y \) can also be written as

\[ A_x^y = \{t^{p-1}x + (1-t)^{p-1}y, s \leq 1 \} \]

In what follows we show that the family of p-convex sets is closed under the operations of the sum and scalar multiplication.

Theorem 2.1. If \( C_1 \) and \( C_2 \) are p-convex sets, then \( C_1 + C_2 \) is also p-convex, where

\[ C_1 + C_2 = \{x_1 + x_2; x_1 \in C_1, x_2 \in C_2 \} \]

Proof.

Let \( x, y \in C_1 + C_2 \) then

Aboubakr Bayoumi, Ahmed fathy Ahmed, Mathematics Deprt., Alazhar University / Faculty of science, Cairo, Naser City, Cairo Egypt, tel.+20 127 65976597585.P.O.Box 11884
p-Convex Functions in Discrete Sets

Theorem 2.2. If C is a p-convex set, then \( \mathcal{O}C \) is also a p-convex set, where \( \mathcal{O}C=\{\mathcal{O}x, x \in C, \mathcal{O} \in \mathcal{K}\} \)

Proof. Let \( x, y \in \mathcal{O}C \), then \( x = \mathcal{O}x_1 \) and \( y = \mathcal{O}y_1 \) for every \( x_1, y_1 \in C \).

Now
\[
(1-t)^{1/p}x + t^{1/p}y = (1-t)^{1/p}(x_1 + x_2) + t^{1/p}(y_1 + y_2)
\]
\[
= (1-t)^{1/p}x_1 + t^{1/p}y_1 + \mathcal{O}(1-t)^{1/p}x_2 + t^{1/p}y_2 \in C_1 + C_2.
\]
Therefore \( C_1 + C_2 \) is a p-convex set. ■

Corollary 2.3 If \( C_1 \) and \( C_2 \) are p-convex sets, then \( \mathcal{O}C_1 + \mathcal{O}C_2 \) is also a p-convex set, where \( \mathcal{O}C_1, \mathcal{O}C_2 \) are scalar. That is, the family of p-convex sets forms a vector space.

Proof. It is obvious. ■

If \( X \) is a topological vector space and \( A \) is a subset of \( X \), the “closed p-convex hull”, denoted by “p-cvx “, is the smallest closed p-convex set containing \( A \).

Let \( \chi_1, \chi_2, \ldots, \chi_n \in A \), where \( A \) is p-convex. Given \( t_j \geq 0 \) such that \( \sum^n_{j=1} t_j = 1 \), then \( \sum^n_{j=1} t_j \chi_j \) is said to be a “p-convex combination of \( \chi_j \).”

III. P-CONVEX FUNCTIONS AND P-EPGRAPH.

In this paper, we extend some concepts and theorems to non convex analysis. In fact, we have proved that the family of p-convex sets form a vector space, (Th.2.3).

The epigraph of \( f \) is defined to be the set of all points lying on or above its graph. We proved that, if \( S \) is a nonempty p-convex set in \( \mathbb{R}^n \) and \( f: S \rightarrow \mathbb{R}^+ \). Then \( f \) is p-convex if and only if the epigraph of \( f \), p-epi \( f \), is a p-convex set (Th. 3.3). As an example of a p-convex function is \( f(x)=\|x\|_p \) defined by a p-norm on a vector space (Example 3.1).

We also proved equivalent properties to A-p-convex sets for some fixed set A of E, (Th.3.5). Here the set B is said to be A-p-convex set if \( B=A \cap C \) for some p-convex set \( C \) in \( E \).

Now for two fixed subsets \( A \) and \( Q \) of a vector space \( E \), if an operator between two power sets,

\[
\gamma = \gamma_{pq} : P(E) \rightarrow P(A),
\]
is defined by
\[
\gamma(B) = p - \text{cvx}(B \cap Q) \cap A
\]
for a real vector space \( E \) and for any subset \( A \) of \( E \) we proved that \( B \) is a A-p-convex if and only if \( B \sqsubseteq \gamma(B) \) for all \( A \sqsubseteq B \), (Th.3.5)

Finally, we proved that the intersection \( \cap B_j, j \in J \) of A-p-convex sets \( B_j \) is A-p-convex, (Th.3.8).

Example 3.1. The following is an example of a p-convex function. Let \( f: B_{1p2}^2 \rightarrow \mathbb{R} \) with \( f(x)=\|x\|_p, x \in B_{1p2} \).

In fact,
\[
f((1-t)^{1/p}x + t^{1/p}y) = \|((1-t)^{1/p}x + t^{1/p}y)\|_p
\]
\[
\leq \|((1-t)^{1/p}x\|_p + \|t^{1/p}y\|_p
\]
i.e. \( f(x) \) is a p-convex function. ■

Example 3.2. The following is an example of half-convex function
\[
f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}, x_1, x_2 \geq 0
\]
In fact,
\[
f((1-t)^{1/2}x_1 + t^{1/2}y_1, ((1-t)^{1/2}x_2 + t^{1/2}y_2))
\]
\[
= f((1-t)^{1/2}x_1, ((1-t)^{1/2}x_2 + t^{1/2}y_1, t^{1/2}y_2))
\]
\[
= f((1-t)^{1/2}x_1 + t^{1/2}y_1, ((1-t)^{1/2}x_2 + t^{1/2}y_2))
\]
\[
= \sqrt{(1-t)^{1/2}x_1 + t^{1/2}y_1} + \sqrt{(1-t)^{1/2}x_2 + t^{1/2}y_2}
\]
\[
\leq (1-t) \sqrt{x_1} + t \sqrt{y_1}
\]
\[
+ (1-t) \sqrt{x_2} + t \sqrt{y_2}
\]
\[
\leq (1-t)(\sqrt{x_1} + \sqrt{x_2}) + t(\sqrt{y_1} + \sqrt{y_2})
\]
\[
\leq (1-t)f(x) + tf(y).
\]

Let \( S \) be a non convex set of a vector space \( E \) and \( f:S \rightarrow \mathbb{R} \); “the epigraph of \( f \)” denoted by p-epi \( f \), is a subset of \( E \times \mathbb{R} \) defined by,
\[
p-epi f = \{(x, t) \in S \times \mathbb{R}, f(x) \leq t \}
\]
That is, the epigraph of \( f \) is the set of all points lying on or above its graph.

Theorem 3.3. Let \( S \) be a nonempty p-convex set in \( \mathbb{R}^n \) and \( f:S \rightarrow \mathbb{R}^+ \). Then \( f \) is p-convex if and only if \( p-epi f \) is a p-convex set.

Proof. Assume \( f \) is a p-convex function. Let \( (a, b) \in S \) and \((a, \alpha), (a, \beta) \in p-epi f \) . Then for any \( t \in [0,1] \), we have
For all \( t_0, t_1, \ldots, t_n \) with \( \sum_{j=0}^n t_j^p = 1 \), if \( \sum_{j=0}^n t_j a_j \in A \), then \( \sum_{j=0}^n t_j a_j \in B \).

**Proof.** This is obvious. As far as property 4 is concerned, we can in view generalize of Caratheodory's theorem, let \( n \) be the dimension of \( E \) if the space is finite dimensional; otherwise we must use all \( n \). \( \blacksquare \)

Let us now fix two subsets \( A \) and \( Q \) of a vector space \( E \) and define an operator between two power sets,

\[
\gamma = \gamma_{PQ} : P(E) \rightarrow P(A),
\]

by

\[
\gamma(B) = p - cvx(B \cap Q) \cap A.
\]

We may consider \( E = \mathbb{R}^n \), \( A = m\mathbb{Z}^n \), \( m = 1, 2, \ldots \) and \( Q = \mathbb{Z}^n \).

Note that \( \gamma(C) \) is \( A \)-p-convex if \( C \) is \( p \)-convex in \( \mathbb{R}^n \).

**Theorem 3.6.** Let \( E \) be a real vector space and \( A \) any subset of \( E \). Then \( B \) is \( A \)-p-convex if and only if \( B = \gamma(B) \) for all \( \mathcal{Q} \mathcal{A} \), also if and only if \( B = \gamma(B) \) for some \( \mathcal{Q} \mathcal{A} \).

**Proof.** If \( B \) is \( A \)-p-convex, then \( B = C \cap A \) for some \( p \)-convex \( C \).

Let \( E \) be a real vector space and fix a subset \( A \) of \( E \). A set \( B \) of \( E \) is said to be \( \textit{\( A \)-p-convex} \) if there exist a \( p \)-convex set \( C \) in \( E \) such that \( B = C \cap A \).

**Corollary 3.7.** If \( B = C \cap A \), then \( \mathcal{Q} \gamma(B) \) for any \( Q \).

**Proof.** Note that in the previous definition of \( A \)-p-convex sets we may take

\[
C = \mathcal{Q} \gamma(B) = p - cvxB
\]

Provided that \( \mathcal{Q} \mathcal{A} \). \( \blacksquare \)

**Theorem 3.8.** Let \( E \) be a vector space and \( A \) any subset of \( E \). If \( B_j, j \in J \), are \( A \)-p-convex sets, then the intersection \( \bigcap B_j \) is an \( A \)-p-convex set. If the index set \( J \) is ordered and filtering to right and if \( (B_j)_j \) is an increasing family of \( A \)-p-convex sets, then the union \( \bigcup B_j \) is also \( p \)-convex.

**Proof.** For each \( B_j \), we have

\[
B_j = C_j \cap A,
\]

where \( C_j = p \)-cvx is a \( p \)-convex set in \( E \). Then

\[
\bigcap B_j = (C_j \cap A) = (\bigcap C_j) \cap A.
\]

For the union we have

\[
\bigcup B_j = \bigcup (C_j \cap A) = (\bigcup C_j) \cap A.
\]

Let us now fix two subsets \( A \) and \( Q \) of a vector space \( E \) and define an operator between two power sets,

\[
\gamma = \gamma_{PQ} : P(E) \rightarrow P(A),
\]

by

\[
\gamma(B) = p - cvx(B \cap Q) \cap A.
\]

We may consider \( E = \mathbb{R}^n \), \( A = m\mathbb{Z}^n \), \( m = 1, 2, \ldots \) and \( Q = \mathbb{Z}^n \).

Note that \( \gamma(C) \) is \( A \)-p-convex if \( C \) is \( p \)-convex in \( \mathbb{R}^n \).

**Theorem 3.6.** Let \( E \) be a real vector space and \( A \) any subset of \( E \). Then \( B \) is \( A \)-p-convex if and only if \( B = \gamma(B) \) for all \( \mathcal{Q} \mathcal{A} \), also if and only if \( B = \gamma(B) \) for some \( \mathcal{Q} \mathcal{A} \).

**Proof.** If \( B \) is \( A \)-p-convex, then \( B = C \cap A \) for some \( p \)-convex \( C \).

Let \( E \) be a real vector space and fix a subset \( A \) of \( E \). A set \( B \) of \( E \) is said to be \( \textit{\( A \)-p-convex} \) if there exist a \( p \)-convex set \( C \) in \( E \) such that \( B = C \cap A \).

**Corollary 3.7.** If \( B = C \cap A \), then \( \mathcal{Q} \gamma(B) \) for any \( Q \).

**Proof.** Note that in the previous definition of \( A \)-p-convex sets we may take

\[
C = \mathcal{Q} \gamma(B) = p - cvxB
\]

Provided that \( \mathcal{Q} \mathcal{A} \). \( \blacksquare \)

**Theorem 3.8.** Let \( E \) be a vector space and \( A \) any subset of \( E \). If \( B_j, j \in J \), are \( A \)-p-convex sets, then the intersection \( \bigcap B_j \) is an \( A \)-p-convex set. If the index set \( J \) is ordered and filtering to right and if \( (B_j)_j \) is an increasing family of \( A \)-p-convex sets, then the union \( \bigcup B_j \) is also \( p \)-convex.

**Proof.** For each \( B_j \), we have

\[
B_j = C_j \cap A,
\]

where \( C_j = p \)-cvx is a \( p \)-convex set in \( E \). Then

\[
\bigcap B_j = (C_j \cap A) = (\bigcap C_j) \cap A.
\]

For the union we have

\[
\bigcup B_j = \bigcup (C_j \cap A) = (\bigcup C_j) \cap A.
\]
\[ \bigcup B_j = \bigcup (C_j \cap A) = (\bigcup C_j) \cap A. \]

\[ \text{IV. CONCLUSION} \]

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

\[ \text{ACKNOWLEDGE} \]

The Author would like to thank Prof. C.O. Kiselman at Uppsala University for his kind interest and fruitful dissections in this work.

\[ \text{REFERENCES} \]


Dr Aboubakr Bayoumi graduated from faculty of Science at Cairo University in 1967. He had got his Ph. D in mathematics in 1979 at Uppsala University. The title of his thesis is “Holomorphic functions in metric spaces”. He obtained a Fellowship from Mittag Leffler Institute in Stockholm for two years (1980-1982).


In 2010 he has was awarded a gold medal in Mathematics from Galileo Academy of Science in London.

Ahmed Fathy Ahmed, Lecturer in Mathematics. Mr. Ahmed has graduated from Faculty of Science, in 2004, and Master in Pure Mathematics in 2014 from Faculty of Science, Al Azhar University, Egypt.