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p-Convex Functions in Discrete Sets

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***Abstract*— We study the new concept of a p-convex function and A-p-convex sets for some set A of a vector space E. These concepts may have applications in convex and non linear analysis and other topics of mathematical sciences.**

***Index Terms*— Non convex analysis, convex and p-convex sets , A-p-convex sets, p-convex functions, discrete sets**

I. INTRODUCTION

In this paper, we extend some concepts and theorems to non convex analysis . In fact, we have proved that the family of p-convex sets form a vector space, (Th.2.3).

The epigraph of f is defined to be the set of all points lying on or above its graph. We proved that, if S is a nonempty p-convex set in R^n and f: S→ R^+. Then f is p-convex if and only if the epigraph of f, p-epi f , is a p-convex set (Th. 3.3). As an example of a p-convex function is f(x)= $||x||_p$ defined by a p-norm on a vector space (Example 3.1).

We also proved equivalent properties to A-p-convex sets for some fixed set A of E, (Th.3.5). Here the set B is said to be A-p-convex set if B=A$\cap C$ for some p-convex set C in E.

Now for two fixed subsets A and Q of a vector space E if an operator between two power sets,

$\gamma = \gamma_{PQ}$:P(E)→ $P(A)$,

is defined by

$\gamma(B) = p - cvx(B \cap Q) \cap A$

for a real vector space E and for any subset A of E we proved that B is a A-p-convex if and only if B=γ (B) for all Q $\supset A$, (Th. 3.5)

II. SOME PROPERTIES OF P-CONVEX SETS :

Throughout this paper we let 0<p$\leq$1. Let us first go to the following definitions and examples

A "**p-norm**" on a vector space E over a field K is a mapping f(x)=$||x||_p$ from E to $R^+ U\ \{0\}$ satisfying the following axioms,

1. $||x||_p$=0 if and only if x=0, x $\in E$
2. $||tx||_p = |t|^p\ \ ||x||_p\ ,\ t \in K, x \in E$
3. $||x+y||_p$= ≤ $||x||_p$ + $||y||_p$= , x,y for all x,y$\in E$.

Let U be a set in a vector space E and x, y $\in$U, s ,t ≥0. The set

A_x^y={sx+ty ; s^p+t^p=1}, x$\in E$

={$s^{1/p}x$+$t^{1/p}$y ; s+t=1}

Is said to be the "**closed arc segment** " joining x, y. A_x^y can also be written as

$A_x^y = \{(1-t)^{1/p}$ x+$t^{1/p}y,;\ s^p + t^p = 1\}$.

A set A in a vector space is said to be "**P-convex**" if for all x ,y $\in A\ , s, t \geq 0$, we have

s x + t y $\in A$ whenever s^p + t^p=1

Equivalently, if we put s=1-t , we obtain $(1-t)^{1lp}$x+t^{1lp}y $\in A$ whenever $0 \leq t \leq 1$.

We note that the singleton set $A = \{x\}$ is convex but it is not p-convex.

As examples of p-convex sets is the arc segment defined above.

Also the closed unit ball in B_{l2}^p , in the metric space l_2^p ,

B_{l2}^p={$x = (x_1, x_2)$; $\mathrm{I}x_1\mathrm{I}^p + \mathrm{I}x_2\mathrm{I}^p\ \ \leq 1$}

Is a p-convex set.

We claim that $(1-t)^{1lp}\mathrm{x} + t^{1lp}\mathrm{y}\ \in B_{l2}^p$, for x ,y $\in B_{l2}^p$

Note that

$(1-t)^{1lp}(\mathrm{x}_1,\mathrm{x}_2\ ) + t^{1lp}(\mathrm{y}_1,\mathrm{y}_2\ )$

= $((1-t)^{1lp}\ \mathrm{x}_1$ + $t^{1lp}\mathrm{y}_1$, $(1-t)^{1lp}x_2$+ $t^{1lp}\mathrm{y}_2$).

Since ($a + b)^2 \leq a^2 + b^2$, a , b> 0 and 1$\geq p > 0$, it follows that

| $((1-t)^{1lp}\ \mathrm{x}_1$ + $t^{1lp}\mathrm{y}_1|^p$ + $|((1-t)^{1lp}\ \mathrm{x}_2$+$t^{1lp}\mathrm{y}_2|^p$

≤($|(1-t)^{1lp}\ \mathrm{x}_1|$ +| $t^{1lp}\mathrm{y}_1|)^p$ +(| $(1-t)^{1lp}\ \mathrm{x}_2|$ +|$t^{1lp}\mathrm{y}_2|)^p$

≤($(1-t)|\ \ \ |x_1|^P$ + $t|y_1|^p$ + $(1-t)|x_2|^P$ +$t|y_2|^p$

≤ (1-t)+t ≤1. ■

In what follows we show that the family of p-convex sets is closed under the operations of the sum and scalar multiplication.

Theorem 2.1 . If C_1 and C_2 are p-convex sets , then $C_1 + C_2$ is also p-convex, where

$C_1 + C_2 = \{x_1 + x_2;\ \ x_1 \in C_1\ , x_2 \in C_2\}$

Proof.

Let x, y $\in C_1 + C_2$ then

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x= x_1+x_2 and y= y_1+y_2, for every $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$,
Now ,
$(1-t)^{1/p}x + t^{1/p}$ y= $(1-t)^{1/p}(x_1+x_2)$ $+t^{1/p}(y_1+y_2)$
=[$(1-t)^{1/p}x_1 + t^{1Lp}y_1$]+ $[(1-t)^{1/p}x_2 + t^{1lp}y_2] \in C_1+C_2$.
Therefore C_1+C_2 is a p-convex set . . ■

Theorem 2.2. If C is a p-convex set, then $\propto$C is also p-convex, where
$\propto$C={$\propto$x, x$\in C, \propto \in K\}$

Proof. Let x, y$\in\propto$C , then $x=\propto x_1$ and y=$\propto y_1$ for every $x_1, y_1 \in C$.
Now
$(1-t)^{1/p}x + t^{1/p}$ y= $(1-t)^{1/p}\propto x_1 + +t^{1/p}\propto y_1$
= $\propto[(1-t)^{1/p}x_1 + +t^{1/p}y_1]$
Then $\propto$C is a p-convex set. ■

Corollary 2.3 If C_1 and C_2 are p-convex sets, then $\propto C_1+\beta C_2$ is also a p-convex set, where $\propto, \beta$ are scalar. That is, the family of p-convex sets form a vector space.

Proof . It is obvious. ■

If X is a topological vector space and A is a subset of X , the "**closed p-convex hull**" , denoted by "**p-cvx** ", is the smallest closed p-convex set containing A.

Let $x_1, x_2, \ldots, x_n \in A$, where A is p-convex . Given $t_j \geq 0$ such that $\sum_1^n t_j^p$ =1, then $\sum_1^n t_j x_j$ is said to be a "**p-convex combination of** x_j."

III. P-CONVEX FUNCTIONS AND P-EPIGRAPH.

In this paper, we extend some concepts and theorems to non convex analysis . In fact, we have proved that the family of p-convex sets form a vector space, (Th.2.3).

The epigraph of f is defined to be the set of all points lying on or above its graph. We proved that, if S is a nonempty p-convex set in R^n and f: S$\to R^+$. Then f is p-convex if and only if the epigraph of f, p-epi f , is a p-convex set (Th. 3.3). As an example of a p-convex function is f(x)= $||x||_p$ defined by a p-norm on a vector space (Example 3.1).

We also proved equivalent properties to A-p-convex sets for some fixed set A of E, (Th.3.5). Here the set B is said to be A-p-convex set if B=A$\cap C$ for some p-convex set C in E.

Now for two fixed subsets A and Q of a vector space E if an operator between two power sets,

$\gamma=\gamma_{PQ}$:P(E)$\to P(A)$,
is defined by
$\gamma(B)=p-cvx(B\cap Q)\cap A$
for a real vector space E and for any subset A of E we proved that B is a A-p-convex if and only if B=γ (B) for all Q$\supset A$, (Th. 3.5)

Finally, we proved that the intersection $\cap B_j$, j$\in J$ of A-p-convex sets B_j is A-p-convex, (Th.3.8).

Example 3.1. The following is an example of a **p-convex function**. let
f: $B_{lp2} \to$R with f(x)=$||x||_p$, x$\in B_{lp2}$,
In fact,
f($(1-t)^{1/p}x + t^{1/p}$y)=$||(1-t)^{1/p}x + t^{1/p}y||_p$
$\leq ||(1-t)^{1lp}x||_p + ||t^{1lp}y||_p$
i.e. f(x) is a p-convex function. ■

Example 3.2. The following is an example of ½ **-convex function**
f(x_1, x_2)) =$\sqrt{x_1}+\sqrt{x_2}$, $x_1, x_2 \geq 0$
In fact,
f((1-t)^{2}x+t^2 y)=f((1-$t)^2$($x_1, x_2) + t^2$ ((y_1, y_2))
=f((1-$t)^2$ x_1, ((1-$t)^2$ x_2)+(t^2 $y_1, t^2 y_2$))
=f((1-$t)^2$ x_1+t^2 y_1), ((1-$t)^2$ x_2+t^2 y_2))
= $\sqrt{(1-t)^2 x_1+t^2 y_1} + \sqrt{(1-t)^2 x_2+t^2 y_2)}$
$\leq (1-t)\sqrt{x_1} + t\sqrt{y_1}$ $+(1-t)\sqrt{x_2} + t\sqrt{y_2}$
$\leq (1-t)(\sqrt{x_1}+\sqrt{x_2})$ +t($\sqrt{y_1}+\sqrt{y_2}$)
$\leq(1-t)f(x)$+t f(y). ■

Let S be a non convex set of a vector space E and f:S$\to R$; **"the epigraph of f "** denoted **by p-epi f** , is a subset of E x R defined by ,
p-epi f ={(x, t); x $\in S$, t$\in R$, $f(x)\leq t\}$
That is, the epigraph of f is the set of all points lying on or above its graph.

. **Theorem 3.3.** Let S be a nonempty p-convex set in R^n and f: S$\to R^+$. Then f is p-convex if and only if p-epi f is a p-convex set.

Proof . Assume f is a p-convex function**.** Let (a ,b)$\in S$ and $(a,\propto),(a,\beta)$ in p-epi f .Then for any t $\in[0,1]$,we have f($(1-t)^{1/p}a$ + $t^{1/p}$ b) $\le (1-t)f(a)+tf(b) \le (1-t)\propto +t\beta$

Since S is a p-convex set, $(1-t)^{1lp}\, a + t^{1lp}\, b \in S.$
Therefore,

($(1-t)^{1/p}a$ + $t^{1/p}$ b) , $((1-t)\propto +t\beta)) \in p-epi\,f$,

i.e., p-epi f is a p-convex set.

To show sufficiency, assume that epi-f is a p-convex set, and consider a, b $\in$S and

(a, f(a)) , (b, f(b)) $\in$ p − epi f

From p-convexity of p-epi f , for t$\in [0,1]$, we have

($(1-t)^{1/p}a$ + $t^{1/p}$ b) , $(1-t))^{1/p}f(a)+t^{1/p}f(b)) \in p-epi\,f$,

Since,

f($(1-t)^{1/p}a$ + $t^{1/p}$b)$\le (1-t))^{1/p}f(a)+t^{1/p}f(b)$,

it follows that f is a p-convex function. ■

Corollary 3.4. If f and g are p-convex functions , then $\propto f+\beta g$ is also a p-convex function, when $\propto,\beta \ge 0$.

Proof.

$(\propto f+\beta g)[(1-t)^{1/p}x+t^{1/p}y]=$

$(\propto f)[(1-t)^{1/p}x+t^{1/p}y]+(\beta g)[(1-t)^{1/p}x+t^{1/p}y$

$\le \propto[(1-t)^{\frac{1}{p}}f(x)+t^{\frac{1}{p}}f(y)]+\beta[(1-t)^{\frac{1}{p}}f(x)+t^{\frac{1}{p}}f(y)]$,

since f and g are p-convex. Therefore $\propto f+\beta g$ is also p-convex. ■

Let E be a real vector space and fix a subset A of E . A set B of E is said to be “ **A-p-convex**” if there exist a p-convex set C in E such that

B=C$\cap$ A.

We are interested in the case E=R^n and A=Z^n.

For digitization of p-convex sets the mapping C$\to C\cap Z^n$ is not always satisfactory, because it yields the empty set for some long and narrow p-convex sets C. One might then want to replace it by a mapping like

C$\to (C+D)\cap Z^n$

where D is some fixed set which grantees that the image is nonempty when C is nonempty.

Theorem 3.5. Given a vector space E and a subset A of E , the following properties are equivalent for any subset B of A.

1. B is A-p-convex.
2. B=A$\cap$ (p-cvx B).
3. B$\sqsupseteq$ A $\cap$ (p − cvx B)
4. For all n, all $a_0, a_1, \ldots, a_n \in A$, and for all nonnegative numbers $t_0, t_1, \ldots, t_n$ with $\sum_{j=0}^{n} t_j^p=1$, If $\sum_{j=0}^{n} t_j a_j \in$A, then $\sum_{j=0}^{n} t_j a_j \in$B,

Proof. This is obvious. As far as property 4 is concerned , we can in view generalize of Caratheodory,s theorem, let n be the dimension of E if the space is finite dimensional; otherwise we must use all n. ■

Let us now fix two subsets A and Q of a vector space E and define an operator between two power sets,

$\gamma=\gamma_{PQ}$:P(E)$\to P(A)$,

by

$\gamma(B)=p-cvx(B\cap Q)\cap A.$

We may consider E=R^n , A=mZ^n, m=1,2,…and Q=Z^n.

Note that $\gamma(C)$ is A-p-convex if C is p-convex in R^n.

Theorem 3.6 .Let E be a real vector space and A any subset of E . Then B is A-p-convex if and only if B= $\gamma(B)$ for all Q$\sqsupseteq A$, also if and only if B= $\gamma(B)$ for some Q$\sqsupseteq A$.

Proof. If B is A-p-convex, then B=C$\cap$A for some p-convex C. Now,

$$\gamma(B)=\gamma(C\cap A)=p-cvx(C\cap A\cap Q)\cap A$$

$$=p-cvx(C\cap A)\cap A$$

$$=p-cvx(B)\cap A=C\cap A=B,$$ for all Q$\sqsupseteq A$.

If B= $\gamma(B)$ for some choice of Q$\sqsupseteq A$, then B= $p-cvx(B\cap Q)\cap A$=C$\cap A$.

Defining C=p-cvx(B$\cap$Q) so that B is A-p-convex. ■

Corollary 3.7. If $B=C\cap A$, then Q$\sqsupseteq\gamma(B)$ for any Q.

Proof. Note that in the previous definition of A-p-convex sets we may take

C= $\gamma(B)=p-cvxB$

Provided that Q$\sqsupseteq A$. ■

Theorem 3.8. Let E be a vector space and A any subset of E. If $B_j, j\in J$, are A-p-convex sets, then the intersection $\cap B_j$ is a A-p-convex set . If the index set J is ordered and filtering to right and if $(B_j)_j$ is an increasing family of A-p-convex sets, then the union $\cup B_j$ is also p-convex.

Proof. For each B_j , we have

$B_j=C_j\cap A,$

where C_j =p-cvx is a p-convex set in E. Then

$\cap B_j=\cap(C_j\cap A)=(\cap C_j)\cap A.$

For the union we have

$$\cup B_j = \cup \left(C_j \cap A\right) = (\cup C_j) \cap A. \; ■$$

IV. Conclusion

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions

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