On the Exponentiated New Weighted Weibull Distribution

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Abstract— **A new model called exponentiated new weighted Weibull distribution has been defined and studied. Some mathematical properties of the proposed model including moments, hazard rate, quantile, Order Statistics and moment generating function are derived. Also, numerical illusteratan to follow the behaviours of estimators are applied. Parameters estimation using maximum likelihood and it's variance covariance matrix are obtained.**

Index Terms— **New weighted Weibull distribution, hazard rate, quantile, order statistics, moment generating function, and maximum likelihood estimation.**

I. INTRODUCTION

In recent time, numerous researchers had used Weibull distribution as an alternative to some distribution such as gamma and log-normal distribution in reliability engineering and life testing. However, researchers continue to develop different generalizations of the Weibull distribution to increase its flexibility in modeling lifetime data. The Weibull distribution is a well-known common distribution and has been a powerful probability distribution in reliability analysis, while weighted distributions are used to adjust the probabilities of the events as observed and recorded. The weighted exponential distribution being a competitor to the Weibull, gamma and generalized exponential distributions has received appreciable usage in the fields of engineering and medicine. Distribution and demonstrated its application using lifetime data.. Mahdy (2013) applied Azzalini's method to the Weibull distribution that produced a new class of weighted Weibull distribution as $WW\ (\lambda,\alpha,\beta)$ distribution with an additional parameter called "Sensitive Skewnes Parameter" and the sensitive skewness parameter governs essentially the shape of the probability density function of $WW\ (\lambda,\alpha,\beta)$ distribution. Recently, many authors have studied the properties of exponentiated distributions. For instance, Gupta et al (2001) for exponential pareto, Nadarajah and Gupta (2007) for exponential gamma distribution, Mudholkar, G. S., Srivastava, D. K. & Friemer, M. (1995) the exponentiated Weibull family: Are analysis of the bus-motor-failure data. Salem and Abo-Kasem (2011) based their research on estimation for the parameters of the exponentiated Weibull distribution. Gupta and Kundu (2001) exponentiated

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exponential family an alternative to gamma and Weibull distributions. Azzalini (1985) first proposed a method of obtaining weighted and the method has been used extensively for several symmetric and non-symmetric distributions. The rest of this article, is organized as follows, the new weighted Weibull distribution is defined in section 2, exponentiated new weighted Weibull distribution is defined in section 3, the basic statistical properties of the new model are derived in section 4, the parameters estimation using maximum likelihood are derived In section 5, the model is applied to four real life data sets to assess its flexibility followed by a concluding remark in section 6.

II. NEW WEIGHTED WEIBULL DISTRIBUTION

In this section, the density of the new weighted Weibull distribution has been derived based on the definition given in (1).A new weighted Weibull distribution has been defined and studied by Nasiru (2015).Some mathematical properties of the distribution have been studied and the method of maximum likelihood was proposed for estimating the parameters of the distribution. The usefulness of the new distribution was demonstrated by applying it to a real lifetime dataset. Badmus, N. Idowu ɲ & Bamiduro, T. Adebayo (2014) had presented an exponentiated weighted Weibull model which is established with a view to obtaining a model that is better than both weighted Weibull and Weibull distribution in terms of the estimate of their characteristics. If $g(x)$ is probability density function (pdf) and $\bar{G}(x)$ is the corresponding survival function such that the cumulative distribution function (cdf), (x), exist;

Then the new weighted distribution is defined as:

$$d\left(x;\alpha,\theta,\lambda\right)=K\ g\left(x\right)\ G\left(\lambda x\right) \quad (1)$$

Where K is a normalizing constant.

Consider a two parameter Weibull distribution with pdf given by:

$$g\left(x\right)=\alpha\theta x^{\theta-1}e^{\left(-\alpha x^{\theta}\right)},\quad x>0,\alpha>0,\theta>0 \quad (2)$$

where α is a scale parameter , θ is a shape parameter.

The cdf is given by:

$$G\left(x\right)=1-e^{\left(-\alpha x^{\theta}\right)},\quad x>0,\alpha>0,\theta>0 \quad (3)$$

The survival function is given by:

$$\bar{G}\left(x\right)=S\left(x\right)=e^{\left(-\alpha x^{\theta}\right)},\quad x>0,\alpha>0,\theta>0 \quad (4)$$

Using equations (1), (2) and (4) the pdf of the new weighted Weibull distribution is defined as:

$$k(x;\alpha,\theta,\lambda)=\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1}e^{\left(-\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)},$$

$$x>0,\alpha>0,\theta>0,\lambda>0 \qquad (5)$$

The corresponding cdf of the new weighted Weibull distribution is given by:

$$K(x;\alpha,\theta,\lambda)=1-e^{\left(-\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)},$$

$$x>0,\alpha>0,\theta>0,\lambda>0 \qquad (6)$$

It can be deduced immediately from (6) that,

$$\lim_{x\to\infty}K(x)=1,\ \lim_{x\to-\infty}K(x)=0, 0\le K(x)\le 1$$

Where α is a scale parameter , θ and λ are shape parameters.

Figure 1 and 2illustrates possible shapes of the pdf and the cdf of the new weighted Weibull distribution for some selected values of the parameters α , θ and λ

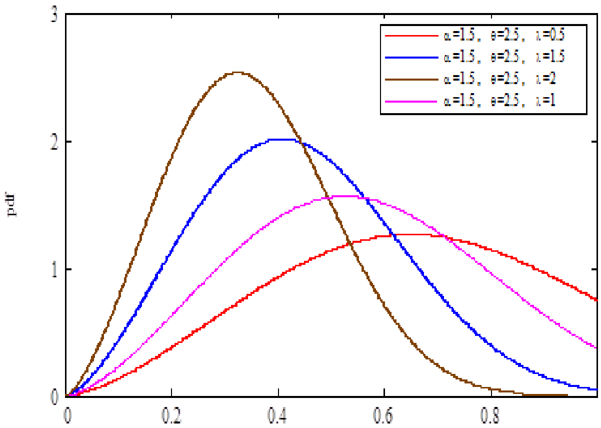

Figure 1: Probability density function of new weighted Weibull distribution

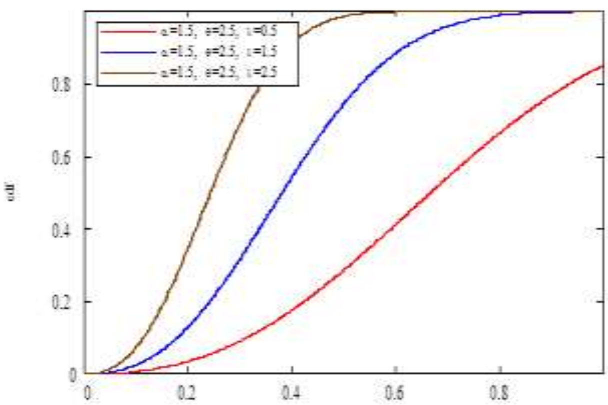

Figure 2: Cumulative distribution function of new weighted Weibull distribution

The survival function is given by:

$$S_1(x;\alpha,\theta,\lambda)=1-K(x;\alpha,\theta,\lambda)=e^{\left(-\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)},$$

$$x>0,\alpha>0,\theta>0,\lambda>0 \qquad (7)$$

And the hazard function is:

$$h_1(x)=\frac{k(x;\alpha,\theta,\lambda)}{S_1(x;\alpha,\theta,\lambda)}=\frac{\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1}e^{\left(-\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}}{e^{\left(-\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}},$$

$$x>0,\alpha>0,\theta>0,\lambda>0 \qquad (8)$$

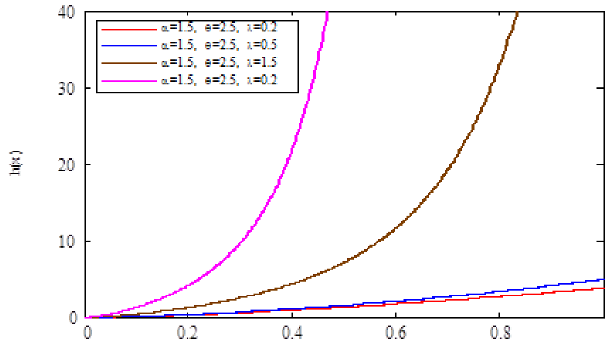

Figure 3: Hazard function of new weighted Weibull distribution

3- The Exponentiated New Weighted Weibull Distribution

The pdf of the exponentiated new weighted Weibull distribution (ENWW) is derived using (5), (6) to give (9).

$$f(x;\alpha,\theta,\beta,\lambda)=\beta k(x;\alpha,\theta,\beta,\lambda).\left[K(x;\alpha,\theta,\beta,\lambda)\right]^{\beta-1}$$

$$=\beta\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1}e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}, x>0,\alpha>0,\theta>0,\lambda>0,\beta>0 \ (9)$$

Where β is a shape parameter

The corresponding cdf of the exponentiated new weighted Weibull distribution (ENWW) is givenby:

$$F(x;\alpha,\theta,\beta,\lambda)=\left[K(x;\alpha,\theta,\beta,\lambda)\right]^{\beta}$$

$$=\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta},$$

$$x>0,\alpha>0,\theta>0,\lambda>0,\beta>0 \qquad (10)$$

It can be deduced immediately from (10) that

$$\lim_{x\to\infty}F(x)=1,\ \lim_{x\to-\infty}F(x)=0, 0\le F(x)\le 1$$

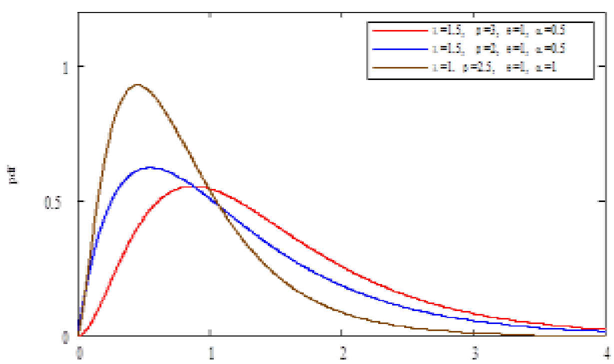

Figure 4: Probability density function of exponentiated new weighted Weibull distribution

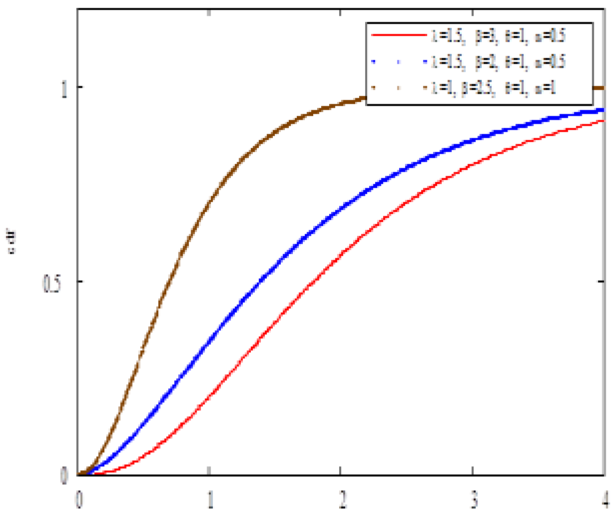

Figure 5: Cumulative distribution function of exponentiated new weighted Weibull distribution

The survival function is given by:

$$S_2(x;\alpha,\theta,\beta,\lambda)=1-F(x;\alpha,\theta,\beta,\lambda)=1-\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta},$$

$$x>0,\alpha>0,\theta>0,\lambda>0,\beta>0 \qquad (11)$$

4-Special Cases:

There are several sub models from ENWW distribution such as exponentiated New weighted exponential distribution(ENWE), New weighted Weibull distribution (NWW), New weighted exponential distribution (NWE), exponentiated Weibull distribution (EW), exponentiated exponential distribution(EE), Weibull distribution(W), exponential distribution(E).

TABLE:Some sub models from the exponentiated New weighted exponential distribution (ENWE).

where α is a scale parameter and, θ,β and λ are shape parameters.

	α	β	θ	λ	Dist n	References
1	—	1	—	—	NWW	Nasiru (2015)
2	—	1	—	0	w	Azzalini 's (1985)
3	—	—	1	—	ENWE	
4	—	1	1	—	NWE	Oguntunde, P.E. Owoloko E.A. and
5	—	1	1	0	E	Balogun O.S.(2016)
6	—	—	—	0	EW	Saralees Nadarajah gauss M.
		—				Cordeiro and Edwin.M.M ortego
7	—		1	0	EE	Saralees Nadarajah (2011)

5-Statistical Properties

In this section, the statistical properties of the exponentiated new weighted Weibull distribution (ENWW) are studied. hazard rate ,quantile, mode, moment, moment generating function have been derived and skewness, kurtosis.

5.1-Hazard function

$$.h_2(x)=\frac{f(x;\alpha,\theta,\beta,\lambda)}{S_2(x;\alpha,\theta,\beta,\lambda)}=\frac{\beta.\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1}e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}.\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}}{1-\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta}}, \qquad (12)$$

Where α is a scale parameter, θ,β and λ are shape parameters.

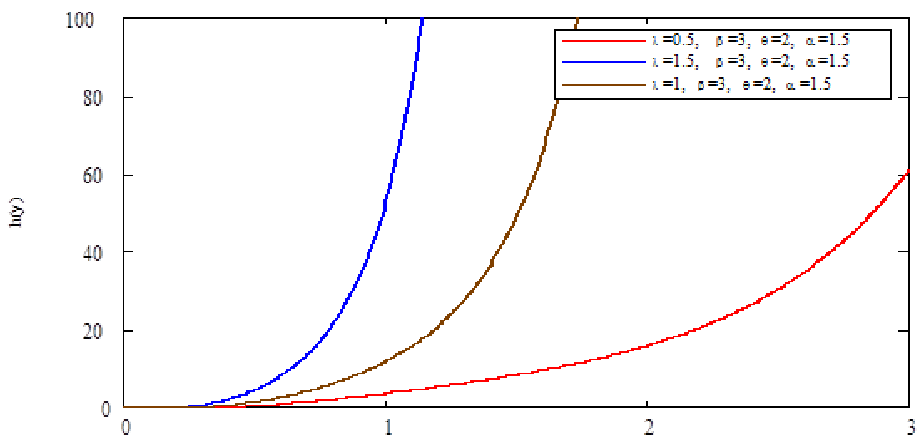

Figure 6: Hazard function of exponentiated new weighted Weibull distribution

5.2- Quantile function and median.

Let (u), $0 < u < 1$ denote the quantile function for the (ENWW). Then $Q(u)$ is given by:

$$Q(u) = F^{-1}(u)$$

$$F(x) = u \qquad \Rightarrow \qquad x = F^{-1}(u)$$

$$submi\ \ from\ (10)$$

$$F(x) = \left[1 - e^{-\left(\alpha x^{\theta} + \alpha(\lambda x)^{\theta}\right)}\right]^{\beta}$$

$$u = \left[1 - e^{-\left(\alpha x^{\theta} + \alpha(\lambda x)^{\theta}\right)}\right]^{\beta}$$

$$u^{\frac{1}{\beta}} = \left[1 - e^{-\left(\alpha x^{\theta} + \alpha(\lambda x)^{\theta}\right)}\right]^{\beta}$$

$$1 - u^{\frac{1}{\beta}} = e^{-\left(\alpha x^{\theta} + \alpha(\lambda x)^{\theta}\right)}$$

$$\ln\left(1 - u^{\frac{1}{\beta}}\right) = \ln\left(e^{-\left(\alpha x^{\theta} + \alpha(\lambda x)^{\theta}\right)}\right) \qquad \Rightarrow \qquad \ln\left[\frac{1}{1 - u^{\frac{1}{\beta}}}\right] = \alpha x^{\theta}(1 + \lambda^{\theta})$$

$$Q(u) = \left[\frac{In\left(\frac{1}{1 - u^{\frac{1}{\beta}}}\right)}{\alpha\left(1 + \lambda^{\theta}\right)}\right]^{\frac{1}{\theta}}, \tag{13}$$

Where, U has the uniform U (0, 1) distribution. The median is obtained directly by substituting u = 0.5 in (13). Therefore, the median is given by (14):

$$\text{Median} = \left[\frac{In\left(\frac{1}{1 - (0.5)^{\frac{1}{\beta}}}\right)}{\alpha\left(1 + \lambda^{\theta}\right)}\right]^{\frac{1}{\theta}}, \tag{14}$$

To simulate from the (ENWW) is straight forward. Let u be a uniform variate on the unit interval $(0,1)$. Thus by means of the inverse transformation method, we consider the random variable X given by (15):

$$X = \left[\frac{In\left(\frac{1}{1 - u^{\frac{1}{\beta}}}\right)}{\alpha\left(1 + \lambda^{\theta}\right)}\right]^{\frac{1}{\theta}}, \tag{15}$$

5.3 -Mode

Consider the density of the ENWW given in (9) The mode is obtained by solving $\frac{\partial \ln f(x; \alpha, \theta, \beta, \lambda)}{\partial x} = 0$ for x Therefore the mode at $x = x_0$ is given by :

$$f\left(x;\alpha,\theta,\beta,\lambda\right)=\beta.k\left(x;\alpha,\theta,\beta,\lambda\right).\left[K\left(x;\ \alpha,\ \theta,\beta,\ \lambda\right)\right]^{\beta-1}$$

$$=\beta.\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1}e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}.\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1},\ x>0,\alpha>0,\theta>0,\lambda>0,\beta>0 \quad (9)$$

let $\mathrm{C}=\alpha\left(1+\lambda^{\theta}\right)$

$$f\left(x;\alpha,\theta,\beta,\lambda\right)=\beta.\mathrm{C}.\theta x^{\theta-1}e^{-Cx^{\theta}}.\left[1-e^{-Cx^{\theta}}\right]^{\beta-1}$$

using $\ln f\left(x;\alpha,\theta,\beta,\lambda\right)$

$$\ln f\left(x;\alpha,\theta,\beta,\lambda\right)=\ln\left(\beta.\mathrm{C}.\theta\right)+\ln\left(x^{\theta-1}\right)+\ln\left(e^{-Cx^{\theta}}\right)+\left(\beta-1\right)\ln\left[1-e^{-Cx^{\theta}}\right]$$

Differentiation both sides with repect to x ,and *let* A= $\ln\left(\beta.\mathrm{C}.\theta\right)$

$$\frac{\partial\ln f\left(x;\alpha,\theta,\beta,\lambda\right)}{\partial x}=A+\frac{(\theta-1)}{x}-Cx^{\theta}+\left(\beta-1\right)\frac{C\theta x^{\theta-1}e^{-Cx^{\theta}}}{1-e^{-Cx^{\theta}}}$$

This equation can be solved numerically by using iteration methods.

5.4 -Moment and Moment Generating Function

In this section, the r^{th} non central moment and the moment generating function have been derived.

Theorem 1. If a random variable X has the exponentiated new weighted Weibull distribution, then the r^{th} non central moment is given by the following:

$$\mu_r'=\left[\frac{1}{\alpha\left(1+\lambda^{\theta}\right)}\right]^{\frac{r}{\theta}}\beta\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{r}{\theta}+1}}\Gamma\left(\frac{r}{\theta}+1\right) \quad (16)$$

Proof.

$$\mu_r'=\int_0^{\infty}x^{r}f\left(x,\alpha,\theta,\lambda,\beta\right)dx$$

This implies

$$\mu_r'=\int_0^{\infty}x^{r}\left[\beta\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1}e^{-\left(\alpha x^{\theta}\left(1+\lambda^{\theta}\right)\right)}\left(1-e^{-\left(\alpha x^{\theta}\left(1+\lambda^{\theta}\right)\right)}\right)^{\beta-1}\right]dx. \quad (17)$$

let $y=\alpha x^{\theta}\left(1+\lambda^{\theta}\right)$ $\quad dy=\alpha\theta\left(1+\lambda^{\theta}\right)x^{\theta-1}dx$, *and* $\quad x=\left(\frac{y}{\alpha\left(1+\lambda^{\theta}\right)}\right)^{\frac{1}{\theta}}$

$$\mu_r'=\int_0^{\infty}\left(\frac{y}{\alpha\left(1+\lambda^{\theta}\right)}\right)^{\frac{r}{\theta}}\beta e^{-y}\left(1-e^{-y}\right)^{\beta-1}dy$$

$$=\left[\frac{1}{\alpha\left(1+\lambda^{\theta}\right)}\right]^{\frac{r}{\theta}}.\beta.\int_0^{\infty}y^{\frac{r}{\theta}}e^{-y}\left[1-e^{-y}\right]^{\beta-1}dy$$

If ' b' is a positive real non-integer and | z |,we consider the power series expansion ;

let $(1-z)^{\mathrm{b}-1}=\sum_{j=0}^{\infty}W_jZ^{j}, W_j=(-1)^{j}\binom{b-1}{j}$,and $\left(1-e^{-y}\right)^{\beta-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}e^{-yj}$

$$\mu_r'=\left(\frac{1}{\alpha\left(1+\lambda^{\theta}\right)}\right)^{\frac{r}{\theta}}\beta\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\int_0^{\infty}y^{\frac{r}{\theta}}e^{-(j+1)y}dy \quad (18)$$

let $\int_0^{\infty}y^{\frac{r}{\theta}}e^{-(j+1)y}dy=\frac{1}{(j+1)^{\frac{r}{\theta}+1}}\int_0^{\infty}\left[y\,(j+1)\right]^{\frac{r}{\theta}}e^{-(j+1)y}dy\,(j+1)$

$$=\frac{1}{(j+1)^{\frac{r}{\theta}+1}}\Gamma\left(\frac{r}{\theta}+1\right)$$

$$\mu_r'=\left(\frac{1}{\alpha\left(1+\lambda^{\theta}\right)}\right)^{\frac{r}{\theta}}\beta\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{r}{\theta}+1}}\Gamma\left(\frac{r}{\theta}+1\right) \quad \square$$

This complete the proof.

put $r=1,2,3,4$ we obtained

$$\mu_1' = E(x) = \left(\frac{1}{\alpha(1+\lambda^\theta)}\right)^{\frac{1}{\theta}} \beta\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{1}{\theta}+1}}\Gamma\left(\frac{1}{\theta}+1\right)$$

$$\mu_2' = E(x^2) = \left(\frac{1}{\alpha(1+\lambda^\theta)}\right)^{\frac{2}{\theta}} \beta\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{2}{\theta}+1}}\Gamma\left(\frac{2}{\theta}+1\right)$$

$$\mu_3' = E(x^3) = \left(\frac{1}{\alpha(1+\lambda^\theta)}\right)^{\frac{3}{\theta}} \beta\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{3}{\theta}+1}}\Gamma\left(\frac{3}{\theta}+1\right)$$

$$\mu_4' = E(x^4) = \left(\frac{1}{\alpha(1+\lambda^\theta)}\right)^{\frac{4}{\theta}} \beta\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{4}{\theta}+1}}\Gamma\left(\frac{4}{\theta}+1\right)$$

There fore the variance is given by:

$$Var(X) = E(X^2) - [E(X)]^2$$

Theorem 2. Let X have the exponentiated new weighted Weibull distribution.The moment generating function of X denoted by $M_X(\mathrm{t})$ is given by:

$$M_x(t) = \sum_{i=0}^{\infty}\frac{t^i}{i!}\left(\frac{1}{\alpha(1+\lambda^\theta)}\right)^{\frac{i}{\theta}} \beta\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{i}{\theta}+1}}\Gamma\left(\frac{i}{\theta}+1\right) \tag{19}$$

Proof .

By definition

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x,\alpha,\theta,\lambda,\beta)dx$$

Using Taylor series

$$M_x(t) = \int_0^{\infty}\left(1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+...............+\frac{t^nx^n}{n!}+...\right).f(x,\alpha,\theta,\lambda,\beta)dx$$

$$=\sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i)$$

$$=\sum_{i=0}^{\infty}\frac{t^i}{i!}\left(\frac{1}{\alpha(1+\lambda^\theta)}\right)^{\frac{i}{\theta}} \beta\sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}\frac{1}{(j+1)^{\frac{i}{\theta}+1}}\Gamma\left(\frac{i}{\theta}+1\right) \qquad \square$$

This completes the proof.

5.5-Skewness and Kurtosis

In this study, the moment about zero based measures of skewness and kurtosis.

Skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$

Kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^2$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$

5.6-Order Statistics

Let X_1 denote the smallest of $\{X_1, X_2, ..., X_n)\}$, X_2 denote the second smallest of $\{X_1, X_2, ..., X_n)\}$ and similarly X_k denote the k^{th} smallest of $\{X_1, X_2, ..., X_n)\}$ Then the random variables $X_1, X_2, ..., X_n$ called the order statistics of the sample $X_1, X_2, ..., X_n$, has probability density function of the k^{th} order statistic, $X_{(k)}$, as:

$$g_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k}, \qquad \text{for} \qquad k = 1,2,3,...,n \qquad (20)$$

The pdf of the k^{th} order statistic is defined as:

$$g_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\beta.\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1} e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}.\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}\left[\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta}\right]^{k-1}$$
$$.\left[1-\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta}\right]^{n-k}, \qquad (21)$$

The pdf of the largest order statistic X_n is therefore:

$$g_{n:n}(x) = n\beta.\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1} e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}.\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}\left[\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta}\right]^{n-1}, \qquad (22)$$

And the pdf of the smallest order statistic $X_{(1)}$ is given by:

$$g_{1:n}(x) = n\beta.\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1} e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}.\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}.\left[1-\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta}\right]^{n-1}, \qquad (23)$$

5.7-Maximum Likelihood Estimation.

In this section, the method of maximum likelihood was considered for the estimation of the parameters of the exponentiated new weighted Weibull distribution (ENWW).

Consider a random sample of size n, consisting of value $\{x_1, x_2, ..., x_n)\}$ from the exponentiated new weighted Weibull distribution (ENWW).

$$f(x;\beta,\theta,\alpha,\lambda) = \beta\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1} e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}, \qquad (9)$$

For the four parameter ENWW distribution the likelihood function of the above density is given by:

$$L(x;\beta,\theta,\alpha,\lambda) = \prod_{i=1}^{n} f(x;\beta,\theta,\alpha,\lambda)$$
$$= \prod_{i=1}^{n} \beta\left(1+\lambda^{\theta}\right)\alpha\theta x^{\theta-1} e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}$$
$$= \beta^{n}\left(1+\lambda^{\theta}\right)^{n}\alpha^{n}\theta^{n}\left(\prod_{i=1}^{n} x^{\theta-1}\right)\left(\prod_{i=1}^{n} e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right)\left(\prod_{i=1}^{n}\left[1-e^{-\left(\alpha x^{\theta}+\alpha(\lambda x)^{\theta}\right)}\right]^{\beta-1}\right), \qquad (24)$$

Where $X=\{X_1,X_2,...,X_n)\}$ .Taking the logarithm, (24)becomes.

$$\ln L(x;\beta,\theta,\alpha,\lambda)=n\ln\beta+n\ln\left(1+\lambda^{\theta}\right)+n\ln\alpha+n\ln\theta+(\theta-1)\sum_{i=1}^{n}\ln x_i-\alpha\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}x_i^{\theta}$$
$$+(\beta-1)\sum_{i=1}^{n}\ln\left[1-e^{-\left(1+\lambda^{\theta}\right)\alpha x_i^{\theta}}\right], \tag{25}$$

Differentiate (25) with respect to the unknown parameters and equating to zero.
Let

$$O=\lambda^{\theta}\ln\lambda, V=\lambda^{\theta}\left(\ln\lambda\right)^2, R_i=x_i^{\theta}\ln x_i, \mathrm{D_i}=x_i^{\theta}\left(\ln x_i\right)^2, Q_i=\alpha x_i^{\theta}\left(1+\lambda^{\theta}\right), \mathrm{B}=\theta\lambda^{(\theta-1)}$$

$$\frac{\partial\ln L}{\partial\hat{\theta}}=\frac{nO}{1+\lambda^{\theta}}+\frac{n}{\theta}+\sum_{i=1}^{n}\ln x_i-\alpha\left[O.\sum_{i=1}^{n}x_i^{\theta}+\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}R_i\right]+(\beta-1)\left(\sum_{i=1}^{n}\frac{e^{-Q_i}.\alpha\left[O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right]}{\left(1-e^{-Q_i}\right)}\right)=0,$$

$$\frac{\partial\ln l}{\partial\hat{\alpha}}=\frac{n}{\alpha}-\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}x_i^{\theta}+(\beta-1)\sum_{i=1}^{n}\frac{\left(e^{-Q_i}\right).\left[\left(1+\lambda^{\theta}\right)x_i^{\theta}\right]}{\left(1-e^{-Q_i}\right)}$$
$$=\frac{n}{\alpha}-\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}x_i^{\theta}+(\beta-1)\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}x_i^{\theta}\left[\frac{\left(e^{-Q_i}\right)}{\left(1-e^{-Q_i}\right)}\right]=0,$$

$$\frac{\partial\ln l}{\partial\hat{\beta}}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left(1-e^{-Q_i}\right)=0,$$

$$\frac{\partial\ln l}{\partial\hat{\lambda}}=\frac{n\mathrm{B}}{1+\lambda^{\theta}}-\alpha\mathrm{B}\sum_{i=1}^{n}x_i^{\theta}+(\beta-1)\sum_{i=1}^{n}\frac{\left(e^{-Q_i}\right).\left[\alpha x_i^{\theta}\mathrm{B}\right]}{\left(1-e^{-Q_i}\right)}$$
$$=\frac{n\mathrm{B}}{\left(1+\lambda^{\theta}\right)}-\alpha\mathrm{B}\sum_{i=1}^{n}x_i^{\theta}+(\beta-1)\alpha\mathrm{B}\sum_{i=1}^{n}x_i^{\theta}\frac{\left(e^{-Q_i}\right)}{\left(1-e^{-Q_i}\right)}=0,$$

$$\frac{-\partial^2\ln L}{\partial\hat{\theta}^2}=\frac{-\left[\left(1+\lambda^{\theta}\right).nV\right]+\left[nO^2\right]}{\left(1+\lambda^{\theta}\right)^2}+\frac{n}{\theta^2}+\alpha\left[O.\sum_{i=1}^{n}R_i+\sum_{i=1}^{n}x_i^{\theta}V+\left(1+\lambda^{\theta}\right).\sum_{i=1}^{n}\mathrm{D_i}+O\sum_{i=1}^{n}\mathrm{R_i}\right]$$
$$-(\beta-1)\sum_{i=1}^{n}\frac{1}{\left(1-e^{-Q_i}\right)^2}\left[\begin{array}{l}\left(1-e^{-Q_i}\right).\left(e^{-Q_i}\right).\alpha\left\{R_i.O+x_i^{\theta}V+\left(1+\lambda^{\theta}\right)\mathrm{D_i}+R_i.O\right\}+\\ \left(\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)\left\{\left(e^{-Q_i}\right).\left[-\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)\right]\right\}\right.\\ \left.-\left\{\left(e^{-Q_i}\right).\left[\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)\right]\left(e^{-Q_i}\right).\left[\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)\right]\right\}\right)\end{array}\right]=0$$

$$-\frac{\partial^2\ln l}{\partial\hat{\alpha}^2}=\frac{n}{\alpha^2}-\left(\begin{array}{l}\left\{(\beta-1)\sum_{i=1}^{n}\frac{\left[\left(1+\lambda^{\theta}\right)x_i^{\theta}\right]}{\left(1-e^{-Q_i}\right)^2}.\left(1-e^{-Q_i}\right).e^{-Q_i}\left[-x_i^{\theta}\left(1+\lambda^{\theta}\right)\right]\right\}\\ \qquad-\left\{e^{-Q_i}.e^{-Q_i}.\left[\left(1+\lambda^{\theta}\right)x_i^{\theta}\right]\right\}\end{array}\right)=0,$$

$$-\frac{\partial^2 \ln l}{\partial \hat{\beta}^2} = \frac{n}{\beta^2} = 0\,,$$

$$-\frac{\partial^2 \ln l}{\partial \hat{\lambda}^2} = -\frac{\left[\left(1+\lambda^{\theta}\right)n\theta(\theta-1)\lambda^{\theta-2}\right]+\left[n\mathrm{B}^2\right]}{\left(1+\lambda^{\theta}\right)^2} + \alpha\theta(\theta-1)\lambda^{\theta-2}\sum_{i=1}^{n} x_i^{\theta}$$

$$-\left((\beta-1)\alpha\mathrm{B}\left[\sum_{i=1}^{n} x_i^{\theta}\frac{\left(1-e^{-Q_i}\right).\left(e^{-Q_i}\right).\left[-\alpha x_i^{\theta}\mathrm{B}\right]}{\left(1-e^{-Q_i}\right)^2} - \frac{\left(e^{-Q_i}\right).\left(e^{-Q_i}\right).\left[\alpha x_i^{\theta}\mathrm{B}\right]}{\left(1-e^{-Q_i}\right)^2}\right]\right)$$

$$-\left[\sum_{i=1}^{n} x_i^{\theta}\left(\frac{e^{-Q_i}}{\left(1-e^{-Q_i}\right)}\right).(\beta-1)(\theta-1)\alpha\theta\lambda^{\theta-2}\right] = 0,$$

$$\frac{\partial \ln L}{\partial \hat{\beta}\partial \hat{\theta}} = \sum_{i=1}^{n}\frac{e^{-Q_i}\,.\,\alpha\left(\left(1+\lambda^{\theta}\right)R + x_i^{\theta}.O\right)}{\left(1-e^{-Q_i}\right)} = 0,$$

$$\frac{\partial \ln L}{\partial \hat{\beta}\partial \hat{\lambda}} = \sum_{i=1}^{n}\frac{e^{-Q_i}\,.\,\alpha x_i^{\theta}\mathrm{B}}{\left(1-e^{-Q_i}\right)} = 0,$$

$$\frac{\partial \ln L}{\partial \hat{\beta}\partial \hat{\alpha}} = \sum_{i=1}^{n}\frac{e^{-Q_i}\;.\,x_i^{\theta}\left(1+\lambda^{\theta}\right)}{\left(1-e^{-Q_i}\right)} = 0,$$

$$\frac{\partial \ln L}{\partial \hat{\theta}\partial \hat{\lambda}} = \frac{1}{\left(1+\lambda^{\theta}\right)^2}\left[\left[\left(1+\lambda^{\theta}\right).nV_i\right]-\left[nO.\mathrm{B}\right]\right] - \alpha\left[V.\sum_{i=1}^{n} x_i^{\theta} + \mathrm{B}\sum_{i=1}^{n} R_i\right]$$

$$+(\beta-1)\sum_{i=1}^{n}\frac{1}{\left(1-e^{-Q_i}\right)^2}\left(1-e^{-Q_i}\right).\left[\begin{array}{l}\left(e^{-Q_i}\right).\alpha\left\{V.x_i^{\theta}+\mathrm{BR_i}\right\}+\left(\begin{array}{l}\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)\\ .\left\{\left(e^{-Q_i}\right).\left[-\alpha x_i^{\theta}\mathrm{B}\right]\right\}\end{array}\right)-\\ \left\{\left(e^{-Q_i}\right).\left[\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)\right].\left(e^{-Q_i}\right).\left[\alpha x_i^{\theta}\mathrm{B}\right]\right\}\end{array}\right] = 0,$$

$$\frac{\partial \ln L}{\partial \hat{\theta}\partial \hat{\alpha}} = -\left[\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n} R_i + \sum_{i=1}^{n} x_i^{\theta}O\right] + (\beta-1)\sum_{i=1}^{n}\frac{1}{\left(1-e^{-Q_i}\right)^2}$$

$$.\left[\begin{array}{l}\left(1-e^{-Q_i}\right)\quad.\left(e^{-Q_i}\right).\left(Ox_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right)+\alpha\left\{O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right\}.\\ \left(e^{-Q_i}\right).\left(\left(1+\lambda^{\theta}\right)x_i^{\theta}\right)-\left(\left(e^{-Q_i}\right).\alpha\left(O.x_i^{\theta}+\left(1+\lambda^{\theta}\right)R_i\right).\left(e^{-Q_i}\right).\left(\left(1+\lambda^{\theta}\right)x_i^{\theta}\right)\right)\end{array}\right] = 0\,,$$

$$\frac{\partial \ln L}{\partial \hat{\alpha}\partial \hat{\lambda}} = -\mathrm{B}\sum_{i=1}^{n} x_i^{\theta} + (\beta-1)\sum_{i=1}^{n}\frac{1}{\left(1-e^{-Q_i}\right)^2}$$

$$.\left[\left(1-e^{-Q_i}\right).\left(e^{-Q_i}\right).\mathrm{B}x_i^{\theta}+\left(1+\lambda^{\theta}\right)x_i^{\theta}\left(e^{-Q_i}\right).\left(\alpha\mathrm{B}x_i^{\theta}\right)-\left\{\left(e^{-Q_i}\right).\left(1+\lambda^{\theta}\right)x_i^{\theta}.\left[\left(e^{-Q_i}\right).\left(\alpha\mathrm{B}x_i^{\theta}\right)\right]\right\}\right] = 0$$

Application

In this section, the application of the exponentiated new weighted Weibull distribution is demonstrated using the lifetime data of 20 electronic components (see Murthy et al., 2004, pp. 83,100). Teimouri and Gupta (2013) studied this data using a three-parameter Weibull distribution. In this study, the weighted Weibull distribution is fitted to this data and the results compared to that of Teimouri and Gupta (2013). The data is shown in Table 1. From Table 2.

Table1: Lifetimes of 20 electronic components

0.03	0.22	0.73	1.25	1.52	1.8	2.38	2.87	3.14	4.72
0.12	0.35	0.79	1.41	1.79	1.94	2.4	2.99	3.17	5.09

Tabel2: Estimates ([a]denotes standard errors),D([b]denotes P values),AIC,BIC,CAICand HQIC

	MLEs								
Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	D	AIC	BIC	CAIC	HQIC
ENWW	0.336 21.663[a]	0.398 0.08[a]	2.221 0.398[a]	0.064 0.315[a]	0.103 0.984[b]	71.387	75.37	74.054	72.165
NWW	0.495 2.128[a]	—	1.596 0.296[a]	0.201 0.314[a]	0.163 0.661[b]	74.292	77.28	75.28	74.876
EE	—	0.05 0.011[a]	—	2.5* 10^-4 2.565* 10^-4[a]	0.997 0.0[b]	146.974	148.965	147.68	147.363
EW	—	0.078 0.018[a]	2.01 0.446[a]	0.025 0.025[a]	0.99 0.0[b]	100.48	103.367	101.98	101.063
E	—	—	—	2.205* 10^-3 0.011[a]	0.989 0.0[b]	377.828	378.824	378.05	378.022

III. Conclusion

We investigated on the statistical properties of the proposed distribution e.g moments, moment generating function, estimation of parameters using (MathCAD2014) for data analysis presented in this article. We also upgraded with an additional parameter to the existingfour parameters in the exponentiated new weighted Weibull distribution results from the estimated parameters show that the exponentiated new Weighted Weibull distribution has a better representation of data than new weighted weibull distribution.

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