The Nature Diagnosability of Bubble-sort Star Graphs under the PMC Model and MM* Model

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Abstract—Many multiprocessor systems have interconnection networks as underlying topologies and an interconnection network is usually represented by a graph where nodes represent processors and links represent communication links between processors. No fault set can contain all the neighbors of any fault-free vertex in the system, which is called the nature diagnosability of the system. Diagnosability of a multiprocessor system is one important study topic. As a famous topology structure of interconnection networks, the $n$-dimensional bubble-sort star graph $BS_n$ has many good properties. In this paper, we prove that the nature diagnosability of $BS_n$ is $4n-7$ under the PMC model for $n \geq 4$, the nature diagnosability of $BS_n$ is $4n-7$ under the MM* model for $n \geq 5$.

Index Terms—Bubble-sort star graph, Diagnosability, Interconnection network.

I. INTRODUCTION

Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies and a network is usually represented by a graph where nodes represent processors and links represent communication links between processors. Some processors may fail in the system, so processor fault identification plays an important role for reliable computing. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system $G$ is said to be $t$-diagnosable if all faulty processors can be identified without replacement, provided that the number of presented faults does not exceed $t$. The diagnosability $t(G)$ of $G$ is the maximum value of $t$ such that $G$ is $t$-diagnosable. For a $t$-diagnosable system, Dahbura and Masson [1] proposed an algorithm with time complex $O(n^{t+1})$, which can effectively identify the set of faulty processors. Several diagnosis models (e.g., Preparata, Metze, and Chien’s (PMC) model [2], Barsi, Grandoni, and Maestrini’s (BGM)model [3], and Maeng and Malek’s (MM) model [4]) have been proposed to investigate the diagnosability of multiprocessor systems. In particular, two of the proposed models, the PMC model and the MM model, are well known and widely used. In the PMC model, the diagnosis of the system is achieved through two linked processors testing each other. In the MM model, to diagnose a system, a node sends the same task to two of its neighbors, and then compares their responses. For this reason, the MM model is also said to be the comparison model. Sengupta and Dahbura [1] proposed a special case of the MM model, called the MM* model, in which each node must test its any pair of adjacent nodes. Numerous studies have been investigated under the PMC model and MM model or MM* model.

In the traditional diagnosis of a multiprocessor system, one generally assumes that any subset of processors may simultaneously fail. If all the neighbors of some node $v$ are faulty simultaneously, it is impossible to determine whether $v$ is faulty or fault-free. As a consequence, the diagnosability of the system is less than its minimum node degree. However, in some large-scale multiprocessor systems, we can safely assume that all neighbors of any node do not fail at the same time. Based on this assumption, in 2005, Lai et al. [5] introduced the restricted diagnosability of the system called the conditional diagnosability. They consider the situation that no fault set can contain all the neighbors of any node in the system. Since the probability that the all neighbors of a fault node fail and create faults is more to the probability that the all neighbors of a fault-free node fail and create faults in the system, we consider the situation that no fault set can contain all the neighbors of any fault-free node in the system, which is called the nature diagnosability of the system. In 2012, Peng et al. [6] proposed a measure for fault diagnosis of the system, namely, the $g$-good-neighbor diagnosability (which is also called the $g$-good-neighbor conditional diagnosability), which requires that every fault-free node contains at least $g$ fault-free neighbors. In [6], they studied the $g$-good-neighbor diagnosability of the $n$-dimensional hypercube under the PMC model. In [7], Wang and Han studied the $g$-good-neighbor diagnosability of the $n$-dimensional hypercube under the MM* model. In 2016, Peng et al. [8] gave some properties of the $g$-good-neighbor diagnosability of a multiprocessor system. In 2017, Wang et al. [9] studied that the $g$-good-neighbor diagnosability of bubble-sort star graph networks under the PMC model and MM* model. Yuan et al. [10,11] studied that the $g$-good-neighbor diagnosability of the k-ary n-cube ($k \geq 3$) under the PMC model and MM* model. As a favorable topology structure of interconnection networks, the Cayley graph $CT_n$ generated by the transposition tree $\Gamma_n$ has many good properties. In [12], Wang et al. studied the $2$-good-neighbor diagnosability of $CT_n$ under the PMC model and MM* model. In 2016, Zhang et al. [13] proposed a new measure for fault diagnosis of the system, namely, the $g$-extra diagnosability, which restrains that every fault-free component has at least ($g+1$) fault-free nodes. In [13], they studied the $g$-extra diagnosability of the $n$-dimensional hypercube under the PMC model and MM* model. In 2016, Wang et al. [14] studied that the $2$-extra diagnosability of the $n$-dimensional bubble-sort star graph under the PMC model and MM* model. In [15], Han and Wang studied that the

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is called a nature faulty set if \( |N(v) \cap (V \setminus F)| \geq 1 \) for every vertex \( v \) in \( V \setminus F \). A nature cut of \( G \) is a nature faulty set \( F \) such that \( G - F \) is disconnected. The minimum cardinality of nature cuts is said to be the nature connectivity of \( G \), denoted by \( k'(G) \). For graph-theoretical terminology and notation not defined here we follow [26].

### B. The PMC model and MM* model

For the PMC model and MM* model, we follow [10]. In a system \( G = (V, E) \), a faulty set \( F \subseteq V \) is called a conditional faulty set if it does not contain all of neighbors of any vertex in \( G \). A system \( G \) is conditional \( t \)-diagnosable if every two distinct conditional faulty subsets \( F_i, F_j \in V \) with \( |F_i| \leq t, |F_j| \leq t \) are distinguishable. The conditional diagnosability \( t(G) \) of \( G \) is the maximum number of \( t \) such that \( G \) is conditional \( t \)-diagnosable. By [27], \( t(G) \geq t(G) \).

**Theorem 1.** ([17]) For a system \( G = (V, E) \), \( t(G) = t_0(G) \leq t_1(G) \leq t(G) \).

In [17], Wang et al. proved that the nature diagnosability of the Bubble-sort graph \( B_n \) under the PMC model is \( 2n - 3 \) for \( n \geq 4 \). In [28], Zhou et al. proved the conditional diagnosability of \( B_n \) is \( 4n - 11 \) for \( n \geq 4 \) under the PMC model. Therefore, \( t_1(B_n) < t_1(B_n) \) when \( n \geq 5 \) and \( t_1(B_n) = t_1(B_n) \) when \( n = 4 \).

### C. The bubble-sort star graph

The bubble-sort star graph has been known as a famous topology structure of interconnection networks. In this section, its definition and some properties are introduced.

Let \( [n] = \{1, 2, \ldots, n\} \), and let \( S_n \) be the symmetric group on \( [n] \), containing all permutations \( p = p_1 p_2 \cdots p_n \) of \( [n] \). It is well known that \( \{1, \ldots, i\} : 2 \leq i \leq n \) is a generating set for \( S_n \). So \( \{1, \ldots, i\} : 2 \leq i \leq n \} \cup \{1, i+1\} : 2 \leq i \leq n-1 \} \) is also a generating set for \( S_n \). The n-dimensional bubble-sort star graph \( BS_n \) [29,30] is the graph with vertex set \( V(BS_n) = S_n \) in which two vertices \( u, v \) are adjacent if and only if \( u = v(i, \ldots, i+1) \), \( 2 \leq i \leq n \) or \( u = v(i, \ldots, i+1) \), \( 2 \leq i \leq n-1 \). It is easy to see from the definition that \( BS_n \) is a \((2n-3)\)-regular graph on \( n! \) vertices.

Note that \( BS_n \) is a special Cayley graph. \( BS_n \) has the following useful properties.

**Proposition 1.** For any integer \( n \geq 1 \), \( BS_n \) is \((2n-3)\)-regular, vertex transitive.

**Proposition 2.** For any integer \( n \geq 2 \), \( BS_n \) is bipartite.

**Proposition 3.** For any integer \( n \geq 3 \), the girth of \( BS_n \) is 4.

**Theorem 2.** ([31]) Let \( H \) be a simple connected graph with \( n = |V(H)| \geq 3 \). If \( H^1 \) and \( H^2 \) are two different labelled graphs obtained by labelling \( H \) with \( \{1, 2, \ldots, n\} \), then \( Cay(H^1, S_n) \) is isomorphic to \( Cay(H^2, S_n) \).

We can partition \( BS_n \) into \( n \) subgraphs \( BS_1, BS_2, \ldots, BS_n \), where every vertex \( u = x_1 x_2 \ldots x_n \in V(BS_n) \) has a fixed integer \( i \) in the last position \( x_i \) for \( i \in [n] \). It is obvious that
$BS_n^i$ is isomorphic to $BS_{n-i}$ for $i \in [n]$. Let $v \in V(BS_n^i)$. Then $\nu(ln)$ and $\nu(n-1,n)$ are called outside neighbors of $v$.

**Proposition 3.** ([29]) Let $BS_n^i$ be defined as above. Then there are $2(n-2)!$ independent cross-edges between two different $H_i$'s.

**Proposition 4.** ([29]) Let $BS_n$ be the bubble-sort star graph. If two vertices $u, v$ are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices $u, v$ are not adjacent, there are at most three common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \leq 3$.

**Lemma 1.** ([9]) The nature connectivity $\kappa'(BS_n)$ of the bubble-sort star graph $BS_n$ is 8.

A connected graph $G$ is super nature connected if every minimum nature cut $F$ of $V(G)$ isolates one edge. If, in addition, $G - F$ has two components, one of which is an edge, then $G$ is tightly $|F|$ super nature connected.

**Theorem 3.** ([14]) For $n \geq 5$, the bubble-sort star graph $BS_n$ is tightly $(4n - 8)$ super nature connected.

**Lemma 2.** Let $A = \{(1), (12)\}$. If $n \geq 4$, $F_1 = N_{BS_n}(A)$, $F_2 = A \cup N_{BS_n}(A)$, then $|F_1| = 4n - 8$, $|F_2| = 4n - 6$, $\delta(BS_n - F_1) \geq 1$, and $\delta(BS_n - F_2) \geq 1$.

**Proof.** By $A = \{(1), (12)\}$, we have $BS_n[A] \cong BS_n = K_2$.

Since $BS_n$ has not 3-cycles, $|N_{BS_n}(A)| = 4n - 8$. Thus from calculating, we have $|F_1| = 4n - 8$, $|F_2| = |A| + |F_1| = 4n - 6$.

Claim 1. For any $x \in S_n \setminus F_2$, $|N_{BS_n}(x) \cap F_2| \leq 2n - 4$.

Since $BS_n$ is a bipartite graph, there is no 5-cycle $(1), (k), x, (12), (l), (1) \in S_n \setminus F_2$. Let $u \in N_{BS_n}(\{(1), (12)\}) \setminus (12)$. If $u$ is adjacent to $x$, then $x$ is not adjacent to each of $N_{BS_n}(\{(1), (12)\}) \setminus (1)$. Since $|N_{BS_n}(\{(1), (12)\})| = 2n - 4$, we have that $x$ is adjacent to at most $2(2n - 4)$ vertices in $F_1$.

By Claim 1, $|N_{BS_n}(x) \cap F_2| \leq 2n - 4$ for any $x \in S_n \setminus F_2$.

Therefore, $\delta(BS_n - F_2) \geq 2n - 3 - (2n - 4) = 1$. $BS_n - F_2$ has two components $BS_n - F_1$ and $BS_n$. Note that $\delta(BS_n) = 1$.

Therefore, $\delta(BS_n - F_2) \geq 1$.

**III. THE NATURE DIAGNOSABILITY OF THE BUBBLE-SORT STAR GRAPH UNDER THE PMC MODEL**

In this section, we shall show the nature diagnosability of the bubble-sort star graph under the PMC model. Let $F_1$ and $F_2$ be two distinct subsets of $V$ for a system $G = (V, E)$.

Define the symmetric difference $\delta F_1 \Delta F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Yuan et al. [10] presented a sufficient and necessary condition for a system to be nature $t$-diagnosable under the PMC model.

**Theorem 4.** ([10]) A system $G = (V, E)$ is nature $t$-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V(F_1 \Delta F_2)$ and $v \in F_2 \Delta F_2$ for each distinct pair of nature faulty subsets $F_1$ and $F_2$ of $V$ with $|F_1| \leq t$ and $|F_2| \leq t$.

**Lemma 3.** A graph of minimum degree 1 has at least two vertices.

The proof of Lemma 3 is trivial.

**Lemma 4.** Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph $BS_n$ under the PMC model is less than or equal to $4n - 7$, i.e., $t(BS_n) \leq 4n - 7$.

**Proof.** Let $A$ be defined in Lemma 2, and let $F_1 = N_{BS_n}(A)$, $F_2 = A \cup N_{BS_n}(A)$. By Lemma 2, $|F_1| = 4n - 8$, $|F_2| = 4n - 6$, $\delta(BS_n - F_1) \geq 1$ and $\delta(BS_n - F_2) \geq 1$. Therefore, $F_1$ and $F_2$ are both nature faulty sets of $BS_n$ with $|F_1| = 4n - 8$ and $|F_2| = 4n - 6$. Since $A = F_1 \cap F_2$ and $N_{BS_n}(A) = F_1 \subset F_2$, there is no edge of $BS_n$ between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Theorem 4, we can deduce that $BS_n$ is not nature $(4n - 6)$-diagnosable under the PMC model. Hence, by the definition of nature diagnosability, we conclude that the nature diagnosability of $BS_n$ is less than $4n - 6$, i.e., $t(BS_n) \leq 4n - 7$.

**Lemma 5.** Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph $BS_n$ under the PMC model is more than or equal to $4n - 7$, i.e., $t(BS_n) \geq 4n - 7$.

**Proof.** By the definition of nature diagnosability, it is sufficient to show that $BS_n$ is nature $(4n - 7)$-diagnosable.

By Theorem 4, to prove $BS_n$ is nature $(4n - 7)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(BS_n)$ with $u \in V(BS_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each distinct pair of nature faulty subsets $F_1$ and $F_2$ of $V(BS_n)$ with $|F_1| \leq 4n - 7$ and $|F_2| \leq 4n - 7$.

We prove this statement by contradiction. Suppose that there are two distinct nature faulty subsets $F_1$ and $F_2$ of $V(BS_n)$ with $|F_1| \leq 4n - 7$ and $|F_2| \leq 4n - 7$, but the vertex set pair $(F_1, F_2)$ is not satisfied with the condition in Theorem 4, i.e., there are no edges between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. Without loss of generality, assume that $F_1 \bigcap F_2 = \emptyset$.

Suppose $V(BS_n) = F_1 \cup F_2$. By the definition of $BS_n$, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 8n - 14$ for $n \geq 4$.

Since $n \geq 4$, we have that $n! = |V(BS_n)| \geq |F_1 \cup F_2| \geq |F_1| + |F_2| \geq 2(4n - 7) = 8n - 14$, a contradiction. Therefore, $V(BS_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, and $F_1$ is a nature faulty set, $BS_n - F_1$ has two parts $BS_n - F_1$ and $BS_n[F_1 \setminus F_2]$ (for convenience). Thus, $\delta(BS_n - F_1) \geq 1$ and $\delta(BS_n[F_1 \setminus F_2]) \geq 1$.

Similarly, $\delta(BS_n[F_1 \setminus F_2]) \geq 1$ when $F_1 \cap F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. When $F_1 \cap F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a nature faulty set. Since there are no edges between $V(BS_n - F_1)$ and $F_1 \Delta F_2$, $F_1 \cap F_2$ is a nature cut. Since $n \geq 4$, by Theorem 3, $|F_1 \cap F_2| \geq 4n - 8$. By Lemma 3, $|F_2 \setminus F_1| \geq 2$. Therefore, $|F_2| = |F_1 \cap F_2| + |F_2 \setminus F_1| \geq 2 + 4n - 8$.
8 = 4n - 6, which contradicts with that |F₁| ≤ 4n - 7. So BSₙ is nature (4n - 7) -diagnosable. By the definition of 
t₁(BSₙ), t₁(BSₙ) ≥ 4n - 7.

Combining Lemmas 4 and 5, we have the following theorem.

**Theorem 5.** Let n ≥ 4. Then the nature diagnosability of the bubble-sort star graph BSₙ under the PMC model is 4n - 7.

IV. THE NATURE DIAGNOSABILITY OF THE BUBBLE-SORT STAR GRAPH BSₙ UNDER THE MM* MODEL

Before discussing the nature diagnosability of the bubble-sort star graph BSₙ under the MM* model, we first give an existing result.

**Theorem 6.** ([1,10]) A system G = (V,E) is nature t -diagnosable under the MM* model if and only if each distinct pair of nature faulty subsets F₁ and F₂ of V with |F₁| ≤ t and |F₂| ≤ t satisfies one of the following conditions.

1. There are two vertices u,w ∈ V \ (F₁ ∪ F₂) and there is a vertex F₁ΔF₂ such that uw ∈ E and vw ∈ E.
2. There are two vertices u,v ∈ F₁ \ F₂ and there is a vertex w ∈ V \ (F₁ ∪ F₂) such that uw ∈ E and vw ∈ E.
3. There are two vertices u,v ∈ F₁ \ F₂ and there is a vertex w ∈ V \ (F₁ ∪ F₂) such that uw ∈ E and vw ∈ E.

**Lemma 6.** Let n ≥ 4. Then the nature diagnosability of the bubble-sort star graph BSₙ under the MM* model is less than or equal to 4n - 7, i.e., t₁(BSₙ) ≤ 4n - 7.

**Proof.** Let A, F₁ and F₂ be defined in Lemma 2. By Lemma 2, |F₁| ≤ 4n - 8, |F₂| ≤ 4n - 6, δ(BSₙ - F₁) ≥ 1 and δ(BSₙ - F₂) ≥ 1. So both F₁ and F₂ are nature faulty sets. By the definitions of F₁ and F₂, F₁ΔF₂ = A. Note F₁ \ F₂ = ∅, F₂ \ F₁ = A and (V(BSₙ \ (F₁ ∪ F₂)) \ A = ∅. Therefore, both F₁ and F₂ are not satisfied with any one condition in Theorem 6, and BSₙ is not nature (3n - 6) -diagnosable. Hence, t₁(BSₙ) ≤ 4n - 7. The proof is complete.

**Lemma 7.** Let n ≥ 5. Then the nature diagnosability of the bubble-sort star graph BSₙ under the MM* model is more than or equal to 4n - 7, i.e., t₁(BSₙ) ≥ 4n - 7.

**Proof.** By the definition of nature diagnosability, it is sufficient to show that BSₙ is nature (4n - 7) -diagnosable.

By Theorem 6, suppose, on the contrary, that there are two distinct nature faulty subsets F₁ and F₂ of BSₙ with |F₁| ≤ 4n - 7 and |F₂| ≤ 4n - 7, but the vertex set pair (F₁,F₂) is not satisfied with any one condition in Theorem 6. Without loss of generality, assume that F₁ \ F₂ ≠ ∅. Similarly to the discussion on V(BSₙ) ≠ F₁ ∪ F₂ in Lemma 5, we can deduce that V(BSₙ) ≠ F₁ ∪ F₂. Therefore, V(BSₙ) ≠ F₁ ∪ F₂.

**Claim 1.** BSₙ - F₁ - F₂ has no isolated vertex.

Suppose, on the contrary, that BSₙ - F₁ - F₂ has at least one isolated vertex w. Since F₁ is a nature faulty set, there is a vertex u ∈ F₁ \ F₂ such that u is adjacent to w. Since the vertex set pair (F₁,F₂) is not satisfied with any one condition in Theorem 6, there is at most one vertex u ∈ F₁ \ F₂ such that u is adjacent to w. Thus, there is just a vertex w ∈ F₁ \ F₂ such that u is adjacent to w. Similarly, we can deduce that there is just a vertex v ∈ F₁ \ F₂ such that v is adjacent to w when F₁ \ F₁ ≠ ∅. Let W ⊆ Sₙ \ (F₁ ∪ F₂) be the set of isolated vertices in BSₙ[Sₙ \ (F₁ ∪ F₂)], and let H be the subgraph induced by the vertex set Sₙ \ (F₁ ∪ F₁ ∪ W). Then for any w ∈ W, there are (2n - 5) neighbors in F₁ ∩ F₂. Since |F₁| ≤ 4n - 7, we have

\[ \sum_{w \in W} |N(BSₙ \ (F₁ ∪ F₁ ∪ W), w)| = |W| \leq (2n - 5) \leq \sum_{u \in F₁ ∩ F₂} d_{BSₙ}(v) \leq |F₁ ∩ F₂| (2n - 3) \leq |F₂| - 1 (2n - 3) \leq (4n - 8)(2n - 3) = 8n² - 28n + 24. \]

It follows that |W| ≤ \( \frac{8n² - 28n + 24}{2n - 5} < 4n - 3 \) for n ≥ 5. Note |F₁ \ F₁ ∩ F₂ \ F₁ ∩ F₂| ≤ 2(4n - 7) - (2n - 5) = 6n - 9. Suppose V(H) = ∅. Then n! \( |Sₙ| = |V(BSₙ)| = |F₁ ∩ F₂| + |W| < 6n - 9 + 4n - 3 = 10n - 11 \). This is a contradiction to n ≥ 5. So V(H) ≠ ∅.

Since the vertex set pair (F₁,F₂) is not satisfied with the condition (1) of Theorem 6, and any vertex of V(H) is not isolated in H, we induce that there is no edge between V(H) and F₁ΔF₂. Thus, F₁ ∩ F₂ is a vertex cut of BSₙ and δ(BSₙ - (F₁ ∩ F₂)) ≥ 1, i.e., F₁ ∩ F₂ is a nature cut of BSₙ.

By Theorem 3, |F₁ ∩ F₂| ≥ 4n - 8. Because |F₁| ≤ 4n - 7, |F₂| ≤ 4n - 7, and neither F₁ \ F₂ nor F₁ \ F₁ is empty, we have |F₁ \ F₂| ≥ |F₂ \ F₁|. Let F₁ \ F₂ = \{v₁\} and F₂ \ F₁ = \{v₂\}. Then for any vertex w ∈ W, w are adjacent to v₁ and v₂. According to Proposition 5, there are at most three common neighbors for any pair of vertices in BSₙ, it follows that there are at most three isolated vertices in BSₙ - F₁ - F₂, i.e., |W| ≤ 3.

Suppose that there is exactly one isolated vertex v in BSₙ - F₁ - F₂.

Let v₁ and v₂ be adjacent to v. Then N₂₈(y) \ (v₁,v₂) \subseteq F₁ ∩ F₂. Since BSₙ contains no triangle, it follows that N₂₈(y) \ (v₁) \subseteq F₁ ∩ F₂ \ , N₂₈(y) \ (v₁) \subseteq F₁ ∩ F₂ \ \{N₂₈(y) \ (v₁) \} \cap N₂₈(y) \ (v₁) = \emptyset \ and [N₂₈(y) \ (v₁,v₂)] \cap [N₂₈(y) \ (v₁)] = \emptyset. By Proposition 5, [N₂₈(y) \ (v₁)] \cap [N₂₈(y) \ (v₂)] ≤ 2. Thus, |F₁ ∩ F₂| ≥ |F₁ \ F₂| + |F₂ \ F₁| ≥ 1 + 6n - 15 = 6n - 14 > 4n - 7 (n ≥ 5), which contradicts |F₁| ≤ 4n - 7.

Suppose that there are exactly two isolated vertices v and w in BSₙ - F₁ - F₂.

Let v₁ and v₂ be adjacent to v and w, respectively. Then N₂₈(y) \ (v₁,v₂) \subseteq F₁ \ F₂. Since BSₙ contains no triangle, it follows that N₂₈(y) \ (v,w) \subseteq F₁ ∩ F₂ \ , N₂₈(y) \ (v,w)

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Lemma 5. There are at most two common neighbors for any pair of vertices in $B_{5}$. Thus, it follows that $|N_{V}(v) \cap \{w\}| = \emptyset$. By the arbitrariness of $d$, it follows that $|F| \leq 4n - 7$.

Suppose there exist exactly three isolated vertices $u,v,w$ in $B_{5} - F_{1} - F_{2}$.

Let $v_{1}$ and $v_{2}$ be adjacent to $u,v,w$, respectively. Then $N_{V}(v) \cap \{v_{1},v_{2}\} \subseteq F_{1} \cap F_{2}$. Since $B_{n}$ contains no triangle, it follows that $N_{V}(v) \cap \{u,v,w\} \subseteq F_{1} \cap F_{2}$. Let $B_{n} - F_{1} - F_{2} = \emptyset$. By Proposition 5, there are at most three common neighbors for any pair of vertices in $B_{n}$. Thus, it follows that $|N_{V}(v) \cap \{u,v,w\}| = 0$. Thus, $|F| \leq 4n - 7$.

The proof of Claim I is complete.

Let $u \in V(B_{n}) \setminus (F_{1} \cup F_{2})$. By Claim I, $u$ has at least one neighbor in $B_{n} - F_{1} - F_{2}$. Since the vertex set pair $(F_{1},F_{2})$ is not satisfied with any one condition in Theorem 6, by the condition (1) of Theorem 6, for any pair of adjacent vertices $u,w \in V(B_{n}) \setminus (F_{1} \cup F_{2})$, there is no vertex $v \in V(B_{n}) \setminus (F_{1} \cup F_{2})$ such that $uv \in E(B_{n})$ and $vw \in E(B_{n})$. It follows that $u$ has no neighbor in $B_{n}$. By the arbitrariness of $u$, there is no edge between $V(B_{n}) \setminus (F_{1} \cup F_{2})$. Since $F_{1} \setminus F_{2} \neq \emptyset$ and $F_{1}$ is a nature faulty set, $\delta_{B_{n}}(\{F_{1} \cup F_{2}\}) \geq 1$.

By Lemma 3, $|F_{1} \setminus F_{1}| \geq 2$. Since both $F_{1}$ and $F_{2}$ are nature faulty sets, and there is no edge between $V(B_{n}) \setminus (F_{1} \cup F_{2})$ and $F_{1} \setminus F_{2}$, $F_{1} \cap F_{2}$ is a nature cut of $B_{n}$. By Theorem 3, we have $|F_{1} \cap F_{2}| \geq 4n - 8$. Therefore, $|F_{1} \setminus F_{1}| + |F_{1} \setminus F_{2}| \geq 4n - 6$, which contradicts $|F_{1}| \leq 4n - 7$. Therefore, $B_{n}$ is nature $(4n - 7)$-diagnosable in the $MM^{*}$ model. The proof is complete.

Combining Lemmas 6 and 7, we have the following theorem.

**Theorem 7.** Let $n \geq 5$. Then the nature diagnosability of the bubble-sort star graph $B_{n}$ under the $MM^{*}$ model is $4n - 7$.

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References


