The Nature Diagnosability of Bubble-sort Star Graphs under the PMC Model and MM* Model

Mujiangshan Wang, Yuqing Lin, Shiyong Wang

Abstract—Many multiprocessor systems have interconnection networks as underlying topologies and an interconnection network is usually represented by a graph where nodes represent processors and links represent communication links between processors. No fault set can contain all the neighbors of any fault-free vertex in the system, which is called the nature diagnosability of the system. Diagnosability of a multiprocessor system is one important study topic. As a famous topology structure of interconnection networks, the n-dimensional bubble-sort star graph BSn has many good properties. In this paper, we prove that the nature diagnosability of BSn is 4n − 7 under the PMC model for \( n \geq 4 \), the nature diagnosability of BSn is 4n − 7 under the MM* model for \( n \geq 5 \).

Index Terms—Bubble-sort star graph, Diagnosability, Interconnection network.

I. INTRODUCTION

Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies and a network is usually represented by a graph where nodes represent processors and links represent communication links between processors. Some processors may fail in the system, so processor fault identification plays an important role for reliable computing. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system \( G \) is said to be \( t \)-diagnosable if all faulty processors can be identified without replacement, provided that the number of presented faults does not exceed \( t \). The diagnosability \( t(G) \) of \( G \) is the maximum value of \( t \) such that \( G \) is \( t \)-diagnosable. For a \( t \)-diagnosable system, Dahbura and Masson [1] proposed an algorithm with time complexity \( O(n^{t+1}) \), which can effectively identify the set of faulty processors. Several diagnosis models (e.g., Preparata, Metze, and Chien's (PMC) model [2], Barsi, Grandoni, and Maestrini's (BGM)model [3], and Maeng and Malek's (MM) model [4] have been proposed to investigate the diagnosability of multiprocessor systems. In particular, two of the proposed models, the PMC model and the MM model, are well known and widely used. In the PMC model, the diagnosis of the system is achieved through two linked processors testing each other. In the MM model, to diagnose a system, a node sends the same task to two of its neighbors, and then compares their responses. For this reason, the MM model is also said to be the comparison model. Sengupta and Dahbura [1] proposed a special case of the MM model, called the MM* model, in which each node must test its any pair of adjacent nodes. Numerous studies have been investigated under the PMC model and MM model or MM* model.

In the traditional diagnosis of a multiprocessor system, one generally assumes that any subset of processors may simultaneously fail. If all the neighbors of some node \( v \) are faulty simultaneously, it is impossible to determine whether \( v \) is faulty or fault-free. As a consequence, the diagnosability of the system is less than its minimum node degree. However, in some large-scale multiprocessor systems, we can safely assume that all neighbors of any node do not fail at the same time. Based on this assumption, in 2005, Lai et al. [5] introduced the restricted diagnosability of the system called the conditional diagnosability. They consider the situation that no fault set can contain all the neighbors of any node in the system. Since the probability that the all neighbors of a fault node fail and create faults is more to the probability that the all neighbors of a fault-free node fail and create faults in the system, we consider the situation that no fault set can contain all the neighbors of any fault-free node in the system, which is called the nature diagnosability of the system. In 2012, Peng et al. [6] proposed a measure for fault diagnosis of the system, namely, the g-good-neighbor diagnosability (which is also called the g-good-neighbor conditional diagnosability), which requires that every fault-free node contains at least \( g \) fault-free neighbors. In [6], they studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model. In [7], Wang and Han studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the MM* model. In 2016, Ren and Wang [8] gave some properties of the g-good-neighbor diagnosability of a multiprocessor system. In 2017, Wang et al. [9] studied that the 2-good-neighbor diagnosability of bubble-sort star graph networks under the PMC model and MM* model. Yuan et al. [10,11] studied that the g-good-neighbor diagnosability of the k-ary n-cube \((k \geq 3)\) under the PMC model and MM* model. As a favorable topology structure of interconnection networks, the Cayley graph \( CT_n \) generated by the transposition tree \( \Gamma_n \) has many good properties. In [12], Wang et al. studied the 2-good-neighbor diagnosability of \( CT_n \) under the PMC model and MM* model. In 2016, Zhang et al. [13] proposed a new measure for fault diagnosis of the system, namely, the g-extra diagnosability, which restrains that every fault-free component has at least \((g+1)\) fault-free nodes. In [13], they studied the g-extra diagnosability of the n-dimensional hypercube under the PMC model and MM* model. In 2016, Wang et al. [14] studied that the 2-extra diagnosability of the n-dimensional bubble-sort star graph under the PMC model and MM* model. In [15], Han and Wang studied that the

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g-extra diagnosability of folded hypercubes. In 2017, Wang and Yang [16] studied the 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model and MM* model. In [17], Wang et al. studied the nature diagnosability of $CT_n$ under the PMC model and MM* model and proved that the nature diagnosability of the system is less than or equal to the conditional diagnosability of the system. Therefore, the nature diagnosability of the system is nature and one important study topic. In 2016, Bai and Wang [18] studied the nature diagnosability of Moebius cubes; Hao and Wang [19] studied the nature diagnosability of augmented k-ary n-cubes; Jirimutu and Wang [20] studied the nature diagnosability of alternating group graph networks; Ma and Wang [21] studied the nature diagnosability of crossed cubes; Zhao and Wang [22] studied the nature diagnosability of augmented 3-ary n-cubes. The star graph and the bubble-sort graph have been proved to be an important viable candidate for interconnecting a multiprocessor system. The feature of the star graph include low degree of node, small diameter, symmetry, and high degree of fault-tolerance. The diagnosabilities of the star graph under the PMC model and MM model were studied in [23,24]. Lin et al. [25] showed that the conditional diagnosability of the star graph under the comparison diagnosis model is $3n-7$. In this paper, the nature diagnosability of the n-dimensional bubble-sort star graph $BS_n$ under the PMC model and MM* model has been studied. It is proved that the nature diagnosability of $BS_n$ is $4n-7$ under the PMC model for $n \geq 4$, the nature diagnosability of $BS_n$ is $4n-7$ under the MM* model for $n \geq 5$.

II. PRELIMINARIES

In this section, some definitions and notations needed for our discussion, the bubble-sort star graph, the PMC model and MM* model are introduced.

A. Definitions and Notations

A multiprocessor system is modeled as an undirected simple graph $G = (V, E)$, whose vertices (nodes) represent processors and edges (links) represent communication links. Given a nonempty vertex subset $V' \subseteq V$, the subgraph induced by $V'$ in $G$, denoted by $G[V']$, is a graph, whose vertex set is $V'$ and the edge set is the set of all the edges of $G$ with both endpoints in $V'$. The degree $d_G(v)$ of a vertex $v$ is the number of edges incident with $v$. We denote by $\delta(G)$ the minimum degrees of vertices of $G$. For any vertex $v$, we define the neighborhood $N_G(v)$ of $v$ in $G$ to be the set of vertices adjacent to $v$. $u$ is called a neighbor or a neighbor vertex of $v$ for $u \in N_G(v)$. Let $S \subseteq V$. We use $N_G(S)$ to denote the set $\cup_{v \in S} N_G(v) \setminus S$. For neighborhoods and degrees, we will usually omit the subscript for the graph when no confusion arises. A graph $G$ is said to be $k$-regular if for any vertex $v$, $d_G(v) = k$. Let $G$ be a connected graph. The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left when $G$ is complete. A fault set $F \subseteq V$ is called a nature faulty set if $|N(v) \cap (V \setminus F)| \geq 1$ for every vertex $v$ in $V \setminus F$. A nature cut of $G$ is a nature faulty set $F$ such that $G - F$ is disconnected. The minimum cardinality of nature cuts is said to be the nature connectivity of $G$, denoted by $\kappa'(G)$. For graph-theoretical terminology and notation not defined here we follow [26].

B. The PMC model and MM* model

For the PMC model and MM* model, we follow [10].

In a system $G = (V, E)$, a faulty set $F \subseteq V$ is called a conditional faulty set if it does not contain all of neighbors of any vertex in $G$. A system $G$ is conditional $t$-diagnosable if every two distinct conditional faulty subsets $F_1, F_2 \subseteq V$ with $|F_1| \leq t, |F_2| \leq t$ are distinguishable. The conditional diagnosability $t(G)$ of $G$ is the maximum number of $t$ such that $G$ is conditional $t$-diagnosable. By [27], $t(G) \geq t(G)$.

Theorem 1. ([17]) For a system $G = (V, E)$, $t(G) = t_0(G) \leq t_1(G) \leq t_2(G)$.

In [17], Wang et al. proved that the nature diagnosability of the Bubble-sort graph $B_n$ under the PMC model is $2n-3$ for $n \geq 4$. In [28], Zhou et al. proved the conditional diagnosability of $B_n$ is $4n-11$ for $n \geq 4$ under the PMC model. Therefore, $t_1(B_n) < t_1(B_n)$ when $n \geq 5$ and $t_1(B_n) = t_1(B_n)$ when $n = 4$.

C. The bubble-sort star graph

The bubble-sort star graph has been known as a famous topology structure of interconnection networks. In this section, its definition and some properties are introduced.

Let $[n] = \{1, 2, \ldots, n\}$ and let $S_n$ be the symmetric group on $[n]$. containing all permutations $p = p_1 p_2 \cdots p_n$ of $[n]$. It is well known that $\{(i) : 2 \leq i \leq n\}$ is a generating set for $S_n$. So $\{(i, i+1) : 2 \leq i \leq n-1\}$ is also a generating set for $S_n$. The n-dimensional bubble-sort star graph $BS_n$ [29,30] is the graph with vertex set $V(BS_n) = S_n$ in which two vertices $u, v$ are adjacent if and only if $u = v(i, i+1), 2 \leq i \leq n$ or $u = v(i, i+1), 2 \leq i \leq n-1$. It is easy to see from the definition that $BS_n$ is a $(2n-3)$-regular graph on $n^2$ vertices.

Note that $BS_n$ is a special Cayley graph. $BS_n$ has the following useful properties.

Proposition 1. For any integer $n \geq 1$, $BS_n$ is $(2n-3)$-regular, vertex transitive.

Proposition 2. For any integer $n \geq 2$, $BS_n$ is bipartite.

Proposition 3. For any integer $n \geq 3$, the girth of $BS_n$ is 4.

Theorem 2. ([31]) Let $H$ be a simple connected graph with $n = |V(H)| \geq 3$. If $H^1$ and $H^2$ are two different labelled graphs obtained by labelling $H$ with $\{1, 2, \ldots, n\}$, then $\text{Cay}(H^1, S_n)$ is isomorphic to $\text{Cay}(H^2, S_n)$.

We can partition $BS_n$ into $n$ subgraphs $BS_1, BS_2, \ldots, BS_n$, where every vertex $u = x_1 x_2 \ldots x_n \in V(BS_n)$ has a fixed integer $i$ in the last position $x_i$ for $i \in [n]$. It is obvious that
$B_{n}^{i}$ is isomorphic to $B_{n-i}$ for $i \in [n]$. Let $v \in V(B_{n}^{i})$. Then $v(ln)$ and $v(n-i,n)$ are called outside neighbors of $v$.

**Proposition 3.** ((29)) Let $B_{n}^{i}$ be defined as above. Then there are $2(n-2)!$ independent cross-edges between two different $H_{i}$'s.

**Proposition 4.** ((29)) Let $B_{n}$ be the bubble-sort star graph. If two vertices $u, v$ are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices $u, v$ are not adjacent, there are at most three common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \leq 3$.

**Lemma 1.** ([9]) The nature connectivity $\kappa'(B_{n})$ of the bubble-sort star graph $B_{n}$ is 8.

A connected graph $G$ is super nature connected if every minimum nature cut $F$ of $V(G)$ isolates one edge. If, in addition, $G - F$ has two components, one of which is an edge, then $G$ is tightly $F$ super nature connected.

**Theorem 3.** ([14]) For $n \geq 5$, the bubble-sort star graph $B_{n}$ is tightly $(4n - 8)$ super nature connected.

**Lemma 2.** Let $A = \{1, (12)\}$. If $n \geq 4$, $F_{1} = N_{B_{n}}(A)$, $F_{2} = A \cup N_{B_{n}}(A)$, then $|F_{1}| = 4n - 8$, $|F_{2}| = 4n - 6$, $\delta(B_{n} - F_{2}) \geq 1$, and $\delta(B_{n} - F_{1}) \geq 1$.

**Proof.** By $A = \{1, (12)\}$, we have $B_{n}(A) \cong B_{n} = K_{2}$. Since $B_{n}$ has no 3-cycles, $|N_{B_{n}}(A)| = 4n - 8$. Thus from calculating, we have $|F_{1}| = 4n - 8$, $|F_{2}| = |A| + |F_{1}| = 4n - 6$.

Claim 1. For any $x \in S_{1} \setminus F_{2}$, $|N_{B_{n}}(x) \cap F_{2}| \leq 2n - 4$.

Since $B_{n}$ is a bipartite graph, there is no 5-cycle (1), (12), (1)(12), (1)(1), (1) of $B_{n}$, where $x, y \in S_{1} \setminus F_{2}$, $x \neq y$.

Thus from calculating, we have that $|S_{1} \setminus F_{2}| \geq 2n - 4$.

By Claim 1, $|N_{B_{n}}(x) \cap F_{2}| \leq 2n - 4$ for any $x \in S_{1} \setminus F_{2}$.

Therefore, $\delta(B_{n} - F_{2}) \geq 2n - 3 - (2n - 4) = 1$. $B_{n} - F_{1}$ has two components $B_{n} - F_{2}$ and $B_{n}$. Note that $\delta(B_{n}) = 1$.

Therefore, $\delta(B_{n} - F_{1}) \geq 1$.

III. THE NATURE DIGNOSABILITY OF THE BUBBLE-SORT STAR GRAPH UNDER THE PMC MODEL

In this section, we shall show the nature diagnosability of the bubble-sort star graph under the PMC model. Let $F_{1}$ and $F_{2}$ be two distinct subsets of $V$ for a system $G = (V, E)$. Define the symmetric difference $F_{1} \Delta F_{2} = (F_{1} \setminus F_{2}) \cup (F_{2} \setminus F_{1})$. Yuan et al. [10] presented a sufficient and necessary condition for a system to be nature $t$ diagnosable under the PMC model.

**Theorem 4.** ([10]) A system $G = (V, E)$ is nature $t$ diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_{1} \cup F_{2})$ and $v \in F_{1} \Delta F_{2}$ for each distinct pair of nature faulty subsets $F_{1}$ and $F_{2}$ of $V$ with $|F_{1}| \leq t$ and $|F_{2}| \leq t$.

**Lemma 3.** A graph of minimum degree 1 has at least two vertices.

The proof of Lemma 3 is trivial.

**Lemma 4.** Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph $B_{n}$ under the PMC model is less than or equal to $4n - 7$, i.e., $t(B_{n}) \leq 4n - 7$.

**Proof.** Let $A$ be defined in Lemma 2, and let $F_{1} = N_{B_{n}}(A)$, $F_{2} = A \cup N_{B_{n}}(A)$. By Lemma 2, $|F_{1}| = 4n - 8$, $|F_{2}| = 4n - 6$, $\delta(B_{n} - F_{2}) \geq 1$ and $\delta(B_{n} - F_{1}) \geq 1$. Therefore, $F_{1}$ and $F_{2}$ are both nature faulty sets of $B_{n}$ with $|F_{1}| = 4n - 8$ and $|F_{2}| = 4n - 6$. Since $A = F_{1} \Delta F_{2}$ and $N_{B_{n}}(A) = F_{1} \subset F_{2}$, there is no edge of $B_{n}$ between $V(B_{n}) \setminus (F_{1} \cup F_{2})$ and $F_{1} \Delta F_{2}$. By Theorem 4, we can deduce that $B_{n}$ is not nature $(4n - 6)$-diagnosable under the PMC model. Hence, by the definition of nature diagnosability, we conclude that the nature diagnosability of $B_{n}$ is less than $4n - 6$, i.e., $t(B_{n}) \leq 4n - 7$.

**Lemma 5.** Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph $B_{n}$ under the PMC model is more than or equal to $4n - 7$, i.e., $t(B_{n}) \geq 4n - 7$.

**Proof.** By the definition of nature diagnosability, it is sufficient to show that $B_{n}$ is nature $(4n - 7)$-diagnosable.

By Theorem 4, to prove $B_{n}$ is nature $(4n - 7)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(B_{n})$ with $u \in V(B_{n}) \setminus (F_{1} \cup F_{2})$ and $v \in F_{1} \Delta F_{2}$ for each distinct pair of nature faulty subsets $F_{1}$ and $F_{2}$ of $V(B_{n})$ with $|F_{1}| \leq 4n - 7$ and $|F_{2}| \leq 4n - 7$.

We prove this statement by contradiction. Suppose that there are two distinct nature faulty subsets $F_{1}$ and $F_{2}$ of $V(B_{n})$ with $|F_{1}| \leq 4n - 7$ and $|F_{2}| \leq 4n - 7$, but the vertex set pair $(F_{1}, F_{2})$ is not satisfied with the condition in Theorem 4, i.e., there are no edges between $V(B_{n}) \setminus (F_{1} \cup F_{2})$ and $F_{1} \Delta F_{2}$. Without loss of generality, assume that $F_{1} \Delta F_{2} = \emptyset$.

Suppose $V(B_{n}) = F_{1} \cup F_{2}$. By the definition of $B_{n}$, $|F_{1} \cup F_{2}| = |F_{1}| = n!$. It is obvious that $n! > 8n - 14$ for $n \geq 4$. Since $n \geq 4$, we have that $n! = |V(B_{n})| = |F_{1} \cup F_{2}| = |F_{1}| + |F_{2}| - |F_{1} \cap F_{2}| = |F_{1}| + |F_{2}| \leq 2(4n - 7) = 8n - 14$, a contradiction. Therefore, $V(B_{n}) \neq F_{1} \cup F_{2}$.

Since there are no edges between $V(B_{n}) \setminus (F_{1} \cup F_{2})$ and $F_{1} \Delta F_{2}$, and $F_{1}$ is a nature faulty set, $B_{n} - F_{1}$ has two parts $B_{n} - F_{1} - F_{2}$ and $B_{n}[F_{1} \setminus F_{2}]$ (for convenience). Thus, $\delta(B_{n} - F_{1} - F_{2}) \geq 1$ and $\delta(B_{n}[F_{1} \setminus F_{2}]) \geq 1$. Similarly, $\delta(B_{n}[F_{1} \setminus F_{2}]) \geq 1$ when $F_{1} \cap F_{2} \neq \emptyset$. Therefore, $F_{1} \cap F_{2}$ is also a nature faulty set. When $F_{1} \cap F_{2} = \emptyset$, $F_{1} \cap F_{2} = F_{1}$ is also a nature faulty set. Since there are no edges between $V(B_{n} - F_{1} - F_{2})$ and $F_{1} \Delta F_{2}$, $F_{1} \cap F_{2}$ is a nature cut. Since $n \geq 4$, by Theorem 3, $|F_{1} \cap F_{2}| \geq 4n - 8$. By Lemma 3, $|F_{2} \setminus F_{1}| \geq 2$. Therefore, $|F_{2} \setminus F_{1}| \geq 2 + 4n -
8 = 4n − 6 , which contradicts with that |F₀| ≤ 4n − 7 . So BS₅₄ is nature (4n − 7) diagnosable. By the definition of t₀(BS₅₄), t₀(BS₅₄) ≥ 4n − 7.

Combining Lemmas 4 and 5, we have the following theorem.

**Theorem 5.** Let n ≥ 4 . Then the nature diagnosability of the bubble-sort star graph BS₅₄ under the PMC model is 4n − 7.

IV. THE NATURE DIAGNOSABILITY OF THE BUBBLE-SORT STAR GRAPH BS₅₄ UNDER THE MM* MODEL

Before discussing the nature diagnosability of the bubble-sort star graph BS₅₄ under the MM* model, we first give an existing result.

**Theorem 6.** ([11, 10]) A system G = (V, E) is nature t-diagnosable under the MM* model if and only if each distinct pair of nature faulty subsets F₁ and F₂ of V with |Fi| ≤ t and |F₁ ∩ F₂| ≤ t satisfies one of the following conditions.

1. There are two vertices u, w ∈ V \ (F₁ ∪ F₂) and there is a vertex F₁ΔF₂ such that uw ∈ E and vw ∈ E .
2. There are two vertices u, v ∈ F₁ \ F₂ and there is a vertex w ∈ V \ (F₁ ∪ F₂) such that uw ∈ E and vw ∈ E .
3. There are two vertices u, v ∈ F₁ \ F₂ and there is a vertex w ∈ V \ (F₁ ∪ F₂) such that uw ∈ E and vw ∈ E .

**Lemma 6.** Let n ≥ 4 . Then the nature diagnosability of the bubble-sort star graph BS₅₄ under the MM* model is less than or equal to 4n − 7 , i.e., t₀(BS₅₄) ≤ 4n − 7.

**Proof.** Let A, F₁, and F₂ be defined in Lemma 2. By Lemma 2, |F₀| = 4n − 8 , |F₁| = 4n − 6 , δ(BS₅₄ − F₀) ≥ 1 and δ(BS₅₄ − F₁) ≥ 1 . So both F₀ and F₁ are nature faulty sets. By the definitions of F₀ and (F₀)ΔF₁ = A . Note F₀ \ F₁ = 0 , F₁ \ F₀ = A , and (V(BS₅₄) \ (F₀ ∪ F₁)) \ A = 0 . Therefore, both F₀ and F₁ are not satisfied with any one condition in Theorem 6, and BS₅₄ is not nature (3n − 6)-diagnosable. Hence, t₀(BS₅₄) ≤ 4n − 7 . The proof is complete.

**Lemma 7.** Let n ≥ 5 . Then the nature diagnosability of the bubble-sort star graph BS₅₄ under the MM* model is more than or equal to 4n − 7 , i.e., t₀(BS₅₄) ≥ 4n − 7.

**Proof.** By the definition of nature diagnosability, it is sufficient to show that BS₅₄ is nature (4n − 7)-diagnosable.

By Theorem 6, suppose, on the contrary, that there are two distinct nature faulty subsets F₁ and F₂ of BS₅₄ with |F₀| ≤ 4n − 7 and |F₁| ≤ 4n − 7 , but the vertex set pair (F₀, F₁) is not satisfied with any one condition in Theorem 6. Without loss of generality, assume that F₁ \ F₀ 0 . Similarly to the discussion on V(BS₅₄) \ (F₀ ∪ F₁) in Lemma 5, we can deduce that V(BS₅₄) \ (F₀ ∪ F₁) . Therefore, V(BS₅₄) \ (F₁ ∪ F₂) .

**Claim 1.** BS₅₄ − F₁ − F₂ has no isolated vertex.

Suppose, on the contrary, that BS₅₄ − F₁ − F₂ has at least one isolated vertex w . Since BS₅₄ is a nature faulty set, there is a vertex u ∈ F₀ \ F₁ such that u is adjacent to w . Since the vertex set pair (F₀, F₁) is not satisfied with any one condition in Theorem 6, there is at most one vertex u ∈ F₀ \ F₁ such that u is adjacent to w . Thus, there is just a vertex u ∈ F₀ \ F₁ such that u is adjacent to w . Similarly, we can deduce that there is just a vertex v ∈ F₀ \ F₁ such that v is adjacent to w when F₁ \ F₀ ∅ . Let W ⊆ S₅₄ \ (F₀ ∪ F₁) be the set of isolated vertices in BS₅₄[S₅₄ \ (F₀ ∪ F₁)] , and let H be the subgraph induced by the vertex set S₅₄ \ (F₀ ∪ F₁ ∪ W) . Then for any w ∈ W , there are (2n − 5) neighbors in F₀ ∩ F₁ . Since |F₀| ≤ 4n − 7 , we have ∑w∈W |N₅₄(F₀, F₁) \ w(w) | |W|| 2(2n − 5) ≤ ∑w∈Wd₅₄(w) ≤ |F₀ ∩ F₁||(2n − 3) ≤ |F₀ ∩ F₁||(2n − 3) ≤ (4n − 8)(2n − 3) = 8n² − 28n + 24 . It follows that |W| ≤ 8n² − 28n + 24 < 4n − 3 for n ≥ 5 . Note |F₀ ∩ F₁||F₀| + |F₁| − |F₀ ∩ F₁| ≤ 2(4n − 7) − (2n − 5) = 6n − 9 . Suppose V(H) = ∅ . Then n! |S₅₄| = |V(BS₅₄)| − |F₀ ∩ F₁| + |W| < 6n − 9 + 4n − 3 = 10n − 11 . This is a contradiction to n ≥ 5 . So V(H) ∅ .

Since the vertex set pair (F₀, F₁) is not satisfied with the condition (1) of Theorem 6, and any vertex of V(H) is not isolated in H , we induce that there is no edge between V(H) and F₁ΔF₂ . Thus, F₀ ∩ F₁ is a vertex cut of BS₅₄ and δ(BS₅₄ − (F₀ ∪ F₁)) ≥ 1 , i.e., F₀ ∩ F₁ is a nature cut of BS₅₄ . By Theorem 3, |F₀ ∩ F₁| ≥ 4n − 8 . Because |F₀| ≤ 4n − 7 , |F₁| ≤ 4n − 7 , and neither F₁ \ F₀ nor F₀ \ F₁ is empty, we have |F₀ ∩ F₁| ≥ |F₀| \ F₁| ≥ 1 . Let F₁ ∩ F₀ = {v₁} and F₁ \ F₀ = {v₂} . Then for any vertex w ∈ W , w are adjacent to v₁ and v₂ . According to Proposition 5, there are at most three common neighbors for any pair of vertices in BS₅₄ , it follows that there are at most three isolated vertices in BS₅₄ − F₀ − F₁ , i.e., |W| ≤ 3 .

Suppose that there is exactly one isolated vertex v in BS₅₄ − F₀ − F₁ .

Let v₁ and v₂ be adjacent to v . Then N₅₄(v₁, v₂) ⊆ F₀ ∩ F₁ . Since BS₅₄ contains no triangle, it follows that N₅₄(v₁) \ {v} ⊆ F₀ ∩ F₁ ; N₅₄(v₂) \ {v} ⊆ F₀ ∩ F₁ ; [N₅₄(v₁) \ {v₁, v₂}] \ [N₅₄(v₂) \ {v₁, v₂}] = ∅ and [N₅₄(v₂) \ {v₁, v₂}] \ [N₅₄(v₁) \ {v₁, v₂}] = ∅ . By Proposition 5, |[N₅₄(v₁) \ {v₁}] \ [N₅₄(v₂) \ {v₂}]| ≤ 2 . Thus, F₀ ∩ F₁ | |F₀| + |F₁| − |F₀ ∩ F₁| ≥ 1 + 6n − 15 = 6n − 14 > 4n − 7 (n ≥ 5) , which contradicts |F₀| ≤ 4n − 7 .

Suppose that there are exactly two isolated vertices v and w in BS₅₄ − F₀ − F₁ .

Let v₁ and v₂ be adjacent to v and w , respectively. Then N₅₄(v₁, v₂) ⊆ F₀ ∩ F₁ . Since BS₅₄ contains no triangle, it follows that N₅₄(v₁) \ {v₁, v₂} ⊆ F₀ ∩ F₁ . Since BS₅₄ contains no triangle, it follows that N₅₄(v₁, v₂) \ {v₁, v₂} ⊆ F₀ ∩ F₁ . Since BS₅₄ contains no triangle, it follows that N₅₄(v₁) \ {v₁, v₂} ⊆ F₀ ∩ F₁ . Since BS₅₄ contains no triangle, it follows that N₅₄(v₁) \ {v₁, v₂} ⊆ F₀ ∩ F₁ .
\( \subseteq F_1 \cap F_2 \), \([N_{BS}(v) \setminus \{v_i, v_j\}] \cap [N_{BS}(v_i) \setminus \{v, w\}] = \emptyset \) and \([N_{BS}(v) \setminus \{v_i, v_j\}] \cap [N_{BS}(v_i) \setminus \{v, w\}] = \emptyset \). By Proposition 5, there are at most two common neighbors for any pair of vertices in \( BS_n \). Thus, it follows that \([N_{BS}(v) \setminus \{v, w\}] \cap [N_{BS}(v_i) \setminus \{v, w\}] \leq 1 \). Thus, \([F_1 \cap F_2 \| N_{BS}(v) \setminus \{v_i, v_j\}] + [N_{BS}(v) \setminus \{v, w\}] + [N_{BS}(v_i) \setminus \{v, w\}] = (2n - 5) + (2n - 5) + (2n - 5) - 1 + (2n - 5) = 8n - 22 \). It follows that \([F_1 \cap F_2 \| F_1 \cap F_2 \| F_1 \cap F_2 \] \( \geq 1 \) and \( 8n - 22 = 8n - 21 > 4n - 7 \) \((n \geq 5)\), which contradicts \([F_1 \| 4n - 7 \).

Suppose there are exactly three isolated vertices \( u, v \) and \( w \) in \( BS_n - F_1 - F_2 \).

Let \( v_i \) and \( v_j \) be adjacent to \( u, v \) and \( w \), respectively. Then \( N_{BS}(v) \setminus \{v_i, v_j\} \subseteq F_1 \cap F_2 \). Since \( BS_n \) contains no triangle, it follows that \( N_{BS}(v) \setminus \{u, v, w\} \subseteq F_1 \cap F_2 \). \( N_{BS}(v) \setminus \{u, v, w\} \supseteq F_1 \cap F_2 \). \( N_{BS}(v) \setminus \{u, v, w\} \cap \{v, w\} = \emptyset \) and \( N_{BS}(v) \setminus \{v_i, v_j\} \cap \{v, w\} = \emptyset \). By Proposition 5, there are at most three common neighbors for any pair of vertices in \( BS_n \). Thus, it follows that \([N_{BS}(v) \setminus \{u, v, w\}] \cap [N_{BS}(v) \setminus \{u, v, w\}] = 0 \). Thus, \([F_1 \cap F_2 \| N_{BS}(u) \setminus \{v_i, v_j\}] + [N_{BS}(v) \setminus \{v_i, v_j\}] + [N_{BS}(w) \setminus \{v_i, v_j\}] + [N_{BS}(v) \setminus \{u, v, w\}] = (2n - 5) + (2n - 5) + (2n - 5) + (2n - 6) = 10n - 30 \). It follows that \([F_1 \| F_2 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \] \( \geq 1 \) and \( 10n - 30 = 10n - 29 > 4n - 7 \) \((n \geq 5)\), which contradicts \([F_1 \| 4n - 7 \).

Suppose \( F_1 \| F_2 \| F_1 \| F_2 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \| F_1 \) \( \| 4n - 7 \).

The proof of Claim I is complete.

Let \( u \in V(\{F_1 \cup F_2 \}) \). By Claim I, \( u \) has at least one neighbor in \( BS_n - F_1 - F_2 \). Since the vertex set pair \( (F_1, F_2) \) is not satisfied with any one condition in Theorem 6, by the condition (1) of Theorem 6, for any pair of adjacent vertices \( u, v \in E(\{F_1 \cup F_2 \}) \), there is no vertex \( v \in E(F_1 \cup F_2) \) such that \( u, v \in E(\{F_1 \cup F_2 \}) \) and \( v \) \( \in \) \( E(\{F_1 \cup F_2 \}) \). It follows that \( u \) has no neighbor in \( F_1 \cup F_2 \). By the arbitrariness of \( u \), there is no edge between \( V(\{F_1 \cup F_2 \}) \) \( \supset \) \( F_1 \cup F_2 \). Since \( F_1 \cup F_2 \) \( \neq \emptyset \) and \( F_1 \) is a nature faulty set, \( \delta_{BS}([F_1 \cup F_2]) \geq 1 \).

By Lemma 3, \( [F_1 \cup F_2] \geq 2 \). Since both \( F_1 \) and \( F_2 \) are nature faulty sets, and there is no edge between \( V(\{F_1 \cup F_2 \}) \) \( \supset \) \( F_1 \cup F_2 \) and \( F_1 \cup F_2 \), \( F_1 \cup F_2 \) is a nature cut of \( BS_n \). By Theorem 3, we have \( [F_1 \cup F_2] \geq 4n - 8 \). Therefore, \( [F_1] \| [F_2] \| [F_1 \cup F_2] \| [F_1 \cup F_2] \| [F_1 \cup F_2] \| 2 + (4n - 8) = 4n - 6 \), which contradicts \([F_1 \| 4n - 7 \).

Therefore, \( BS_n \) is nature \((4n - 7)\) diagnosable and \( t_1(\{F_1 \cup F_2 \}) \geq 4n - 7 \). The proof is complete.

Combining Lemmas 6 and 7, we have the following theorem.

**Theorem 7.** Let \( n \geq 5 \). Then the nature diagnosability of the bubble-sort star graph \( BS_n \) under the MM* model is \( 4n - 7 \).


