On The Negative Pell Equation $y^2 = 112x^2 - 7$

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***Abstract*— *Abstract*— The negative pell equation represented by the binary quadratic equation is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. Employing the solutions of the equation under consideration, the integer solutions for a few choices of hyperbola and parabola are obtained.**

***Index Terms*— Binary quadratic, Hyperbola, Integral solutions, Parabola, Pell equation. 2010 Mathematics subject classification:11D09**

I. INTRODUCTION

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interesting mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solution whereas $y^2 = 54x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-17]. More specifically, one may refer "The on-line encyclopedia of integer sequences" (A031396, A130226, A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative pell equation $y^2 = 112x^2 - 7$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

II. METHOD OF ANALYSIS

The negative pell equation representing hyperbola under consideration is

$$y^2 = 112x^2 - 7 \quad (1)$$

whose smallest positive integer solution is

$$x_0 = 2, y_0 = 21 \quad (2)$$

To obtain the other solutions of (1), consider the pell equation

$$y^2 = 112x^2 + 1 \quad (3)$$

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whose general solution is given by

$$\tilde{y}_n = \frac{f_n}{2}$$

$$\tilde{x}_n = \frac{g_n}{2\sqrt{112}}$$

where

$$f_n = (127 + 12\sqrt{112})^{n+1} + (127 - 12\sqrt{112})^{n+1}$$

$$g_n = (127 + 12\sqrt{112})^{n+1} - (127 - 12\sqrt{112})^{n+1}, n = 0,1,2,3,...$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = f_n + \frac{21}{2\sqrt{112}} g_n \quad (4)$$

$$y_{n+1} = \frac{21}{2} f_n + \sqrt{112}\, g_n \quad (5)$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 254x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 254y_{n+2} + y_{n+1} = 0, n = -1,0,1,2,...$$

Some numerical examples of x and y satisfying (1) are given in the table below:

Table1 : Numerical Examples

n	x_n	y_n
0	2	21
1	506	5355
2	128522	1360149
3	32644082	345472491
4	8291468306	$8.774865257*10^{10}$
5	$2.106000306*10^{12}$	$2.228781227*10^{13}$

From the above table , we observe some interesting relations among the solutions which are presented below:

1. x_n is always even
2. y_n is always odd
3. Relations among the solutions:

- $9408x_{n+2} = 7y_{n+3} - 889y_{n+2}$

- $y_{n+1} = 254y_{n+2} - y_{n+3}$
- $341376x_{n+1} = y_{n+3} - 32257y_{n+1}$
- $225799x_{n+1} = 7x_{n+3} - 21336y_{n+1}$
- $889y_{n+1} = 225799y_{n+2} - 9408x_{n+3}$
- $225799x_{n+2} = 889x_{n+3} - 84y_{n+1}$
- $x_{n+1} = x_{n+3} - 24y_{n+2}$
- $889x_{n+2} = 7x_{n+3} - 84y_{n+2}$
- $y_{n+3} = 254y_{n+2} - y_{n+1}$
- $1344x_{n+3} = 32257y_{n+2} - 127y_{n+1}$
- $254y_{n+2} = y_{n+1} + y_{n+3}$
- $341376x_{n+3} = 32257y_{n+3} - 4y_{n+1}$
- $1344x_{n+2} = 127y_{n+2} - y_{n+1}$
- $341376x_{n+2} = 127(y_{n+3} - y_{n+1})$
- $\frac{1}{2}(255x_{3n+3} - x_{3n+4}) + 3\begin{bmatrix}(127+12\sqrt{112})^{n+1} + \\ (127-12\sqrt{112})^{n+1}\end{bmatrix}$ is a cubical integer.

4. Each of the following expressions represents a nasty number:

- $\frac{12}{7}[224x_{2n+2} - 21y_{2n+2} + 7]$
- $\frac{2}{7}[y_{2n+3} - 253y_{2n+2} + 42]$
- $\frac{12}{889}[56672x_{2n+2} - 21y_{2n+3} + 889]$
- $3[255x_{2n+2} - x_{2n+3} + 4]$
- $\frac{1}{7}[1360149x_{2n+3} - 5355x_{2n+4} + 84]$
- $\frac{12}{225799}\begin{bmatrix}14394464x_{2n+2} \\ -21y_{2n+4} + 225799\end{bmatrix}$
- $\frac{12}{889}\begin{bmatrix}14394464x_{2n+3} \\ -5355y_{2n+4} + 889\end{bmatrix}$
- $\frac{12}{7}[56672x_{2n+3} - 5355y_{2n+3} + 7]$
- $\frac{12}{889}[224x_{2n+3} - 5355y_{2n+2} + 889]$
- $\frac{3}{254}[64769x_{2n+2} - x_{2n+4} + 1016]$
- $\frac{12}{225799}[224x_{2n+4} - 1360149y_{2n+2} + 225799]$
- $\frac{2}{7}[253y_{2n+4} - 64261y_{2n+3} + 42]$
- $\frac{3}{52}[14394464x_{2n+4} - 1360149y_{2n+4} + 208]$

$\frac{12}{889}[141680x_{2n+4} - 1360149y_{2n+3} + 889]$

III. REMARKABLE OBSERVATIONS

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table below.

Table 2: Hyperbolas

S.No:	Hyperbola	(X_n, Y_n)
1.	$448Y_n^2 - 441X_n^2 = 790272$	$(255y_{n+1} - y_{n+2}, 253y_{n+1} - y_{n+2})$
2.	$4Y_n^2 - 448X_n^2 = 196$	$(2y_{n+1} - 21x_{n+1}, 21y_{n+1} - 224x_{n+1})$ $(506y_{n+2} - 5355x_{n+2}, 5355y_{n+2} - 50672x_{n+2})$
3.	$4Y_n^2 - 448X_n^2 = 3161284$	$(506y_{n+1} - 21x_{n+2}, 5355y_{n+1} - 224x_{n+2})$ $(2y_{n+2} - 5355x_{n+1}, 21y_{n+2} - 50672x_{n+1})$ $(1360149x_{n+2} - 506y_{n+3}, 14394464x_{n+2} - 5355y_{n+3})$ $(5355x_{n+3} - 128522y_{n+2}, 141680x_{n+3} - 1360149y_{n+2})$
4.	$441Y_n^2 - 448X_n^2 = 7056$	$(253x_{n+1} - x_{n+2}, 255x_{n+1} - x_{n+2})$
5.	$Y_n^2 - 448X_n^2 = 7056$	$(64261x_{n+2} - 253x_{n+3}, 1360149x_{n+2} - 5355x_{n+3})$
6.	$4Y_n^2 - 448X_n^2 = 203940753600$	$(1360149x_{n+1} - 2y_{n+3}, 14394464x_{n+1} - 21y_{n+3})$ $(21x_{n+3} - 128522y_{n+1}, 224x_{n+3} - 1360149y_{n+1})$
7.	$Y_n^2 - 112X_n^2 = 43264$	$(1360149x_{n+3} - 128522y_{n+3}, 14394464x_{n+3} - 1360149y_{n+3})$
8.	$448Y_n^2 - X_n^2 = 790272$	$(1360149y_{n+2} - 5355y_{n+3}, 64261y_{n+2} - 253y_{n+3})$
9.	$7112889Y_n^2 - 7225792X_n^2$ $= 7342322348000$	$(64261x_{n+1} - x_{n+3}, 64769x_{n+1} - x_{n+3})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table below

Table 3: Parabolas

S.No	Parabola	(X_n, Y_n)
1.	$21X_n^2 = -448Y_n - 37632$	$(255y_{n+1} - y_{n+2}, 253y_{2n+2} - y_{2n+3} - 42)$
2.	$448X_n^2 = -14Y_n - 196$	$(2y_{n+1} - 21x_{n+1}, 21y_{2n+2} - 224x_{2n+2} - 7)$ $(506y_{n+2} - 5355x_{n+2}, 5355y_{2n+3} - 56672x_{2n+3} - 7)$
3.	$448X_n^2 = -1778Y_n - 3161284$	$(506y_{n+1} - 21x_{n+2}, 5355y_{2n+2} - 224x_{2n+3} - 889)$ $(2y_{n+2} - 5355x_{n+1}, 21y_{2n+3} - 56672x_{2n+2} - 889)$ $(1360149x_{n+2} - 506y_{n+3}, 14394464x_{2n+3} - 5355y_{2n+4} + 889)$ $(5355x_{n+3} - 128522y_{n+2}, 141680x_{2n+4} - 1360149y_{2n+3} + 889)$
4.	$224X_n^2 = 441Y_n - 3528$	$(253x_{n+1} - x_{n+2}, 255x_{2n+2} - x_{2n+3} + 4)$
5.	$224X_n^2 = 21Y_n - 3528$	$(64261x_{n+2} - 253x_{n+3}, 1360149x_{2n+3} - 5355x_{2n+4} + 84)$
6.	$448X_n^2 = 451598Y_n - 203940753600$	$(1360149x_{n+1} - 2y_{n+3}, 14394464x_{2n+2} - 21y_{2n+4} + 225799)$ $(21x_{n+3} - 128522y_{n+1}, 224x_{2n+4} - 1360149y_{2n+2} + 225799)$
7.	$112X_n^2 = 104Y_n - 43264$	$(1360149x_{n+3} - 128522y_{n+3}, 14394464x_{2n+4} - 1360149y_{2n+4} + 208)$
8.	$X_n^2 = -9408Y_n - 790272$	$(1360149y_{n+2} - 5355y_{n+3}, 64261y_{2n+3} - 253y_{2n+4} + 42)$
9.	$112X_n^2 = 56007Y_n - 113806224$	$(64261x_{n+1} - x_{n+3}, 64769x_{2n+2} - x_{2n+4} + 1016)$

IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 112x^2 - 7$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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