Research of an Adaptive Cubature Kalman Filter for GPS/SINS Tightly Integrated Navigation System

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Abstract—In view of the characteristics of Ballistic missile, the GPS/SINS tightly integrated navigation algorithm based on pseudo range/pseudo range rate in the Launch inertial coordinates is studied in this paper. The coordinate transformation method from ECEF coordinates to the Launch inertial coordinates is derived. The strap-down inertial navigation error equation and GPS error equation were derived in this coordinates. The state equations and measurement equations for GPS/SINS tightly integrated navigation system are established. Then an adaptive cubature kalman filter (ACKF) is employed to improve the position performance. The numerical simulation shows the better accuracy performance of the integrated navigation system and control the quality of the vector tracking loop.

Index Terms—Adaptive cubature kalman filter, High dynamics, Launch inertial coordinates, Tightly integrated navigation.

I. INTRODUCTION

Global positioning system (GPS) and inertial navigation system (INS) has an important role in the field of navigation. GPS can provide the global position, velocity and time information all-time and all-weather. And its precision does not change with time. INS is a completely autonomous navigation system. It has less susceptible to external electromagnetic interference, high data rate, posture information, etc. However, in practical applications, the respective shortcoming of GPS and INS also limits their application. Because of GPS and INS have complementary, combination of the two can achieve performance transcendence. The GPS/SINS tightly integrated navigation algorithm generally based on geographic coordinates [1-5]. In view of the characteristics of Ballistic missile, the GPS/SINS tightly integrated navigation algorithm based on pseudo range/pseudo range rate in the Launch inertial coordinates is studied in this paper. An adaptive cubature kalman filter is employed as the integration filter [6-8]. And the numerical simulation indicates that this GPS/SINS tightly integrated navigation algorithm can improve navigation accuracy effectively.

II. ERROR MODEL IN THE LAUNCH INERTIAL COORDINATES

A. State error model of the SINS

The error angle between SINS mathematical simulation platform and the navigation coordinate is attitude misalignment angle. It is

$$\varphi = [\varphi_x, \varphi_y, \varphi_z]^T$$

(1)

The symmetric matrix of the attitude misalignment angle is

$$\begin{bmatrix}
0 & -\varphi_x & \varphi_y \\
\varphi_x & 0 & -\varphi_z \\
-\varphi_y & \varphi_z & 0
\end{bmatrix}$$

(2)

The update equations for the attitude transformation matrix is

$$\dot{C}_b^n = C^n_b (\omega_{nb}^b \times)$$

(3)

Where \(\omega_{nb}^b\) is inertial navigation system sensitive to the angular velocity.

Because of the inertial device measurement and calculation process exist error. The following relationship exists in the inertial navigation system between the actual output attitude transformation matrix \(\dot{C}_b^n\) and the ideal attitude transformation matrix \(C^n_b\).

$$\dot{C}_b^n = C^n_b + \Delta C^n_b = (I + \varphi \times)C^n_b$$

(4)

Therefore,

$$\Delta C^n_b = (\varphi \times)C^n_b$$

(5)

Derivative (5) to obtain:

$$\dot{\Delta C^n_b} = (\varphi \times)\dot{C^n_b} + (\varphi \times)\dot{C^n_b} = (\varphi \times)\dot{C^n_b} + (\varphi \times)C^n_b (\omega_{nb}^b \times)$$

(6)

Differentiating both sides of (3) results in:

$$\ddot{\Delta C^n_b} = \ddot{C^n_b} (\omega_{nb}^b \times) + C^n_b (\ddot{\omega}_{nb}^b \times)$$

(7)

Comparing (6) and (7) can be obtained that:

$$\dot{\varphi} = C^n_b (\omega_{nb}^b \times)$$

(8)

Combined with vector operation \((ab) \times = a(b \times) a^T\) results in:

$$\dot{\varphi} = C^n_b \delta \omega_{nb}^b$$

(9)

Where \(\delta \omega_{nb}^b = [\varepsilon_x, \varepsilon_y, \varepsilon_z]^T\) are gyro drift error.

In the launch inertial coordinate, speed differential equation is:

$$\dot{V} = C^n_b f^b + g_e$$

(10)

Differentiating both sides of (10) results in:
\[ \dot{\mathbf{V}} = (\delta C_b^n) f^b + C_b^n \delta f^b + \delta g_e \]  

(11)

The position error equation

\[
\begin{bmatrix}
\delta \dot{x} \\
\delta \dot{y} \\
\delta \dot{z}
\end{bmatrix}
= 
\begin{bmatrix}
\delta V_x \\
\delta V_y \\
\delta V_z
\end{bmatrix}
\]  

(12)

Inertial device error equation

\[ \dot{e} = 0, \mathbf{\dot{V}} = 0 \]  

(13)

The system state equation of SINS in the launch inertial coordinate system can be obtained from 1) to 4) results in

\[ \dot{X}_t = F_t X_t + G_t w_t \]  

(14)

In the equation above

\[ X_t(t) = [\varphi_x, \varphi_y, \varphi_z, \partial V_x, \partial V_y, \partial V_z] \]  

(15)

\[
F_t = 
\begin{bmatrix}
0_{3 \times 3} & 0_{3 \times 3} & C_b^n & 0_{3 \times 3} & \vdots \\
0_{3 \times 3} & F_1 & F_2 & 0_{3 \times 3} & C_b^n \\
0_{3 \times 3} & I & 0_{3 \times 9}
\end{bmatrix}
\]

\[
G_t = 
\begin{bmatrix}
C_b^n & 0_{3 \times 3} & \vdots \\
0_{3 \times 3} & C_b^n & \vdots \\
0_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix}_{15 \times 6}
\]

\[ w_t = [\omega_{gx}, \omega_{gy}, \omega_{gz}, \omega_{ax}, \omega_{ay}, \omega_{az}] \]  

In the equation above, \( w \) is system noise variance, \( \omega_{gx}, \omega_{gy}, \omega_{gz} \) are gyro drift noise variances of three axis; \( \omega_{ax}, \omega_{ay}, \omega_{az} \) are accelerometers noise variances of three axis; \( I \) is identity matrix;

\[ F_1 = 
\begin{bmatrix}
0 & -f_x & \vdots \\
-f_z & 0 & \vdots \\
-f_y & f_x & 0
\end{bmatrix}
\]

\[ F_2 = 
\begin{bmatrix}
\frac{\partial g_x}{\partial x} & \frac{\partial g_x}{\partial y} & \frac{\partial g_x}{\partial z} \\
\frac{\partial g_y}{\partial x} & \frac{\partial g_y}{\partial y} & \frac{\partial g_y}{\partial z} \\
\frac{\partial g_z}{\partial x} & \frac{\partial g_z}{\partial y} & \frac{\partial g_z}{\partial z}
\end{bmatrix}
\]

B. GPS error equations

When usually we takes two time-related errors to represent the GPS error condition: the distance error \( \Delta l_u \), which is equivalent to the clock error \( \Delta l_{ru} \); the distance error, which is equivalent to clock frequency error \( \Delta l_{ru} \). Error model expression as follows:

\[
\begin{bmatrix}
\delta l_u = \delta l_{ru} + \omega_u \\
\delta l_{ru} = -\delta l_{ru} + \omega_{ru}
\end{bmatrix}
\]  

(16)

Where \( T_{ru} \) is the relevant time, \( \omega_u \) is white noise of GPS clock error, \( \omega_{ru} \) is GPS clock frequency white noise.

GPS error equation can be expressed as follows:

\[ \dot{X}_G(t) = F_G(t) X_G(t) + G_G(t) W_G(t) \]  

(17)

In the equation above,

\[ X_G(t) = [\delta l_{ru}, \delta l_{ru}]^T, \quad W_G(t) = [\omega_u, \omega_{ru}]^T \]

\[ F_G(t) = 
\begin{bmatrix}
0 & 1 \\
0 & -\frac{1}{T_{ru}}
\end{bmatrix}, \quad G_G(t) = 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Therefore, tightly integrated navigation system error equation can be get from integrating the SINS and GPS error equations:

\[
\begin{bmatrix}
\dot{X}_I(t) \\
\dot{X}_G(t)
\end{bmatrix} = 
\begin{bmatrix}
F_I(t) & 0 \\
0 & F_G(t)
\end{bmatrix} 
\begin{bmatrix}
X_I(t) \\
X_G(t)
\end{bmatrix}
\]

(18)

C. Measurement equation

The pseudo-range from GPS receiver to the GPS satellite numbered \( j \) can be expressed as follows:

\[ \rho_j^G = \rho^j + \delta l_{ru} + \nu_j \]  

(19)

In the equation above, \( \rho^j \) is the ideal pseudo range, \( \delta l_{ru} \) is the equivalent distance corresponding to clock error. \( \delta l_{ru} \) is main error in pseudo range measurement. Therefore, when building the pseudo range measurement model, the impact of the error is the primary consideration; \( \nu_j \) is pseudo range measurement noise, caused by multipath effect, tropospheric delay, ionospheric error. And it can be regarded as white noise.

Satellite position get from the ephemeris is in ECEF coordinate system, assuming that the position of the Satellite numbered \( j \) in the ECEF coordinate system is \( (x_j^e, y_j^e, z_j^e) \). To compare with the INS information, we need to convert it to the launch inertial coordinate system:

1) Convert Satellite position in ECEF (\( e \) coordinate) coordinate to launch coordinate (\( l \) coordinate), the conversion formula is as follows:

\[ 
\begin{bmatrix}
X_l \\
Y_l \\
Z_l
\end{bmatrix} = 
\begin{bmatrix}
C_{el} & C_{el} & r_{0x} \\
C_{el} & C_{el} & r_{0y} \\
C_{el} & C_{el} & r_{0z}
\end{bmatrix} 
\begin{bmatrix}
X_e \\
Y_e \\
Z_e
\end{bmatrix}
\]  

(20)

In the equation above, \( r_0 \) is the geocentric vector emission points, \( C_{el} \) is the transpose matrix of \( C_{el} \), \( C_{el} \) is as follows:
\[
C_l^T = \begin{bmatrix}
-sin\lambda_0 \cos\lambda_0 - \cos\lambda_0 \cos\lambda_0 \\
sin\lambda_0 \cos\lambda_0 - \cos\lambda_0 \sin\lambda_0 \\
\cos\lambda_0 \cos\lambda_0 - \sin\lambda_0 \sin\lambda_0 \\
-cos\lambda_0 \sin\lambda_0 + sin\lambda_0 \sin\lambda_0 \\
\cos\lambda_0 \cos\lambda_0 + sin\lambda_0 \sin\lambda_0 \\
-\sin\lambda_0 \sin\lambda_0 - \cos\lambda_0 \sin\lambda_0 
\end{bmatrix} 
\]

\[
(21)
\]

(2) Convert the position of the Satellite in Launch coordinate system (l coordinate) to launch inertial coordinate (g coordinate), the conversion formula is as follows:

\[
\begin{bmatrix}
X_g + r_{0x} \\
Y_g + r_{0y} \\
Z_g + r_{0z}
\end{bmatrix} = C_l^g \cdot \begin{bmatrix}
X_l + r_{0x} \\
Y_l + r_{0y} \\
Z_l + r_{0z}
\end{bmatrix} \quad (22)
\]

In the equation above, \(C_l^g\) is the transpose matrix of \(C_l\).

\[
C_l^g
\]

is as follows:

\[
\begin{bmatrix}
b_b^2(1-\cos\omega_l t) + b_s \sin\omega_l t \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
b_b^2(1-\cos\omega_l t) - b_s \sin\omega_l t \\
-1 - b_b^2(1-\cos\omega_l t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
b_b^2(1-\cos\omega_l t) - b_s \sin\omega_l t \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
b_b^2(1-\cos\omega_l t) + b_s \sin\omega_l t \\
1 - b_b^2(1-\cos\omega_l t)
\end{bmatrix}
\]

In the equation above, \(b_x = \cos B_0 \cos A_0\), \(b_y = \sin B_0\), \(b_z = -\cos B_0 \sin A_0\); \(B_0\) is the latitude of the emission point, \(\lambda_0\) is the longitude of the emission point, \(A_0\) is the azimuth, \(\omega_0\) is the angular velocity of Earth rotation.

Finally, the position of the Satellite in launch inertial coordinate is \( (x_g^l, y_g^l, z_g^l) \).

So the pseudo-range from the carrier to the GPS satellite numbered \(j\) can be get from INS:

\[
\rho_{ls}^j = \sqrt{(x_g^l - x_s^l)^2 + (y_g^l - y_s^l)^2 + (z_g^l - z_s^l)^2} \quad (24)
\]

Assumed that the real value of the carrier position is \((x, y, z)\), expand the equation above at \((x, y, z)\) in taylor formula and discard second and higher order terms:

\[
\rho_{ls}^j = \rho_l^j + \frac{\partial\rho_l^j}{\partial x} \delta x + \frac{\partial\rho_l^j}{\partial y} \delta y + \frac{\partial\rho_l^j}{\partial z} \delta z + \delta l_n - \nu_{\rho j} \quad (25)
\]

In the equation above,

\[
\frac{\partial\rho_l^j}{\partial x} = \frac{x_s^l - x_s^j}{\rho_{ls}^j} = e_{\mu} , \\
\frac{\partial\rho_l^j}{\partial y} = \frac{y_s^l - y_s^j}{\rho_{ls}^j} = e_{\nu} , \\
\frac{\partial\rho_l^j}{\partial z} = \frac{z_s^l - z_s^j}{\rho_{ls}^j} = e_{\zeta} .
\]

The pseudo range measurement equations of tightly integrated navigation system can be expressed as follows:

\[
\delta\rho_l^j = e_{\mu} \delta x + e_{\nu} \delta y + e_{\zeta} \delta z - \delta l_n - \nu_{\rho j} \quad (26)
\]

Therefore,

\[
(27)
\]

Suppose that the number of effective GPS satellites is \(m\), so:

\[
Z_{\rho}^j(t) = H_{\rho}^j(t) X(t) + V_{\rho}^j(t)
\]

The measurement equation of tightly integrated navigation system can be get as follows:

\[
(28)
\]

\[
(29)
\]

Suppose that the velocity of Satellite numbered \(j\) in ECEF coordinates is \( (v_{gx}, v_{gy}, v_{gz}) \), convert it to launch coordinate, the velocity of Satellite in launch coordinate can be obtained as follows: \( (v_{gx}, v_{gy}, v_{gz}) \).

According to carrier speed obtained by INS in launch coordinate \( (v_{gx}, v_{gy}, v_{gz}) \) the pseudo-range rate can be get as follows:

\[
\rho_{ls}^j = \rho_l^j + \frac{\partial\rho_l^j}{\partial x} \delta v_x + \frac{\partial\rho_l^j}{\partial y} \delta v_y + \frac{\partial\rho_l^j}{\partial z} \delta v_z + \delta l_n - \nu_{\rho j} \quad (30)
\]

So the pseudo range measurement equation is

\[
\delta\rho_l^j = e_{\mu} \delta v_x + e_{\nu} \delta v_y + e_{\zeta} \delta v_z - \delta l_n - \nu_{\rho j} \quad (31)
\]

Therefore,

\[
Z_{\rho}^j(t) = H_{\rho}^j(t) X(t) + V_{\rho}^j(t)
\]

Similarly, suppose that the number of effective GPS satellites is \(m\), so:

\[
\delta\rho_l^j = e_{\mu} \delta v_x + e_{\nu} \delta v_y + e_{\zeta} \delta v_z - \delta l_n - \nu_{\rho j} \quad (32)
\]

The measurement equation of tightly integrated navigation system can be get as follows:

\[
Z_{\rho}^j(t) = H_{\rho}^j(t) X(t) + V_{\rho}^j(t)
\]
\[ Z(t) = \begin{bmatrix} H_{\rho}(t) \\ H_{\phi}(t) \end{bmatrix} X(t) + \begin{bmatrix} V_{\rho}(t) \\ V_{\phi}(t) \end{bmatrix} = H(t) X(t) + V(t) \] (33)

III. The Adaptive Cubature Kalman Filter

Considering a discrete nonlinear system as
\[ x_{k+1} = f(x_k, k) + w_k \]
\[ z_k = h(x_k, k) + v_k \]
Where the state vector \( x_k \in \mathbb{R}^n \), the process noise vector is \( w_k \in \mathbb{R}^n \), the measurement noise vector is \( v_k \in \mathbb{R}^m \), the measurement vector is \( z_k \in \mathbb{R}^m \). The vectors \( w_k \) and \( v_k \) are zero mean Gaussian white sequences with zero cross-correlation with each other:
\[ E[w_k w_i^T] = \begin{cases} Q_{ik}, & i = k \\ 0, & i \neq k \end{cases} \]
\[ E[v_k v_i^T] = \begin{cases} R_{ik}, & i = k \\ 0, & i \neq k \end{cases} ; E[w_k v_i^T] \]
Where \( Q_{ik} \) is the process noise covariance matrix, and \( R_{ik} \) is the measurement noise covariance matrix.

The details of the implementation of ACKF are as follows:
Step One: Initialize state vector \( \hat{x}_{k|0} \) and state covariance matrix \( P_{k|0} \).
Step Two: Time update
Factorize the covariance:
\[ P_{k|k|k-1} = S_{k|k|k-1} S_{k|k|k-1}^T \] (34)
Evaluate the cubature points through the process model
\[ X_{k|k|k-1} = S_{k|k|k-1} \tilde{\xi}_k + \tilde{X}_{k|k|k-1} \] (35)
Estimate the propagated cubature points through the process model
\[ X_{k|k|k-1}^* = f(X_{k|k|k-1}) \] (36)
Estimate the predicted mean
\[ \hat{\tilde{X}}_{k|k|k-1} = \sum_{i=1}^{2n} a_i X_{k|k|k-1} \] (37)
Estimate the predicted error covariance
\[ P_{k|k|k-1} = \sum_{i=1}^{2n} a_i X_{k|k|k-1}^* X_{k|k|k-1}^T - \hat{\tilde{X}}_{k|k|k-1} \hat{\tilde{X}}_{k|k|k-1}^T + Q_{i|k-1} \] (38)
Step Three: Measurement update
Factorize the covariance
\[ P_{k|k} = S_{k|k} S_{k|k}^T \] (39)
Evaluate the cubature points
\[ X_{k|k|k-1} = S_{k|k|k-1} \tilde{\xi}_k + \tilde{X}_{k|k|k-1} \] (40)
Estimate the propagated cubature points through observation model
\[ Z_{k|k|k-1} = h(X_{k|k|k-1}) \] (41)
Evaluate the propagated observation
\[ \hat{\tilde{X}}_{k|k|k} = \sum_{i=1}^{2n} a_i Z_{k|k|k} \] (42)
(5) Evaluate the innovation covariance
\[ P_{z_z} = \frac{1}{a_k} \sum_{i=1}^{2n} a_i Z_{k|k|k} Z_{k|k|k}^T \] (43)
(6) Estimate the cross-covariance
\[ P_{z_x} = \frac{1}{a_k} \sum_{i=1}^{2n} a_i X_{k|k|k} X_{k|k|k}^T - \hat{\tilde{X}}_{k|k|k} \hat{\tilde{X}}_{k|k|k}^T + R_k \] (44)
\[ K_k = P_{z_z}^{-1} P_{z_x} \] (45)
\[ x_{k|k} = \hat{\tilde{X}}_{k|k|k} + K_k (z_k - \hat{\tilde{X}}_{k|k|k}) \] (46)
\[ P_{k|k} = \frac{1}{a_k} P_{k|k|k} - K_k R_k K_k^T \] (47)
Where:
\[ a_k = \frac{\text{tr}(P)}{\text{tr}(e(k) e(k)^T)} \] (48)
\[ P = \sum_{i=1}^{2n} a_i Z_{k|k|k} Z_{k|k|k}^T - \hat{\tilde{X}}_{k|k|k} \hat{\tilde{X}}_{k|k|k}^T \] (49)

IV. The Simulation And Results

In order to verify the correctness of the algorithm, ballistic trajectory generator platform and numerical simulation are designed in this paper. The missile is fired vertically, the total flight time is 200s, the time of powered phase is 60s, the thrust acceleration in powered phase is 40 m/s^2, the constant drift of gyro is 0.5°/h, the white noise of gyro is 0.05°/h, the constant drift of accelerometer is 1mg, the white noise of accelerometer is 0.5mg. The start point: latitude: 31.98°N, longitude:118.8°E, height: 0m;
Initial speed: forward speed: 394.8917m/s, upward speed 0m/s, lateral speed 0m/s. White noise of pseudo-range is 10m, the white noise of pseudo-range rate is 0.1m/s.
Initial attitude: pitch angle: 90°, roll angle: 0°, yaw angle: 0°. The initial attitude error is 0.1°. The numerical simulation results are as follows:
Figure 2. The velocity error

Figure 3. The attitude error

According to figure 1 to figure 3, when received eight stars, position error can converge to within 2m, speed error can converge to 0.1m/s or less, attitude error can converge to within 0.05 °.

V. CONCLUSIONS

According to the characteristics of ballistic missiles, a kind of GPS/SINS tightly integrated navigation algorithm in Launch inertial coordinate is designed. And the state equations and measurement equations for GPS/SINS tightly integrated navigation system are established. An adaptive cubature kalman filter was applied to the system. The numerical simulation indicates that this GPS/SINS tightly integrated navigation algorithm can improve navigation accuracy effectively.

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