Zeros of Polynomials

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***Abstract*— In this paper we find bounds for the number of zeros of a polynomial with certain conditions on its coefficients .The results thus obtained generalize many results known already.**

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***Index Terms*— Bound, Coefficient, Polynomial, Zeros.**

I. Introduction

Cauchy found a bound for all the zeros of a polynomial and proved the following result known as Cauchy's Theorem [1,3] :

Theorem A. All the zeros of the polynomial

$P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z|<1+M$,

where $M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

The bound given by the above theorem depends on all the coefficients of the polynomial. A lot of such results is available in the literature [1-4] .In this connection Shah and Liman [4] proved the following results:

Theorem B. If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a complex polynomial satisfying

$$\sum_{j=1}^{n}|a_j|<|a_0|,$$

Then P(z) does not vanish in $|z|<1$.

Theorem C. If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a complex polynomial satisfying

$$\sum_{j=0}^{n-1}|a_j|<|a_n|,$$

then P(z) has all its zeros in $|z|<1$.

Mezerji and Bidkham [2] generalized Theorems B and C by proving

Theorem D. Let $P(z)=a_0+\sum_{i=\mu}^{n} a_j z^j$ be a complex polynomial of degree n. If for some $R\ge 1$,

$$R^{n-\mu}\sum_{i=0,i\neq j\notin A}^{n}|a_i|<|a_k|,$$

where $A=\{1,2,......,\mu-1\}$, then P(z) has exactly μ zeros in $|z|<R$.

II. Main Results

In this paper we prove the following result:

Theorem 1. Let

$P(z)=a_0+a_1z+a_2z^2+......+a_pz^p+a_nz^n, 1\le p\le n-1$

be a complex polynomial of degree n. If for some $R\ge 1$,

$$R^{n-p}\sum_{j=0,i\neq p}^{n}|a_i|<|a_p|,$$

then P(z) has exactly p zeros in $|z|<R$.

Remark 1. For R=1 and p=n, Theorem 1 reduces to Theorem C.

For p=1, R=1, Theorem 1 reduces to the following result:

Corollary 1. Let $P(z)=a_0+a_1z+a_nz^n$ such that $|a_0|+|a_n|<|a_1|$. Then P(z) has exactly 1 zero in $|z|<1$.

For p=n-1, we get the following result from Theorem 1:

Corollary 2. Let

$P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n$ be a complex polynomial of degree n. If for some $R\ge 1$,

$$R\sum_{j=0,i\neq n-1}^{n}|a_i|<|a_{n-1}|,$$

then P(z) has exactly n-1 zeros in $|z|<R$.

For R=1, Cor.2 gives the following result:

Corollary 3. Let

$P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n$ be a complex polynomial of degree n. If

$$\sum_{j=0,i\neq n-1}^{n}|a_i|<|a_{n-1}|,$$

then P(z) has exactly n-1 zeros in $|z|<1$.

III. Proof of Theorem1

Let

$$g(z)=\frac{1}{a_p}\sum_{j=0,j\neq p}^{n}a_jz^j.$$

Then for $|z|=R$, $R\ge 1$,

$$|g(z)|\le\frac{1}{|a_p|}\sum_{j=0,j\neq p}^{n}|a_j||z|^j$$

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$$= \frac{1}{|a_p|} \sum_{j=0, j \neq p}^{n} |a_j| R^j$$

$$\leq \frac{1}{|a_p|} . R^n \sum_{j=0, j \neq p}^{n} |a_j|$$

$$\leq R^p$$

$$= |z|^p$$

$$= |z^p|$$

Hence , by Rouche's Theorem z^p and g(z)+ $z^p = \dfrac{P(z)}{a_p}$ have the same number of zeros in $|z| < R$.

Since z^p has p zeros there, it follows that P(z) has exactly p zeros in $|z| < R$. That proves the result.

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