Zeros of Polynomials

M. H. Gulzar

Abstract — In this paper we find bounds for the number of zeros of a polynomial with certain conditions on its coefficients. The results thus obtained generalize many results known already.

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I. INTRODUCTION

Cauchy found a bound for all the zeros of a polynomial and proved the following result known as Cauchy’s Theorem [1, 3]:

**Theorem A.** All the zeros of the polynomial

\[ P(z) = \sum_{j=0}^{n} a_j z^j \]

of degree n lie in the circle \( |z| < 1 + M \),

where \( M = \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right| \).

The bound given by the above theorem depends on all the coefficients of the polynomial. A lot of such results is available in the literature [1–4]. In this connection Shah and Liman [4] proved the following results:

**Theorem B.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a complex polynomial satisfying

\[ \sum_{j=1}^{n} |a_j| < |a_0|, \]

then \( P(z) \) does not vanish in \( |z| < 1 \).

**Theorem C.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \) is a complex polynomial satisfying

\[ \sum_{j=0}^{n-1} |a_j| < |a_n|, \]

then \( P(z) \) has all its zeros in \( |z| < 1 \).

Mezerji and Bidkham [2] generalized Theorems B and C by proving

**Theorem D.** Let \( P(z) = a_0 + \sum_{i=\mu}^{n} a_i z^i \) be a complex polynomial of degree n. If for some \( R \geq 1 \),

\[ R^{-\mu} \sum_{j=0, \mu \leq p \neq k}^{n} |a_j| < |a_k|, \]

where \( A = \{ 1, 2, \ldots, \mu - 1 \} \), then \( P(z) \) has exactly \( \mu \) zeros in \( |z| < R \).

II. MAIN RESULTS

In this paper we prove the following result:

**Theorem 1.** Let

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_p z^p + a_n z^n, 1 \leq p \leq n - 1 \]

be a complex polynomial of degree n. If for some \( R \geq 1 \),

\[ R^{n-p} \sum_{j=0, j \neq p}^{n} |a_j| < |a_p|, \]

then \( P(z) \) has exactly \( p \) zeros in \( |z| < R \).

**Remark 1.** For \( p=n-1 \) and \( p=n \), Theorem 1 reduces to Theorem C.

For \( p=1 \), \( R=1 \), Theorem 1 reduces to the following result:

**Corollary 1.** Let \( P(z) = a_0 + a_1 z + a_n z^n \) such that

\[ |a_0| + |a_n| < |a_1| \].

Then \( P(z) \) has exactly 1 zero in \( |z| < 1 \).

For \( p=n-1 \), we get the following result from Theorem 1:

**Corollary 2.** Let \( P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n \) be a complex polynomial of degree n. If for some \( R \geq 1 \),

\[ R \sum_{j=0, j \neq n-1}^{n} |a_j| < |a_{n-1}|, \]

then \( P(z) \) has exactly \( n-1 \) zeros in \( |z| < R \).

For \( R=1 \), Cor. 2 gives the following result:

**Corollary 3.** Let

\[ P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + a_n z^n \]

be a complex polynomial of degree n. If

\[ \sum_{j=0, j \neq n-1}^{n} |a_j| < |a_{n-1}|, \]

then \( P(z) \) has exactly \( n-1 \) zeros in \( |z| < 1 \).

III. PROOF OF THEOREM I

Let

\[ g(z) = \frac{1}{a_p} \sum_{j=0, j \neq p}^{n} a_j z^j. \]

Then for \( |z| = R \), \( R \geq 1 \),

\[ |g(z)| \leq \frac{1}{|a_p|} \sum_{j=0, j \neq p}^{n} |a_j| |z|^j \].

M. H. Gulzar, Post Graduate Department of Mathematics, University of Kashmir, Srinagar J&K, India 190006

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\[ \frac{1}{|a_p|} \sum_{j=0, j \neq p}^{n} |a_j| R^j \]
\[ \leq \frac{1}{|a_p|} R^n \sum_{j=0, j \neq p}^{n} |a_j| \]
\[ \leq R^p \]
\[ = |z|^p \]
\[ = |z|^p \]

Hence, by Rouche’s Theorem \( z^p \) and \( g(z) + \)
\[ z^p = \frac{P(z)}{a_p} \] have the same number of zeros in \( |z| < R \).

Since \( z^p \) has \( p \) zeros there, it follows that \( P(z) \) has exactly \( p \) zeros in \( |z| < R \). That proves the result.

**REFERENCES**