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Results On Strongly Continuous Semigroup

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***Abstract*—Some results of strongly continuous semigroup (C_0-semigroup) of functional analysis of bounded linear operator defining on a separable Banach spaces like hypercyclic, topologically transitive, chaotic, mixing, weekly mixing and topologically ergrodic have been discussed. The generalization of hypercyclic, topologically transitive, chaotic, mixing, weakly mixing and topologically ergrodic for the direct sum of two and /or hence to *n* C_0-semigroup strongly continuous dynamic system of semigroups in infinite dimensional separable Banach space have been developed with proofs and discusses**

***Index Terms*—semigroup, ergrodic, dynamic system.**

I. INTRODUCTION

Let X be a separable infinite dimensional Banach space. A one-parameter family $(T_t)_{t\geq 0}$ of continuous (bounded) linear operators on *X* is a strongly continuous semigroup (C_0-semigroups) if $T_0 = I$, $T_t \circ T_s = T_{t+s}$, for every $t, s \geq 0$ and $\lim_{t\to s} T_t x = T_s x$, for every $x \in X$. A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be topologically transitive if for any nonempty open subsets U and V of *X*, there exists some $t \geq 0$, such that $(T_t)(U) \cap V \neq \emptyset$ [1], and it is said to be mixing if there exists some $t_0 \geq 0$ such that the condition hold for all $t \geq t_0$[2]. On the other hand a C_0-semigroup $(T_t)_{t\geq 0}$ is said to be hypercyclic if there exists some $x \in X$ whose orb$(x, T_t) = \{T_t : t \geq 0\}$ is dense in X in this case we say that x is called hypercyclic vector for this semigroup [2]. Note that the hypercyclicity and transitivity for a C_0-semigroup are equivalent on $(T_t)_{t\geq 0}$ on a separable Banach space [3]. It's clear that if $(T_t)_{t\geq 0}$ is a hypercyclic operator for some $t > 0$, then the C_0 − semigroup $(T_t)_{t\geq 0}$ is hypercylic. On the other hand if the C_0 − semigroup $(T_t)_{t\geq 0}$ is hypercylic then T_t is hypercylic operator for all $t \geq 0$ [1]. Also a C_0-semigroup $(T_t)_{t\geq 0}$ is said to be weakly mixing if $(T_t \oplus T_t)_{t\geq 0}$ is topologically transitive [1], and it's said to be a topologically ergodic if for every non-empty open subsets U and V of *X*, the set $R(U,V) = \{t \geq 0 : (T_t)(U) \cap V \neq \emptyset\}$ is syndetic, that is $R_+ \backslash R(U,V)$ does not contains arbitrarily long intervals[2]. A point $x \in X$ is called a periodic point of $(T_t)_{t\geq 0}$ if there is some $t > 0$ such that $T_t x = x$ [3]. A C_0-semigroup $(T_t)_{t\geq 0}$ on *X* is called chaotic if it is hypercyclic and its set of periodic points is dense in *X*. Note that if X and Y are separable Banach spaces, then the space $X \oplus Y := \{(x,y) : x \in X, y \in Y\}$ is a Separable Banach space, and if $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are C_0-semigroups on *X* and *Y* respectively, then the direct sum of $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus Y$ defined by $(S_t \oplus T_t)_{t\geq 0}(x,y) = (S_t(x), T_t(y))_{t\geq 0}$ $\forall x \in X, y \in Y$[2].

II. SOME PROPERTIES THAT PRESERVED UNDER QUASICONJUAGACY:

<u>**Definition(2.1) [2]:**</u>

Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be C_0-semigroups on X and *Y*, respectively then $(T_t)_{t\geq 0}$ is called quasiconjugate to $(S_t)_{t\geq 0}$ if there exists a continuous map $\emptyset : Y \to X$ with dense range such that $T_t \circ \emptyset = \emptyset \circ S_t$, $t \geq 0$. If $\emptyset$ can be chosen to be a homeomorphism then $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are called conjugate.

$$\begin{tikzcd} Y \arrow[r, "(S_t)_{t\geq 0}"] \arrow[d, "\emptyset"'] & Y \arrow[d, "\emptyset"] \\ X \arrow[r, "(T_t)_{t\geq 0}"] & X \end{tikzcd}$$

<u>**Definition(2.2) [2]:**</u>

A property P of C_0-semigroup is said to be preserved under (quasi) conjugacy if any C_0-semigroup that is (quasi)conjugate to a C_0-semigroup with property P also possesses property P.

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Proposition (2.3):

Topologically transitive for a C_0-semigroup is preserved under quasiconjugacy.

Proof: Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are C_0-semigroups on X and Y, respectively such that $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t)_{t\geq 0}$ by $\emptyset: Y \to X$ and $(S_t)_{t\geq 0}$ is topologically transitive Let U, V be non-empty and open subsets of X, since $\emptyset$ is continuous and has dense range, $\emptyset^{-1}(U), \emptyset^{-1}(V)$ are open and non-empty in Y, and since $(S_t)_{t\geq 0}$ is topologically transitive, $S_t(\emptyset^{-1}(U)) \cap \emptyset^{-1}(V) \neq \emptyset$ for some $t \geq 0$,thus $\exists\, y \in \emptyset^{-1}(U)$ such that $S_t(y) \in \emptyset^{-1}(V)$, so $\emptyset(y) \in U$ and $\emptyset\,(S_t(y)) \in \emptyset(\emptyset^{-1}(V)) \in V$, then $T_t(\emptyset(y)) = \emptyset\big(S_t(y)\big) \in V$ for some $t \geq 0$ which mean that $T_t(U) \cap V \neq \emptyset$

Proposition (2.4):

The property of having dense set of periodic points of C_0-semigroup is preserved under quasiconjugacy .

Proof: Let $(T_t)_{t\geq 0}$ be a C_0 -semigroup on *X* that is quasiconjugate to the C_0 -semigroups $(S_t)_{t\geq 0}$ on Y by $\emptyset: Y \to X$, and $(S_t)_{t\geq 0}$ has dense set of periodic points. Let U be a non-empty open subset of X since $\emptyset$ is continuous and has dense range then $\emptyset^{-1}(U)$ is open and non-empty subset of *Y*, thus $\exists\, y \in Y$ is periodic point for $(S_t)_{t\geq 0}$ that is $\exists t > 0$ such that $S_t(y) = y$ such that $y \in \emptyset^{-1}(U)$, $\emptyset(y) \in U$ Now, $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t)_{t\geq 0}$, then

$$(T_t \circ \emptyset)(y) = (\emptyset \circ S_t)(y) = \emptyset\big(S_t(y)\big) = \emptyset(y)$$

,therefore

$(T_t \circ \emptyset)(y) = (T_t(\emptyset(y)) = \emptyset(y)$ for some $t > 0$, then $(T_t)_{t\geq 0}$ has dense set of periodic point.

The following theorem was state in [2] without prove, we now proved it:

Theorem (2.5):

The following properties of a C_0-semigroup are preserved under quasigonjugat:

1) Hypercyclicity
2) Mixing
3) Weakly mixing
4) Chaotic

Proof: **1)** Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are C_0-semigroups on X and Y, respectively such that $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t)_{t\geq 0}$ by $\emptyset: Y \to X$ and $(S_t)_{t\geq 0}$ is hypercyclic C_0-semigroup. Let $y \in Y$ have dense orbit under $(S_t)_{t\geq 0}$ (since $(S_t)_{t\geq 0}$ is hypercyclic).

If U is a non-empty and open subset of *X* then $\emptyset^{-1}(U)$ is non-empty and open in *Y* since $\emptyset$ is continuous and has dense range

$\exists t \geq 0 \ni$ $S_t(y) \in \emptyset^{-1}(U)$, $\emptyset(S_t(y)) \in U$ and since $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t)_{t\geq 0}$, thus $(T_t \circ \emptyset)(y) = \big(\emptyset \circ S_t(y)\big) = \emptyset(S_t(y)) \in U$

, then $\emptyset(y)$ has dense orbit, therefore $(T_t)_{t\geq 0}$ is hypercyclic C_0-semigroup.

2) Let $(T_t)_{t\geq 0}$ be a C_0 -semigroup on *X* that is quasiconjugate to the C_0-semigroups $(S_t)_{t\geq 0}$ on Y by $\emptyset: Y \to X$. Let U, V are non-empty and open subsets of X, since $\emptyset$ is continuous and has dense range then $\emptyset^{-1}(U), \emptyset^{-1}(V)$ are non-empty and open subsets of Y, and since $(S_t)_{t\geq 0}$ is mixing then there exists $t_0 \geq 0$ such that $S_t\big(\emptyset^{-1}(U)\big) \cap \emptyset^{-1}(V) \neq \emptyset$ $\forall\, t \geq t_0$, then

$\emptyset(S_t\big(\emptyset^{-1}(U)\big)) \cap \emptyset(\emptyset^{-1}(V)) \neq \emptyset$, thus $\emptyset(S_t\big(\emptyset^{-1}(U)\big)) \cap V \neq \emptyset$ therefore $\emptyset \circ S_t\big(\emptyset^{-1}(U)\big)) \cap V \neq \emptyset$ and since $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t)_{t\geq 0}$ so $T_t \circ \emptyset\big(\emptyset^{-1}(U)\big)) \cap V \neq \emptyset,$ thus $T_t(U) \cap V \neq \emptyset \quad \forall\, t \geq t_0$ therefore $(T_t)_{t\geq 0}$ is mixing

3) If $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are C_0-semigroup on *X* and Y, respectively such that $(T_t)_{t\geq 0}$ quasiconjugate to $(S_t)_{t\geq 0}$ by $\emptyset: Y \to X$, and $(S_t)_{t\geq 0}$ is weakly mixing, then $(T_t \oplus T_t)_{t\geq 0}$ be a C_0 -semigroup on *X* that is quasiconjugate to the C_0-semigroups $(S_t \oplus S_t)_{t\geq 0}$ on *Y* by $(\emptyset \oplus \emptyset): Y \oplus Y \to X \oplus X$, since $(S_t)_{t\geq 0}$ is weakly mixing then $(S_t \oplus S_t)_{t\geq 0}$ is topologically transitive and topologically transitive is preserved under quasiconjugate then $(T_t \oplus T_t)_{t\geq 0}$ is topologically transitive that is mean $(T_t)_{t\geq 0}$ is weakly mixing and $(T_t)_{t\geq 0}$ is topologically transitive.

4) From Propositions (2.3) and (2.4).

Proposition(2.6):

Topologically ergodic for a C_0 -semigroup is preserved under quasiconjugacy.

Proof: Let $(T_t)_{t\geq 0}$ be a C_0 -semigroup on X that is quasiconjugate to the C_0 -semigroups $(S_t)_{t\geq 0}$ on Y by $\emptyset: Y \to X$, where $(S_t)_{t\geq 0}$ is topologically ergodic. Let U, V be non-empty open subsets of X, since $\emptyset$ is continuous and has dense range, thus $\emptyset^{-1}(U), \emptyset^{-1}(V)$ are non-empty open subsets of Y since $(S_t)_{t\geq 0}$ is topologically ergodic and

$R(\emptyset^{-1}(U), \emptyset^{-1}(V)) = \{t \geq 0: S_t(\emptyset^{-1}(U)) \cap \emptyset^{-1}(V) \neq \emptyset\}$ is synditic that is $R_+ \backslash R(\emptyset^{-1}(U), \emptyset^{-1}(V))$ does not contains long interval

Now let $y \in \emptyset^{-1}(U)$, such that $S_t(y) \in \emptyset^{-1}(V)$ for some $t \geq 0$,

Since $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t)_{t\geq 0}$, then $(T_t \circ \emptyset)(y) = (\emptyset \circ S_t)(y) = \emptyset(S_t(y)) \in V$ for some $t \geq 0$, thus $T_t(U) \cap V \neq \emptyset$ for some $t \geq 0$, $R(U, V) = \{t \geq 0: S_t(U) \cap V \neq \emptyset\}$ is synditic that is $R_+ \backslash R(U, V)$ does not contains long interval, therefore $(T_t)_{t\geq 0}$ is topologically ergodic.

III. The Direct Sum of C_0-Semigroups and the Main Theorem

Before we state the main theorem we consider the following:

Let $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ be C_0-semigroups on X and Y, respectively

to prove that $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to the C_0 -semigroup $(S_t \oplus T_t)_{t\geq 0}$, define $\emptyset: (X \oplus Y) \to X$ by $\emptyset(X \oplus Y) = X$, then $\emptyset$ is continuous and has dense range and

$$(S_t \circ \emptyset)(X \oplus Y) = S_t\big(\emptyset(X \oplus Y)\big) = S_t(X)$$

,

$$\emptyset \circ (S_t \oplus T_t)(X \oplus Y) = \emptyset(S_t \oplus T_t)(X \oplus Y) = \emptyset(S_t(X) \oplus T_t(Y)) = S_t(X)$$

so $(S_t) \circ \emptyset = \emptyset o (S_t \oplus T_t)$ therefore $(S_t)_{t\geq 0}$ is quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$

By the same way we prove that $(T_t)_{t\geq 0}$ is quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$

Main Theorem (3.1):

Let $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ be C_0-semigroups on X and Y respectively, then

i) If $(S_t \oplus T_t)_{t\geq 0}$ is hypercyclic then $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are hypercyclic.

ii) If $(S_t \oplus T_t)_{t\geq 0}$ is topologically transitive then $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are topologically transitive.

iii) If $(S_t \oplus T_t)_{t\geq 0}$ is chaotic then $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are chaotic.

iv) $(S_t \oplus T_t)_{t\geq 0}$ is mixing if and only if $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are mixing.

v) If $(S_t \oplus T_t)_{t\geq 0}$ is weakly mixing then $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are weakly mixing.

vi) $(S_t \oplus T_t)_{t\geq 0}$ is topological ergodic if and only if $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are topological ergodic. In particular, every topologically ergodic C_0 -semigroups is weakly mixing.

Proof:

i) Let $(S_t \oplus T_t)_{t\geq 0}$ is hypercyclic, since $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$, then by theorem (2.5) $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are hypercyclic

ii) Let $(S_t \oplus T_t)_{t\geq 0}$ is topologically transitive, since $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$ then by proposition (2.3) $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are topologically transitive.

iii) Let $(S_t \oplus T_t)_{t\geq 0}$ is chaotic, since $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$ then by theorem (2.5) $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are chaotic.

iv) $\Longrightarrow$ Let $(S_t \oplus T_t)_{t\geq 0}$ is mixing, since $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$, then by theorem (2.5) $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are mixing

$\Longleftarrow$ Let $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are mixing then for any two non-empty open subsets U_1, V_1 of X there exists $t_0 \geq 0$ such that $S_t(U_1) \cap V_1 \neq \emptyset \quad \forall t \geq t_o$ and for any two non-empty open subsets U_2, V_2 of *Y* there exists $l_0 \geq 0$ such that $T_t(U_2) \cap V_2 \neq \emptyset \quad \forall t \geq t_o$. Let $w_0 = max\{t_0, l_0\}$, then we have

$$((S_t \oplus T_t)_{t\geq 0}(U_1 \oplus U_2)) \cap (V_1 \oplus V_2) \quad \forall t \geq w_0$$

$$= ((S_t(U_1) \oplus T_t(U_2)) \cap (V_1 \oplus V_2)$$
$$= S_t(U_1) \cap V_1 \oplus T_t(U_2) \cap V_2$$

since $S_t(U_1) \cap V_1 \neq \emptyset$ and $T_t(U_2) \cap V_2 \neq \emptyset$, thus $S_t(U_1) \cap V_1 \oplus T_t(U_2) \cap V_2 \neq \emptyset$, $((S_t \oplus T_t)(U_1 \oplus U_2)) \cap (V_1 \oplus V_2) \neq \emptyset$

, and therefore $(S_t \oplus T_t)_{t\geq 0}$ is mixing.

v) Let $(S_t \oplus T_t)_{t\geq 0}$ is weakly mixing, since $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$, then by theorem (2.5) $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are weakly mixing .

vi) $\Longrightarrow$ Let $(S_t \oplus T_t)_{t\geq 0}$ is topological ergodic and since $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are quasiconjugate to $(S_t \oplus T_t)_{t\geq 0}$ and by proposition (2.6) then $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are topological ergodic .

$\Longleftarrow$ Let $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are topological ergodic

thus for any two non-empty open subsets U_1, V_1 of *X*

$$R_1(U_1, V_1) = \{\exists t \geq 0 \text{ such that } S_t(U_1) \cap V_1 \neq \emptyset\}$$

is synditic such that $R_+ \backslash R_1(U_1, V_1)$ does not contains long interval. And for any two non-empty open subsets U_2, V_2 on Y

$$R_2(U_2, V_2) = \{\exists\, t \geq 0 \text{ such that } T_t(U_2) \cap V_2 \neq \emptyset\}$$

is synditic such that $R_+ \backslash R_2(U_2, V_2)$ does not contains long interval.

Let $R((U_1 \oplus U_2) \cap (V_1 \oplus V_2)) = R_1(U_1, V_1) \cap R_2(U_2, V_2)$.

Now,

$$(S_t \oplus T_t)(U_1 \oplus U_2) \cap (V_1 \oplus V_2) = S_t(U_1) \cap V_1 \oplus T_t(U_2) \cap V_2 \neq \emptyset$$

Then $R_+ \backslash R((U_1 \oplus U_2) \cap (V_1 \oplus V_2))$ does not contain long intervals.

Proposition (3.2) [4],[5]:

There exists an operator T on a sequence space $\ell^2(\mathbb{Z})$ where such that T and its adjoint T^* are hypercyclic and the direct sum of T and T^* is not hypercyclic.

Remark (3.3):

The direct sum of two hypercyclic C_0 −semigroup need not be hypercyclic C_0 −semigroup as the following example: $(T_t)_{t\geq 0}$ be the C_0 − semigroup of operators on $\ell^2(\mathbb{Z})$, then the there exists $t \geq 0$ such that T_t and it's adjoint T_t^* are hypercyclic (but $T_t \oplus T_t^*$ is not hypercyclic from proposition (3.2)), thus $(T_t)_{t\geq 0}$ and $(T_t^*)_{t\geq 0}$ are C_0 − semigroup but the direct sum of $(T_t)_{t\geq 0}$ and $(T_t^*)_{t\geq 0}$ is not hypercyclic since if $(T_t)_{t\geq 0} \oplus (T_t^*)_{t\geq 0}$ is hypercyclic C_0 −semigroup, then every operator in $(T_t)_{t\geq 0} \oplus (T_t^*)_{t\geq 0}$ is hypercyclic, thus $T_t \oplus T_t^*$ is hypercyclic operator which is contradiction with proposition (3.2).

Consider the same example to see that the direct sum of two topologically transitive C_0 − semigroup need not be topologically transitive C_0 −semigroup (since $\ell^2(\mathbb{Z})$ is separable Banach space).

Proposition (3.4)[2]:

Let $(T_t)_{t\geq 0}$ be a chaotic C_0 -semigroup on *X*, then $(T_t)_{t\geq 0}$ is topologically ergodic.

Theorem (3.5):

The direct sum of two chaotic C_0-semigroup is chaotic.

Proof : Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are two chaotic C_0-semigroup on X and Y respectively, then by proposition (3.4) $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are topological ergodic and by the main theorem $(S_t \oplus T_t)_{t\geq 0}$ is topological ergodic, then by the same theorem (vi) $(S_t \oplus T_t)_{t\geq 0}$ is hypercyclic.

Now to prove $(S_t \oplus T_t)_{t\geq 0}$ has dense set of periodic points. Consider that the sets of all points $(x, y) \in X \oplus Y$ with periodic points x for $(T_t)_{t\geq 0}$ and y for $(S_t)_{t\geq 0}$ provides a dense set of periodic points for $(S_t \oplus T_t)_{t\geq 0}$, then $(S_t \oplus T_t)_{t\geq 0}$ is chaotic.

By the main theorem (3.1) and theorem (3.5), one can proof the generalization of the direct sum of chaotic C_0-semigroup as follow:

Remark (3.6):

Denote by $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ the C_0 -semigroup $((T_1)_t)_{t\geq 0} \oplus ((T_2)_t)_{t\geq 0} \oplus \dots \oplus ((T_n)_t)_{t\geq 0}$

Proposition (3.7):

Let $(T_i)_t : X_i \to X_i$ $(i = 1,2,3,\dots,n)(t \geq 0)$ is C_0-semigroup then the direct sum $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ is chaotic (

By the main theorem (3.1) , one can proof the generalization of the direct sum of mixing C_0-semigroup as follow:

Proposition(3.8):

Let $(T_i)_t : X_i \to X_i$ $(i = 1,2,3,\dots)(t \geq 0)$ is C_0 -semigroup then $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ is mixing C_0 -semigroup if and only if $(T_i)_t$ is mixing C_0-semigroup.

By the main theorem (3.1) , one can proof the generalization of the direct sum of topological ergodic C_0-semigroup as follow:

Proposition(3.9):

Let $(T_i)_t : X_i \to X_i$ $(i = 1,2,3,\dots,n)(t \geq 0)$ is C_0 -semigroup then $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ is topological ergodic C_0-semigroup if and only if $((T_i)_t)_{t\geq 0}$ is topological ergodic C_0-semigroup

By the main theorem (3.1) , one can proof the generalization of the direct sum of hypercyclic C_0-semigroup as follow:

Proposition (3.10):

Let $(T_i)_t : X_i \to X_i$ $(i = 1,2,3,\dots,n), (t \geq 0)$ be a C_0 -semigroup such that $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ is hypercyclic C_0 -semigroup then $((T_i)_t)_{t\geq 0}$ is hypercyclic C_0-semigroup for each $(i = 1,2,3,\dots,n)$.

By the main theorem (3.1) , one can proof the generalization of the direct sum of topologically transitive C_0-semigroup as follow:

Proposition (3.11):

Let $(T_i)_t : X_i \to X_i$ $(i = 1,2,3,\dots,n), (t \geq 0)$ be a C_0 -semigroup such that $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ is topologically transitive C_0-semigroup then $((T_i)_t)_{t\geq 0}$ is topologically transitive C_0-semigroup for each $(i = 1,2,3,\dots,n)$.

By the main theorem (3.1) , one can proof the generalization of the direct sum of weakly mixing C_0-semigroup as follow:

Proposition (3.12):

Let $(T_i)_t : X_i \to X_i$ $(i = 1,2,3,\dots,n), (t \geq 0)$ be a C_0 -semigroup such that $\bigoplus_{i=1}^{n} ((T_i)_t)_{t\geq 0}$ is weakly mixing C_0 -semigroup then $((T_i)_t)_{t\geq 0}$ is weakly mixing C_0-semigroup for each $(i = 1,2,3,\dots,n)$.

IV. Conclusions:

Let $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be a strongly continuous semigroups(C_0-semigroup) on X and Y, respectively then we have:

1- If the C_0 -semigroup $(S_t \oplus T_t)_{t\geq 0}$ is hypercyclic (topologically transitive, weakly mixing, respectively) then $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are hypercyclic (transitive, weakly mixing, mixing, chaotic, topologically transitive, topologically ergodic, respectively) C_0-semigroup.

2- The C_0 -semigroup $(S_t \oplus T_t)_{t\geq 0}$ is chaotic (mixing, topologically ergodic, respectively) if and only if $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are chaotic (mixing, topologically ergodic, respectively) C_0-semigroup.

And then proved it for n strongly continuous semigroup.

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